ICM 2006 Closing round table

Are pure and applied mathematics drifting apart?

Transcription by

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Mathematics is broadening its scope, developing in many new directions and interacting with a wide range or other disciplines, from information technology, social sciences and politics to engineering, biology and neurology, just to mention a few of them. Such an extraordinary expansion is also fostering a fruitful cross-fertilization between different fields of mathematics. With the ubiquity of computers, many fields of pure mathematics are incorporating experimental methodologies which in the past were only used in applied mathematics. In this new landscape, how do pure and applied mathematics interact with each other?

This was the topic of the panel discussion organized as a closing activity of the ICM 2006 on Tuesday, August 29, between 6 and 8 p.m. It was moderated by John Ball and organized by Marta Sanz-Solé.

This article consists of a genuine transcription of the presentations by the panellists and excerpts of some of the contributions by participants in the discussion.

Introduction of the panellists by John Ball, moderator of the round table

Our panel consists of five very distinguished mathematicians:

Lennart Carleson, Professor Emeritus at the University of Uppsala and a former President of IMU. His research interests are in Harmonic Analysis and Dynamical Systems.

All of our panellists are recipients of many awards, and I have decided that it would take far too much time to list them all, but I make an exception in reminding you that Lennart Carleson was awarded this year's Abel Prize, for which we offer many congratulations. I would like to say how much IMU values its collaboration on several fronts with the Norwegian Academy of Sciences and Letters and the Abel Fund.

Ronald Coifman, who is Professor of Mathematics and Computer Science at Yale University. His research interests are in Analysis, in particular Harmonic Analysis and Wavelets, and applications to Information Processing.

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Yuri Manin, who is Professor of Mathematics at Northwestern University, a former director of the Max Planck Institute of Mathematics in Bonn, and a former chair of the Fields Medal and ICM Program Committees. His research interests lie in Algebra and Geometry, in Number Theory, Differential Equations and Mathematical Physics.

Helmut Neunzert, Professor Emeritus at the University of Kaiserslautern. He is a founding member and former President of the European Consortium for Mathematics in Industry, and his research interests are in Kinetic Theory and Fluid Dynamics.

And finally

Peter Sarnak, Professor of Mathematics at Princeton University, and whose research interests are in Number Theory and Analysis.

Just in case I betray at some point my own views on the subject of the round table, I work in Nonlinear Analysis, especially the Calculus of Variations and its applications to Materials Science.

By way of introduction, perhaps I can show the two earliest instances I know of in which the terms "Pure" and "Applied" Mathematics feature in the literature. Here is the first one, the first issue of the *Journal für die reine und angewandte Mathematik*, Crelle's Journal, which appeared in 1826; in the contents of the first issue you can see several papers by Abel. Ten years later, the first volume of the *Journal de Mathématiques Pures et Appliquées*, Liouville's Journal, appeared, and here you see the papers are much more applied. The authors include Coriolis, Liouville, Ampère, Lamé, Jacobi and Sturm, so these were not bad for first issues of these journals! The old volumes of these journals, incidentally, are retro-digitized and freely accessible, which is where I obtained these images.

So pure and applied mathematics have been explicitly mentioned for nearly two hundred years, and were doubtless recognized as being in some way different before that, and our topic is whether they are drifting apart.

Each of our panellists will give their presentations, and then the subject will be open to the floor, and I hope we will have a lively discussion.

Contributions by the panellists

Lennart Carleson

Mathematics really has three different faces. The first concerns general education, and mathematics is of course just as important as learning to read. This is a very important part of society. The second relation to the outside world is mathematics as the language of science, and this is the way in which I am going to use the term "applied mathematics". The third aspect is of course a subject in its own right – a logical system. This is what most of us who are here right now represent. We must clearly understand that of the three, we are the weak part, and that it is absolutely vital for the continuation of our science that we love so much, to stay with good relations to the other two aspects.

So the answer to the question if pure mathematics and applied mathematics are drifting apart, I would say that we should make every effort that it doesn't happen. I would like to object somehow to the word "drift", because we are not really jellyfish and we can do something about this ourselves. So, I should like to concentrate on the aspect of the issue as far as it concerns the teaching of mathematics.

We like to talk about mathematics and applied mathematics in this order, which seems to indicate that applied mathematics is some kind of corollary of mathematics, and that we are looking for ways of applying this. This of course is completely wrong from the point of view of history. Through the years, mathematics has slowly been built from nature, and we have observed the remarkable fact that the laws of nature can be co-ordinated into groups and they follow rules. This started with geometry, of course, and numbers, and then we all know how difficult it has been to make movements into something logically reasonable. It has been around only for like two hundred years in a logical setting. If we take a subject like Probability – well, it may show that I am old, but anyway, it has been built as a mathematical subject in my lifetime really, and looking into the future, we can see new areas emerging where the mathematics is missing, and the most spectacular there is probably Computer Science.

Nevertheless, teaching of mathematics has always been done in a deductive way, that one goes from the general to the special, either as a logical system or as being applied. This of course is contrary to the traditional way of how things should be taught. Let me mention to you that this also has happened in my lifetime. When I started studying at the University of Uppsala in 1945, the first lecture was devoted to the Dedekind cut; we defined continuous functions with epsilons and deltas, we had axioms and we had definitions, and we had Riemann integrability and I don't know what!

As the number of students has increased, and their interest in the logical structure of the field has decreased, one has successively been cutting off these typically mathematical aspects of the mathematics teaching. To put it in a striking way, I would like to say that it is only applied mathematics that remains.

We have all of us, I guess, experienced how there has been pressure from other fields, from Physics, technical subjects, or even Biology, that they want to teach their own mathematics that we don't teach in the relevant way. I would like to say that I can somehow see their point, because we have not made any real effort to implement any kind of inductive way of teaching, that is, going from examples and cases and applications to the concept. You would think that the use of computers would have changed this in a drastic way, but that doesn't seem to be the case at all. We are still fumbling for ways of using computers in the teaching.

My thesis here today would be to say that we should make a really concentrated effort to make our teaching into inductive teaching. One can think of different ways of accommodating students with different interests. I have made a short list of what one could possibly do to change this. One of the essential points is clearly the attitude of ourselves, so to say, and also of our colleagues. Everybody knows that most – or many – mathematicians are really uninterested in things which are not leading to theorems or new statements, and there is a scepticism among our colleagues in other areas, that anything useful can come out of contact with mathematicians. And it is my real wish that we would all try to remedy this situation.

So what could be done? I have made three points here. The first is that one should have closer contact between basic mathematics teaching and the applied areas; at least in Sweden most departments which are applied have separate buildings and we don't really see them. Computing stays in one area and the applied people stay in another area. It would be my wish that people with applied interests would be involved already in the construction and the teaching of the basic courses. One would need to change the curricula in some suitable way, and try to speed up the use of computers in the teaching. We should really accept the fact that most students are not really interested in mathematics. Well, many are interested in their lives, but some of them are also interested in other areas, and one should accept that. We should not try to put our values on people who do not really want them.

One could compare these people with how you learn how to drive. Most people have no idea how a car works or why it works, but you can still use it. It is similar with people who learn mathematics; they only want to be able to read books and understand the formulas that they are taught in the other courses. I think one should not criticize this; one should accept that this is a really natural attitude. After all, mathematics as we know it is a rather sophisticated and not really applicable field. Also there should be for example, something like partial differential equations, everybody should have heard about that – they are going to meet it somewhere else.

Finally, there is a movement in the world around us to apply different sections of mathematics. There is pure mathematics and there is industrial mathematics, and there is applied mathematics and there is the teaching of mathematics, which have different organizations, and different meetings, and lead their own lives. I think that makes sense. But nevertheless there should be places where they meet, where the people from these different areas come together and can exchange experiences.

Ronald Coifman

I will try to address some of the issues of "drifting apart". Mathematics is a big ecological system of different species of mathematicians, and each species likes to think of itself as better than the others. The issue, though, is that the world of mathematics has expanded dramatically. Our universe is so much bigger, that everybody is drifting apart from everybody else, but in reality we enrich our lives substantially. What we have seen, I would say, over the last two decades is the insertion of the computer into our lives – of the digital age. Now that insertion is occurring at a variety of levels, I mean on the sort of everyday ability to collect numbers, and collect data, to the ability of the mathematician to actually run experiments in mathematics, and I would say that if Gauss were here he would probably run experiments like crazy. Leibniz too, and all of those. And if you asked them the question "Are they pure or applied?" they would just laugh at you.

In a way, the drift that we seem to see is mostly social, but not necessarily intellectual. We have seen in this congress many, many people, and many of their talks are related to outside scientific fields, or inspired by outside scientific fields, and so on. The way I see it now is that in fact the need for mathematicians, pure mathematicians, not necessarily in the areas of applications, is actually much greater than it ever was. This is sort of a pre-Newtonian time, anyway, and we don't have the mathematics to do the simplest of all things. We don't have a descriptive language to describe various things, and we don't even have the ability to define the geometries that need to be defined in the real world.

I think there is a serious opportunity here for mathematicians. That opportunity, to realize it, we need to follow what Lennart just said: revamp our teaching style. I am not advocating changing what we teach, just the way that we do it, in a way that makes it more transparent for people who don't necessarily want to invest the same effort as somebody who was born with mathematics in his blood. The opportunity is really the same that occurred in the physical scientific revolution in the time of Newton and Leibniz, which is that there is a need to quantify and describe specifically and precisely all kinds of phenomena that surround us. And the number of phenomena and their complexity is really growing exponentially, just because we can't, and so digital data is generated in overwhelming quantities all over the place, whether this is web data, document data, sensor data… and we're stuck!

Let me give you an example. The data may be that you have the results of some medical tests, blood tests, or some number that you get, and you want to evaluate the function, which is how healthy you are, what health score you can have. We are dealing with a very simple object, which depends on ten or twenty parameters, and we don't have the tools to approximate them. We have heard around the board today, telling us something about some potential tools, but this is a most elementary object of mathematics, which is a function, except that unfortunately the function depends on many more parameters than we used to do before computers – the number of parameters may be ten, twenty – in reality we may have ten thousand or ten million of them. And the tools are not there. So what is needed in this context is for somebody to think very deeply and come up with potential solutions – so mathematicians, pure mathematicians, and their modes of thought are necessary. Computer scientists are not trained for the job. I know of a multitude of examples of that, having to do with acoustic calculations, electromagnetic calculations. Unless you completely revamp the mathematics and reorganize everything you need to do, rebuild the language for describing the objects, you can't go anywhere. It doesn't do us any good to just throw a big matrix at some problem and say this is a linear problem, we can invert the matrix and do this or that – it doesn't do anything.

The obstacles confronting us are actually much more monumental than they ever were, and they require the ability to build the language, to organize very complex objects, to organize them in a variety of geometries. I just described a minute ago the acoustics in this hall. That's a problem that, say, twenty years ago nobody could calculate, and even now I doubt if there are more than maybe ten people in the world who can actually calculate anything, because the object – you hear the echo and everything – and the acoustics here are so complex that, unless you build a language – you cannot use formulas, because formulas will not deal with that – unless you build a new language to describe it, you are dead. So that's one opportunity.

Similarly, by the way, if you go to the social sciences, or to, say, just documents, or machine learning, or other fields of that sort, the language and the geometry to describe the objects that you want to manipulate and their internal relations between them, all of that is yet to be invented. We need people of the kind we had at the beginning, a few of them we had last century, like Shannon, von Neumann, Benoît Mandelbrot, who is here, who recognized certain geometries that people consistently ignored, all of those are opportunities for mathematics, and that mathematics is pure, although the opportunities and the challenges are coming from the outside world, but in the past it has always been that the outside world was probably the most inspirational in actually pushing us towards discovering structures.

It's very nice to be motivated by internal ideas, but I don't think one should be that arrogant in thinking that we know everything that needs to be done – we should let the world tell us. As I said, invention is really what's needed. And that's the crafting of tools, and the people who craft the mathematical tools are people who are interested by the tool and the applications – the test, if you wish, that the tool is effective. But the people who build tools are mathematicians. They may be working, like Shannon, as an engineer, but he built mathematics, and it is pure mathematics, no matter what we say. In fact, it's being used consistently everywhere in pure mathematics. Is Probability an applied field? Of course not; it is motivated by application.

We see in various communities, like the machine learning community, the bio-informatic community, the computer science community, we see emerging a variety of methods which are mysterious, somewhat ad hoc, but extraordinarily successful. The question really is: what are the underlying structures that enable us to assert that certain methods will work or will not work, and what they are capable of achieving? And what are the real deep structures underlying it – this is the job of the pure mathematician.

Yuri Manin

I am certainly a pure mathematician, and what I would like to discuss here is the implicit presupposition that lies at the base of our distinction between pure and applied mathematics; namely that mathematics can tell us something about the external world, that mathematics can be a cognitive tool, although it doesn't look like a cognitive tool. It doesn't study anything specific in the surrounding world.

So in order to understand how mathematics is applied to the understanding of the real world, it will be convenient for me to subdivide it into the following three modes of functioning: model, theory and metaphor. A mathematical model describes a certain range of phenomena, qualitatively or quantitatively, but feels uneasy pretending to be something more. Probably one of the most successful early models is Ptolemy's model of epicycles describing planetary motions, about 150 years of our era, and one of the latest models which does call itself a model is the standard model describing the interaction of elementary particles, around 1960. Generally quantitative models cling to the observable reality by adjusting numerical values of sometimes dozens of free parameters – at least 20 in the standard model. And such models can be remarkably precise, and there are of course qualitative models offering insights into stability, instability, attractors, critical phenomena.

As an example, I quote a recent report which is dedicated to predicting a surge of homicides in Los Angeles. As a methodology it uses pattern recognition of infrequent events. Result: "We have found that the upward turn of the homicide rate is preceded within eleven months by a specific pattern of the crime statistics; both burglaries and assaults simultaneously escalate, while robberies and homicides decline. Both changes – the escalation and the decline – are not monotonic, but rather occur sporadically, each lasting some 2 to 6 months."

Now the age of computers has seen the proliferation of models which are now produced on an industrial scale, so numerically, and very often used as black boxes with hidden computerized input procedures and oracular outputs prescribing behaviour of human users; for example, in financial transactions.

What distinguishes a mathematically formulated theory from a model is primarily its higher aspirations. A theory, so to speak, is an aristocratic model, or if you wish a model is a democratic theory. A modern physical theory – and also all physical theories – generally postulate that it would describe the world with absolute precision, if and only if the world consisted of some restricted variety of stuff, massive point particles obeying only the law of gravity – things like that. The recurring driving force in generating theories is a concept of reality beyond and above the material world; reality which may be grasped only by mathematical tools, from Plato's solids to Galileo's language of nature, to quantum superstrings.

A mathematical metaphor, when it aspires to be a cognitive tool, postulates that some complex range of phenomena might be compared to a mathematical construction. Probably the most known mathematical metaphor now is the artificial intelligence. We know very complex systems which are processing information because we have constructed them, and we are trying to compare them with the human brain, which we do not understand very well – we do not understand almost at all. So at the moment it is a very interesting mathematical metaphor, and what it allows us to do mostly is to sort of cut out our wrong assumptions. If we start comparing them with some very well-known reality, it turns out that they would not work.

My feeling is that mathematical metaphors… more often than not some models and theories also are used as mathematical metaphors, and as such they then contribute to changing our value systems, or at least influence our value systems. I am a little bit concerned about the proliferation of both mathematical models which are hidden inside computer hardware and software, and also I am concerned about the moral issues that are not often addressed too in discussing implications and in discussing the utility of mathematics for society.

Just to very briefly show you what I am concerned about, I will quote a recent sentence – two sentences, actually – from a recent book "Mathematics and War". I think the sentences were written with bitter irony. "Mathematics can also be an indispensable tool. Thus when the effect of fragmentation bombs on human bodies was to be tested, but humanitarian concerns prohibited testing on pigs, mathematical simulation was put into place."

Helmut Neunzert

Now you get a little bit of a contrast programme. After a meta-theory of applied mathematics, we go back down to Earth. Maybe that is the difference between a pure and an applied mathematician, and you see it now live. But I must say we are not drifting apart. With respect to Yuri Manin's last sentence, I totally agree with him. But from the point of view, I would like to change a little bit our point of view now.

When I have spoken with people – Are pure and applied mathematics drifting apart? – some said. "Oh, this is this old question…". Some say "Yes", some say "No". I believe it is really the question of the department – if the people in each department like each other, then it's fine. If they don't, you have a drifting apart. But I would really like to change…. We always do as if mathematics would be the mathematics we do. We academic mathematicians are the world of mathematics. Are we really?

There is a second world, in my opinion. There's a second world of mathematics, and in this second world of mathematics almost all our graduates live. Those people we educate; they are not in general entering our world of academic mathematics. They go somewhere else. They go into industry, banks, insurance companies, R&D departments. There is a second world of mathematics outside of our world, outside of academia – in industry. And this is what I would call mathematics as a technology. And we should all be very happy that mathematics has become a technology, as Ronald Coifman has already described. It is really, for us also – even if we are pure mathematicians – it helps us a lot. I will come to this point later.

This mathematics as a technology, this second world of mathematics, is it pure, or is it applied? Let me describe to you a little bit the results of a project I had together with a psychologist and a historian. It is nice for a mathematician to work with other people. It was a Volkswagen Foundation project, and we were trying to find out what happened to all the graduates in Germany, in mathematics, in 1998. This is eight years ago. These psychologists are unbelievable. They have really asked in questionnaires these people unbelievable questions. I would have never dared to ask "Are you planning to get children?" and "How is the relation between your profession and your family?", and so on. But she did, and the people answered. And the question is: what have they done in the next eight years? What happened to them? Did their dreams, wishes, come true or not?

We had 3,000 graduates in Germany in 1998. That's quite a lot. I think the number today is even higher. Mathematics is very attractive in Germany – you may ask why.

And of this 3,000, 1, 400 went into high schools, so they normally become high school teachers. The other ones – 1,600 – made their diploma (or nowadays, a Master); 600 of these 1,600 were willing to answer a questionnaire. This is a very good sample. We asked these people again in the following years – in 2001, 2003, 2006. And what happened? What came out?

First of all, of these 1,600 – if you take this as a sample – only 10% became academic people. They entered universities or research centres. So we speak always about this 10%, and we forget the other ones. 80% (10% disappeared somehow) work as software designers in R&D, in banks, in insurance, in consulting, and so on. Do they do mathematics? They don't do much pure mathematics, I must say. I have asked 20 former PhD students of mine who work now in industry, and they were laughing and saying "Are you kidding me? If you ask us, do we do pure mathematics or applied, of course, we do not metaphors but models, and algorithms, if we do mathematics at all".

Not all of them do real mathematics. So if I see it correctly, 25% of all our graduates are doing mathematics in industry. The rest have changed – they do management, they do something which is not really mathematics. Now compare 25% to the 10% who go into academia. I claim that the second world of mathematics is a little bit larger than the first world, and we should keep that in mind.

There was a citation in the German Mathematical Society News from a mathematician who works at IBM. He said we should not overestimate the value of mathematics in industry. It is the midwife but not the mother of innovation. But maybe it's good to be a midwife, a very active midwife, which gives so many births to so many good innovations. So, you see what is the result; in this second world, at least as many people do mathematics as in the first world. But of course they do mainly applied mathematics. Now are these people drifting apart from pure mathematics? What would you say? There is this second world and the first world of pure mathematics – I don't think they know whether they are drifting apart. They don't see each other. It's so far away. How many of this second world are at this conference? If I am very optimistic, I would say 10 (out of 4,000). So you see there is a real, big difference between this applied mathematics in industry and the pure mathematics happening here. And this, of course, is very, very bad. I think this is a damage for both worlds.

It's a damage for the second world, for the world of mathematics in industry. Of course, there were some arguments already – we heard from Coifman and others – of course, we need more mathematics to make it better. It is not at all good in many things. In medicine, for example, we are totally missing good models which really describe the complex system of a body. So we would urgently need good mathematics which deals really with their problems. But are mathematicians really dealing with their problems? Yes, if they fit, in their own way. If not, I doubt it.

And for the first world, for our academic world, mathematics as a technology offers a lot. I think it offers new challenges – that was also already said. Many, many good problems come from this outer world. They add, certainly, public prestige. This second world adds money, if we have contacts with them, and it attracts students.

That's not such a minor thing. I think really that both worlds need each other very urgently, but we have to do something. We in the first world have to have open minds. We have to go into industry and see their problems, to speak with them, to get in contact with them, so they know we care about them, and vice-versa – they would be interested in what we are doing.

Peter Sarnak

I speak as a pure mathematician. I have a very keen interest in mathematics broadly, so I have tried to follow most mathematical topics, and I've tried applied maths too – it is much more difficult. My main credentials here being that I was a double major in math and applied math and in fact I had a difficult time choosing between pure and applied math. My views, I think, are going to be a little extreme towards the pure math side, but I think that a good proportion of the people in the audience are on the pure side, so maybe I shall try that angle. As Helmut says, one's views are highly influenced by one's local daily interactions. What happens in your department, and the discussions you have with your colleagues impacts you, and you'll see my views are impacted by my colleagues.

So firstly, are they (pure and applied math) drifting apart? Well, certainly we have to take into account this inflationary process of everything drifting apart, but even given that, it is my feeling that they are moving apart. I have been in mathematics for thirty years, and in this very short period my own experience is that it's not exactly what it used to be thirty years ago, and I think one of the big impacts is the computer which has changed how we go about our business.

Is the drifting apart a problem? I think it is a problem, but not one that's too serious. I think that math/applied math should evolve naturally like science, with good science surviving, and not such good science going away, and that is what should be allowed to happen here too. However, there are alarms, and we have heard a few suggestions, which sound very good. I will mention some alarms sounded by some of my younger colleagues, who I think we should definitely listen to.

Anyway, I'm going to take what may be a very controversial way of dealing with this question of whether they are drifting apart, by trying to see what is good math and what is good applied math, and if there is anything in common in fact between these two activities.

To give a formal definition of what pure math is would be very dangerous. I am sure I wouldn't get out of the hall by the end! But without defining good mathematics – and I am talking here about pure mathematics – we can all recognize it when we see it, like a fox when it sees a rabbit. You can see something that is really good, exciting and cuts to the bottom of a problem. I think the key ingredients, the cycle of ingredients in mathematics are firstly insight, mathematical insights, which often become conjectures, theories, language – these are crucial. But to me, the Holy Grail of mathematics, and something we can never give up, is that of proof. To me, once there is no proof I am not sure it's mathematics. At least, that is my take on it. That's the difference between mathematics and any other science.

Given the exciting week we have had here, let me give Thurston's geometrization conjecture as the epitome of this sort of good conjecture. It is a conjecture which when put forward immediately clarifies what one is looking for, as with all great conjectures – if they're true, they're great; of course, if it turned out to be false it would be much less interesting – but it appears to have turned out to be true. It's a unifying conjecture. It clarifies the shapes of 3-dimensional topological spaces, and it was not something that was obvious, but something that was built up with many examples and theories that he developed in order to come to that conjecture. So conjecturing is, of course, a major part in our subject, but Thurston, in thinking about this – I have not spoken to him recently – but I think you could say he was driven internally rather than by applications, and it is a damn good conjecture,even if it is driven internally.

Many fields have such powerful conjectures that unify the theories. But in describing this cycle I am not only talking about these special great conjectures, or only about mathematics that is unique and comes once in a blue moon. So there's that part of the cycle, which is conjecture, and then there is proof of the conjecture or more often proofs of approximations to the conjectures. As I have said before, and I will repeat, without proof, it's not our subject. So we really need that part, and as it seems clear now with Thurston's conjecture that Perelman has indeed proved it. This is as good as it gets. There have been other successes of this magnitude but we cannot judge all of mathematics by such high standards. This is what we strive for, and I think many people – young people – very strong people, go into math with such high aims in mind. Of course all of us except very few are disappointed, if our aims are so lofty.

I believe these central conjectures are what drive the subject. So we have a cycle of conjecture, theories built around the conjecture, solutions with good solutions generating good problems, and these develop further conjectures and further theories, and this cycle seems to repeat.

It looks – for someone from the outside – like a recipe for disaster, something completely internally driven. A recipe for a sterile subject. In fact, even within pure mathematics, subjects that are introspective, that interact with no other area, that only three experts in the world can talk to each other about (and one of them submits a paper to the Annals, and you get the second expert's opinion, and it of course says that this is the best thing ever written, but you can't get a third opinion) that's a problem. And such subjects naturally shrink. I think that allowing their natural evolution is the best way to let these things run.

Having indicated that good pure mathematics might seem to be driven solely internally, I want to argue that it is not. In fact I would argue that pure mathematics needs other sciences as badly as they need mathematics.

Now we have heard that we need to develop more mathematical theories for more applications and that there is much demand for such. These are very important to make mathematics as active as it is, but we happen to be living in a golden era of pure mathematics, as is witnessed by the very striking successes that we have seen in recent years. I don't think such success can happen in a purely introspective world. Are we at this stage only because of the some special giants that have graced our field in recent times? I don't think so; I think that we are impacted from the outside and often in subtle ways.

Let me continue with the Perelman example a little. To repeat what Hamilton said in his talk here last week. Perelman's work depends heavily on Hamilton's work, which in turn is based on Ricci flow, and as Hamilton explained, the Ricci flow was motivated to him by Einstein's equations. In fact, the process he went through was the very process that Einstein went through in writing down his gravitational equations in equating the only invariant second order tensors that are around. So Hamilton when he was forming his Ricci flow equation,25 years ago, and at that point everything was very experimental, relied on this thing that he knew about Einstein's equations. This gave him a lot of confidence that he was on the right track. So this is a very – indirect, you might say – means of saying Physics impacted this particular programme, which on the face of it seems very internal. But it did give Hamilton the confidence to set off in the right direction and also as Perelman has indicated, his entropy idea which is one of his critical breakthroughs was inspired by a physics paper. I could give you many, many examples of similar things, where the input comes often from Physics, but also from other fields, for example Computer Science. Of course, in more complex applied math and engineering, such an impact is a little harder to see, but we do live in a world where we impact each other, and I don't believe we are a closed cycle, and we do need the applied side.

Just on a sociological level let me tell you, based on my experience,how to tell the difference between a theoretical physicist and a pure mathematician (I am not sure where the applied mathematician fits here). A mathematician will come into you office and tell you how complicated what he is doing is "My proof is highly nontrivial it is a thousand pages long" It is a strange discipline where to convince someone of something you have to write or make use of thousands of pages of complex arguments. Probably it means that one hasn't yet really understood the issue at hand. A Physicist comes into your office and he is always trying to tell you how simple and short what he is doing is and moreover it is universal and explains everything. He is lying because he is hiding 50 of more pages of calculations that he declares are trivial. This difference in presentation explains some of the difference in culture between these disciplines. The truth is somewhere in between. The idea that for something to be good it must be long and complicated is something that has evolved in certain quarters of mathematics, and it seems strange and wrong to me. In the end, we are always looking for the simple thing, and the real truth is somewhere in between these extreme views.

So my point is that while it is well recognized that science requires mathematics, progress in mathematics relies directly or indirectly on its applications and sub-areas of mathematics on their interaction with each other as well as with outside applications. When we turn to good applied math, I have very little right to talk, so I asked a few people for their opinions. But let me first take a completely extreme view. I always remember this article by its title. This is an article by Halmos called 'Applied Mathematics is Bad Mathematics'. I did not read it until I was asked to be on this panel, at which point I thought "I wonder what he's got to say?". So I went and read it, and it is very entertaining. He is a good writer, but I think he is misguided. There are some bad points in the article, even if it is entertaining, and there are some interesting points. One bad point (in my opinion) is very relevant to what I am saying. He argues that mathematics can exist without applications – I am talking about applications generally, not just necessarily applied math but all other sciences. And he says mathematics can exist and will exist without applications, but the converse, he would argue, is false. I don't agree with him at all. I think mathematics cannot exist and flourish without the applications.Even the most pure math would not be where it is today if it were not for the applications.

Now, if you look back far enough, if you talk about Leibniz or Newton, they are philosophers, mathematicians and applied mathematicians simultaneously. But today, with everything requiring people to be very specialized, it's much harder to be universal. Even so, I strongly believe that the impact of applied math or applications is crucial to the development of math. Now I asked a colleague of mine, Weinan E, quite an opinionated young applied mathematician,and whose opinion I value, to give me a definition of what is good applied mathematics. He responded as follows: "It has to be relevant to application areas, whether the application area is in science, engineering, technology or industry". That's one thing he demands. The second thing – and this I found interesting – "It has to help in putting the relevant application area on a solid scientific foundation. This typically requires laying out the mathematical foundation". So he is emphasizing this foundational aspect that the mathematician is supposed to do in another science. Then he added – and this worries me: "Personally I'm very worried that mathematics and applied mathematics are gradually drifting apart", and he says this is particularly a worry in areas in which he works. He works in computational PDE, scientific computation.

Let me end here by saying there is obviously a common ground – and it was always the common ground for mathematics and anything else – and that is the search for those breakthrough ideas and insights. When I was young, I felt it was this common ground that made me feel there was no real difference between pure and applied mathematics. However, I am beginning to feel – and maybe I am just getting old – that there are differences. So as I said, in pure math, I cannot imagine mathematics without proof – or rather I cannot imagine it where proof is not important – where people say, well I do not even care about a proof. That would bother me. In applied math, the big issues or insights in the explanation of some phenomenon are central. It's not clear to me that proof is valued so much in applications. I often go to a lecture and the person ends by saying, especially if it's someone who has got a code or something: "My code works. Why do I need a proof that it works?". Well, it is a little hard to argue with something that works, that it requires a proof, although presumably in an ideal world the proof will give further insight,or an applied insight might lead to a proof. And that was the kind of ideal world that 25 years ago was what I thought it was all about. But now I think this drifting apart is occurring, and I think you see this with the scientists involved, and I'm just an observer.

So let me end by saying that while it seems that the goals and the requirements of pure and applied math are diverging, even taking into account inflation,I think myself that evolution will take care of things. But I am quite concerned by the comments of Weinan E, and the comments of my fellow panellists, who also seem to be quite concerned (well, maybe not all of them – but some of them).

Contributions from the floor with answers by the panellists

John Neuberger opened the discussion by saying that pure and applied mathematics are drifting apart. "Mathematicians are badly needed in industry, but one should begin to connect with industry", he said. He gave some practical suggestions, like knowing what students will be faced with, and then letting the teaching be influenced by this. For example, since most mathematical questions from industry are phrased in terms of computing, students need to understand about computing. In his opinion, fruitful consulting arrangements are not so easy to come by, but individual efforts to make some connections with industry would help to modify the imbalance. He felt suspicious of a bureaucratic solution trying to pair mathematicians to industry, although this is a possibility. Hard frontier scientific problems demand the abilities of pure mathematicians, and they become applied once they get involved in them.

László Lovász claimed he did not feel so much that these sides of mathematics are drifting apart. He said that "Maybe that's because I grew up in a branch which was considered applied". He considered his fields – discrete mathematics and graph theory – an area of pure mathematics which has good applications. However, he pointed out that applications of mathematics arise in many different ways. "In the programme of this congress one could find excellent examples. For instance, Professor Itô won the Gauss prize for work which he did by motivations that I would consider completely pure and internal mathematical motivations, and it became extremely important in very real life activities, like for stock option pricing. Another kind of application is where the mathematician looks at some phenomenon and begins to think about it – what kind of mathematical phenomena could mimic this, could help to understand this. This is like the Nevanlinna prize-winner Jon Kleinberg – how he was looking at the internet and how it relates to the eigenvalues of the corresponding matrix, or Shannon by looking at channels of communication came up with the fundamental ideas of information theory. And then there is also applied math, which Professor Neunzert was talking about", answering direct questions from the real world. As another example, he mentioned Martin Grötschel's lecture at the congress, describing where one actually has to produce applicable results. To ban any of these different types of research, or to consider any of these as inferior would be a very serious mistake. It is the intellectual content of the work that should matter, and not its particular form. All three are terribly important for us. The level of mathematics that other scientists need varies very much, and sometimes such a simple thing as solving a quadratic equation could be extremely useful. And in other cases, of course, you really need very sophisticated and new mathematics. But he does not feel pure and applied mathematics are drifting apart, really. He concluded by saying that "in mathematical areas which are thriving, there is always a lot of exciting connections with applications and with areas that come from the real world".

A participant from the floor, who introduced himself as a pure mathematician, was keen to have definitions and in particular one of an applied mathematician. He claimed not to be able to understand who was drifting from him. He asked whether people who are considered applied mathematicians should have a mathematical education.

Peter Sarnak responded: "One of the things that Weinan E was most concerned about was that people that he defines as doing applied math be educated mathematically in the traditional way. He felt this was really important. When he said he was very concerned, he in fact added that one of his concerns in this direction of education of students is that somehow the applied math community was not attracting the very best mathematically talented people. So I am just personally answering your question in connection with what he felt. I agree with you – who is drifting from who? That's a good question, but I think there is a difference in what an applied mathematician does and what a pure mathematician does. Lovász mentioned Itô – he had a major impact on the world, but he was not motivated in what he was doing by applications. Most pure mathematicians feel they are working on problems purely because of trying to understand numbers, geometry, the theory of equations more deeply, but the application which we all hope will come – and if it wasn't for that, it would not be that important a subject – but we do have a different way of going about things. Applied mathematics, I think, has to have applications in mind, and the style is very different. If you pick up a journal in pure mathematics, there is a theorem, there is a proof, or there is an attempt at making a certain kind of discussion. Many applied math or scientific journals you pick up, they are talking about a phenomenon, and there are pictures and phenomenology, which is all very interesting science, but we do things very differently. And I think that where these things surface is in your own department. So I would like to say that I agree with you, that we are all the same, but I think the way we go about things is very different".

Martin Grötschel started by saying that he would like to address one issue that has been implicit in the previous discussion: psychology, or more precisely, the psychology of mathematical institutions. He said: "Often, pure and applied mathematicians are located in different buildings. Lennart Carleson, for instance, stated 'in Sweden most departments which are applied have separate buildings'. This contributes considerably to the feeling of them and us. Many of us have lived alternating mathematical lives. I have been a pure mathematician for a while and now I am very applied, but I value both sides. One of the great experiences in my mathematical life was that, when I moved to TU Berlin, I noticed that there were no separate institutes of applied and pure mathematics – they were all together. Such an organization is actually something very precious, you can observe that people at TU Berlin – research mathematicians and students, scientists from other disciplines – float between the various areas, depending on their current interests, and the distinctions between pure and applied vanish. I find this exchange extremely positive for all sides. This will help keep the various parts of mathematics together. I believe that the process of "cleaning" mathematics – which many universities went through in the last 50 years – by driving certain areas from mathematics into applied mathematics, and from applied mathematics into other institutions had really negative effects. If I look at the USA situation, applied mathematics, by and large, is defined by "dealing with differential equations". The mathematical optimizers are mostly in industrial engineering or management science departments, many discrete mathematicians belong to computer science departments, statisticians are everywhere but rarely in mathematics, and so on. Can one find really good arguments for such a distribution? What is the reason for this? The current separation of pure and applied mathematics is not a "logical consequence" of different ways of doing mathematics. An unbiased look at the historical development reveals that, in most cases, power games, financial considerations, and personal conflicts within the mathematical community considerably contributed to this effect. It is easier to separate people than solving conflicts. This has been a bad evolution resulting in clustering processes which in turn have led to the psychological situation we are faced with at present. It is my belief that we should try to bring these separate groups/clusters back together. I think that the "institutional unification" would resolve many of the issues we are discussing here. I believe that the institutional and spatial separation contributes a lot to the feeling expressed here by many that there are trenches between various parts of mathematics, in particular between pure and applied."

David Levermore thanked the panel for a very thoughtful discussion and commented as follows. "I have a hat that is a pure hat and an applied hat, so if I can try and speak for both sides of the issues. I think Martin raised a very important point – institutionally what can we do? I think the issue is not so much we drift apart, I think that criticism is valid because we do have control of this. I think the phenomenon has to do with the expansion of human knowledge and endeavour, and that all disciplines to some measure are confronted with this, in particular universities, but also all institutions, not just academic". Then he mentioned some models growing in the USA in response of what he termed balkanization: "One model that does exist in the US, and is thriving in some institutions, is the development of centres – centres that focus around maybe an application or an idea, that brings together people, a mathematical paradigm or an engineering paradigm, or whatever, to work together, learn from each other and stimulate each other. Just, for example, the mathematics department at Maryland is tied to a Norbert Wiener centre in applied harmonic analysis, which involves pure mathematicians and engineers. We have a list of several institutes like that. And I think that if we put our minds to it, we can overcome these sort of intellectual barriers that separate us artificially, because ultimately I think the picture the whole panel has painted is that this is a human endeavour and is really the right one, and I look forward to a very good future".

The moderator of the round table, John Ball, described something about his own experiences of applied mathematics. "I work in the calculus of variations, but also in its applications to materials, and I've written papers with electron microscopists. Now I believe in the value of theorems in applied mathematics, and I believe in the value of theorems for telling us when computer codes work. To me, it seems that the three elements of modelling, analysis and computation all feed on each other to improve what goes on. But I think there is an interesting process when you start working in a new area, a new scientific or some new application area, and something gets you interested in it and then you see that there is something mathematical and you learn a bit more about it, and at some point you have to have some confidence that you can offer something to this field. At the same time, you have not done a degree in the bio-sciences or materials or whatever subject it is, and so you have to somehow be humble and put in a lot of work to learn at least a little piece of this area so that you can break down these language barriers. I think that is a really exciting process, but to start with you may encounter some resistance from the people in the area, and one piece of advice I have is always to cut out the middleman – or woman – and talk directly to the person who is doing the experiments and try maybe to avoid some of the intervening theory", he said.

The colleague who previously asked for a definition of an applied mathematician asked the panellists if they believe that an applied mathematician is a mathematician and if he or she should have the qualification of a mathematician.

Peter Sarnak: Well, my answer would definitely be "Yes", but I think I'd better let some of the other people on the panel give their view of it.

Helmut Neunzert: Would you say "Yes" with respect to the question that he must have a mathematical qualification to be a mathematician?

Peter Sarnak: To a certain extent… he needs to know certain basics, absolutely.

Helmut Neunzert: How many mathematicians in this room do not have a mathematical education? I mean, I know many physicists who have become very good mathematicians later on. Would you not count them?

Ronald Coifman: If we follow Peter Sarnak here, he would tell you that anybody who can prove theorems qualifies, right? The only issue is: What do you mean by proving theorems? I know what you mean but I think you did not think about it enough. An applied mathematician would come up, say, with a computational algorithm, then the theorem involved in that algorithm is that he can by a certain scheme compute something to some precision. That's a theorem, right? And the goal there is not to climb the Everest and prove some old conjectures or do something that will impress your colleagues. The goal is to achieve, to solve difficult problems, to find the tools to do it, and it's really the intellectual challenge involved which maybe will qualify him as being a good mathematician or a good applied mathematician – I don't think it makes any difference. It's really the intellectual novelty and content that will allow you to think of the person as a mathematician. I think Shannon was an engineer, right? And there is no way you could say he was not a mathematician.

Robert Kohn found very reassuring that there was some difficulty in defining, in separating the applied mathematicians and the pure mathematicians. "I think that something that nobody took the time to do was to talk about how mathematicians pure and applied are really rather different in our mission, in our world view and in our functioning, from the other sciences. The most important thing here is not that we worry about creating separations between pure and applied mathematicians, or defining those two. It's really more about making sure we don't leave a big gap between mathematics on the one hand and other areas of science on the other. In the areas I work in, which tend mainly to be close to the physical sciences or finance, the mathematician's job is to think about whether the algorithm really works, to think about what are the properties of this model, to solve problems, to look at whether opportunities have been missed by not bringing to bear the right set of tools, to develop new tools sometimes if they are called for in the application area. And there is nobody else out there who is going to do that if we don't. Fortunately, I think that to a large extent we are not drifting apart. I disagree with many panel members, and I think that the talks at this meeting are the best possible proof of that", he said.

Two participants asked whether there is some need to create a new type of mathematics and even a new type of science for investigation of the reality, as was suggested by the Russian mathematician Andrei Kolmogorov in the last years of his life. For example, to deal with self-references, self-organization, as one could find in biological systems.

Ronald Coifman answered: "It's a terrific question. In terms of dealing with the kinds of mathematics that you need to deal with, say, biology or social sciences, or the more complex structures where every piece of information you measure is linked to the others. I think what seems to be emerging is something like what emerged in physics a long time ago, when Einstein decided that physics is geometry, and that you can describe the physical equations as basically the geometry of space-time, and later on in Yang–Mills and gauge field theories, somehow the physicists got to the point that the geometry encapsulates the relationship between all objects around us. I think we see this emerging in the analysis of actual data of various kinds, whether it's data on the web where you can actually do a fast search by relating every unit of the web to each other and doing something on the global geometry of the web, in order to get the Google rank, or some others. It is a subtle and profound idea, possibly, and we see it happen in biology and neurology and everywhere else. It is a web of relations between objects which encapsulates their internal geometry or their content. That's my view at the moment. I think it is just emerging".

Jean Pierre Bourguignon, director of the Institut des Hautes Études Scientifiques in Bures-sur-Yvette, France, and a specialist in differential geometry and global analysis, stressed the need of conducting the discussion at three different levels to avoid further confusion. "Three levels need to be distinguished: the first one is really the science itself, and if you speak about the science itself, I think the terminology separating applied mathematics and mathematics is not a good one, as was pointed out already by some people. The second level concerns us as professionals; most of us are really making a living by teaching, so it means that what happens to our students is something that should be of primary importance to us, and from that point of view I find the remarks by Professor Neunzert very adequate. That is, if so many of our graduates are really working in the world of technology, I think we, as professionals, need to know more than a little bit what will be our graduates' environment. And the third level takes us as scientists, and because of the growing impact of science on society, we also have another role, namely to provide answers to problems posed by society at large. In this context, we are also asked to interact with other scientists, and more generally with people working in the world of technology. It seems to me that, depending on the level one considers, then the drifting that is the subject of the Round Table has to be measured by different means. It is certainly true that if one takes the first angle, I think there is no drift, because – and this congress provided a very good proof of it – we have ample evidence of the fantastic impact of new questions coming from technology on mathematical research. If one takes the second point of view, then the growing number of our graduates who work for companies, and therefore really using the skills we give them in a very applied way, forces us to get a better knowledge of these applications. And the third level raises the question of the way in which we can contribute to the scientific enterprise, that is more and more shaping the world, and shaping the world means good and bad things at the same time, but certainly new responsibilities – and that is probably the one in which the ethical dimension of our profession is very important –", he said.

Anatole Joffe joined the discussion by pointing out that though the subject under discussion was far from new, the matter was still of great interest. Plutarch (Parallel Lives: Marcellus) in the context of Archimedes' involvement with the defence of Syracuse, had already described the separation between mechanics (applied mathematics) and geometry (pure mathematics) which for Plato was the key to knowledge. He quoted Marc Kac who about forty years ago, mentioned that pure mathematics deals with deep questions in simple situations, while applied mathematics deals with simple questions in extremely complicated models. In Joffe 's opinion, it is very hard to find definitions which will please everybody. The distinction is more likely to be between the pure and the applied mathematician than between pure and applied mathematics. He argued that the pure mathematician is somebody whose motivation comes from inside the subject, while the applied mathematician answers questions asked by other scientists, in order to try to be useful to society. He recommended encouraging all mathematicians to be more receptive to dialogue with scholars of other fields.

Bernhelm Booss-Bavnbek spoke by referring to Harald Bohr and Hardy about the distinction between two phases in sciences: the extending and the consolidating phase. Clearly physics has been in a consolidating phase in the first half of the last century, after a previous period of extension with many new single results at the end of the 19th century. One could claim that mathematics in the first half of the last century still was in a phase of extension, and that what mathematics needed was a new phase of consolidation. Booss addressed Professor Manin asking him whether he would agree with Peter Sarnak who said that we have had a golden period of about 30 years for pure mathematics, in Booss' terms a phase of truly consolidation, where various fields in pure mathematics showed and proved to be interconnected. Could this be a good starting point to make real valuable contributions to other fields like biology, which in spite of the great achievements of Watson and Crick 50 years ago still is in this phenomenology? More concretely, if on the basis of these last 30 years of mathematics consolidation, there is a new impulse for mathematicians to do something towards contributing to consolidation in these more phenomenologically expanding sciences?

Yuri Manin answered: "Sarnak said the last 30 years were years of great consolidation and maturing of the mathematics of the 20th century. I'm less sure – I mean emotionally less sure – about how to characterize in such admittedly simplistic terms the development that is connected with computers, computer science and internet. Kolmogorov, whose name was mentioned here, introduced the notion of Kolmogorov complexity. Kolmogorov complexity, very roughly speaking, of a piece of information is the length of the shortest programme which can be then used to generate this piece of information. In this respect one can say that classical laws of physics – such fantastic laws as Newton's law of gravity of Einstein's equations – are extremely short programmes to generate a lot of descriptions of real physical world situations. I am not at all sure that Kolmogorov's complexity of data that were uncovered by, say, genetics in the human genome project, or even modern cosmology data – I am not at all sure that their Kolmogorov complexity is sufficiently small that they can be really grasped by the human mind. One should be aware that if a certain large piece of information has very large Kolmogorov complexity, then we are bound not to understand it. We are bound to relegate the processing of this data to computers or computer nets, or whatever. And I have a very strong suspicion that this is a new situation in natural sciences, with which we really do not yet know how to cope. We produce technology, and it might happen that this technology is absolutely indispensable to deal with this data.

The discussion ended with a few words by the moderator, John Ball, who said that he wouldn't dare summarize the discussion, very interesting though it had been. He thanked all those who had participated, Marta Sanz-Solé, who organized the round table, and especially the panel.