The work of Wendelin Werner

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1. Introduction

It is my great pleasure to briefly report on some of Wendelin Werner's research accomplishments that have led to his being awarded a Fields Medal at this International Congress of Mathematicians of 2006. There are a number of aspects of Werner's work that add to my pleasure in this event. One is that he was trained as a probabilist, receiving his Ph.D. in 1993 under the supervision of Jean-François Le Gall in Paris with a dissertation concerning planar Brownian Motion – which, as we shall see, plays a major role in his later work as well. Until now, Probability Theory had not been represented among Fields Medals and so I am enormously pleased to be here to witness a change in that history.

I myself was originally trained, not in Probability Theory, but in Mathematical Physics. Werner's work, together with his collaborators such as Greg Lawler, Oded Schramm and Stas Smirnov, involves applications of Probablity and Conformal Mapping Theory to fundamental issues in Statistical Physics, as we shall discuss. A second source of pleasure is my belief that this, together with other work of recent years, represents a watershed in the interaction between Mathematics and Physics generally. Namely, mathematicians such as Werner are not only providing rigorous proofs of already existing claims in the Physics literature, but beyond that are providing quite new conceptual understanding of basic phenomena – in this case, a direct geometric picture of the intrinsically random structure of physical systems at their critical points (at least in two dimensions). One simple but important example is percolation – see Figure 1.

Permit me a somewhat more personal remark as director of the Courant Institute for the past four years. We have a scientific viewpoint, as did our predecessor institute in Göttingen – namely, that an important goal should be the elimination of artificial distinctions between the Mathematical Sciences and their applications in other Sciences – I believe Wendelin Werner's work brilliantly lives up to that philosophy.

Yet a third source of pleasure concerns the collaborative nature of much of Werner's work. Beautiful and productive mathematics can be the result of many different personal workstyles. But the highly interactive style, of which Werner, together with Lawler, Schramm and his other collaborators, is a leading exemplar, appeals to many of us as simultaneously good for the soul while leading to work stronger than the sum of its parts. It is a promising sign to see Fields Medals awarded for this style of work.

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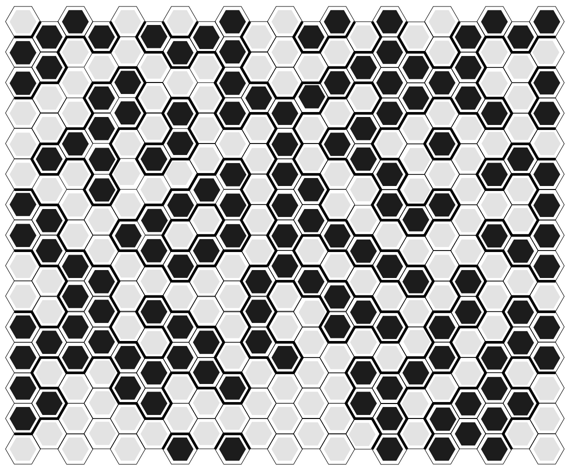

Figure 1. Finite portion of a (site) percolation configuration on the triangular lattice. Each hexagon represents a site and is assigned one of two colors. In the critical percolation model, colors are assigned randomly with equal probability. The cluster interfaces are indicated by heavy lines.

2. Brownian paths and intersection exponents

The area of Probability Theory which most strongly interacts with Statistical Physics is that involving stochastic processes with nontrivial spatial structure. This area, which also interacts with Finance, Communication Theory, Theoretical Computer Science and other fields, has long combined interesting applications with first-class Mathematics. Recent developments however have raised the perceived mathematical status of the best work from "merely" first-class to outstanding. Let me mention two pieces of Werner's work from 1998–2000. These are not only of intrinsic significance, but also were precursors to the breakthroughs about to happen in the understanding of two-dimensional critical systems with (natural) conformal invariance. (There were of course other significant precursors, such as Aizenman's path approach to scaling limits – see, e.g., [1], [2] – and Kenyon's work on loop-erased walks and domino tilings – see, e.g., [13].)

The first of these two pieces of work is a 1998 paper of Bálint Tóth and Werner [36]. The motivation was to construct a continuum version of Tóth's earlier lattice "true self-repelling walk" and this led to a quite beautiful mathematical structure (an extended version of a mostly unpublished and nearly forgotten construction done almost 20 years earlier by Arratia) of coalescing and "reflecting" one-dimensional Brownian paths, running forward and backward in time and filling up all of two-dimensional space-time. There is a (random) plane-filling curve within this structure that is analogous to one that arises in scaling limits of uniformly random spanning trees and was one of Schramm's motivations in his 2000 paper introducing SLE [33]. SLE is an acronym for what was originally called the Stochastic Loewner Evolution and is now often called the Schramm–Loewner Evolution; more about SLE shortly.

The second piece of work consists of two papers with Lawler in 1999 and 2000 involving planar Brownian intersection exponents [25], [26]. In the second of these, it was shown that the same set of exponents must occur providing only that certain locality and conformal invariance properties are valid. This was a key idea which, combined with the introduction of SLE for the analysis of two-dimensional critical phenomena, led to a remarkable series of three papers in 2001–2002 by Lawler, Schramm and Werner [17], [18], [20] which yielded a whole series of intersection exponents.

For example, let $W^1(t), W^2(t), \dots$ be independent planar Brownian motions starting from distinct points at $t = 0$. Then the probability that the random curve segments $W^1([0,t]), \dots, W^n([0,t])$ are all disjoint is $t^{-\zeta_n + o(1)}$ as $t \to \infty$ for some constant ζ_n.

Theorem 1 ([18]). *The intersection exponents ζ_n, for $n \geq 2$, are given by*

$$\zeta_n = \frac{4n^2 - 1}{24}. \tag{1}$$

This formula had been conjectured earlier by Duplantier and Kwon [12] and derived later by Duplantier [11] in a nonrigorous calculation based on two-dimensional quantum gravity. Despite the simplicity of the formula, prior to the introduction of SLE-based methods, its derivation by conventional stochastic calculus techniques appeared to be quite out of reach.

3. Conformal probability theory

The period from 2001 until the present has seen an explosion of interest in and applications of the SLE approach. To discuss this, we first give a very brief introduction to SLE; some good general references are [32], [38], [16]. For, say, a Jordan domain D in the complex plane with distinct points a, b on its boundary ∂D, and κ a positive parameter, (chordal) SLE with parameter κ, denoted SLE_κ, is a certain random continuous path (a curve, modulo monotonic reparametrization) in the closure $\overline{D}$, starting at a and ending at b. When $\kappa \leq 4$, SLE_κ is (with probability one) a simple path that only touches ∂D at a and b. Loewner, in work dating back to the 1920s [28], studied the evolution from a to b of nonrandom curves in terms of a real-valued "driving function." By conformally mapping D to the upper half plane $\mathbb{H}$ and suitably reparametrizing, one obtains (for say $\kappa \leq 4$) a simple curve $\gamma(t)$ in $\mathbb{H}$ for $t \in (0, \infty)$ and conformal mappings g_t from $\mathbb{H} \backslash \gamma([0,t])$ to $\mathbb{H}$ with g_t satisfying Loewner's evolution equation,

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U(t)}, \tag{2}$$

with driving function $U(t) = g_t(\gamma(t))$. SLE_κ corresponds to the choice of $U(t)$ as the random function $B(\kappa t)$ where B is standard one-dimensional Brownian motion. When $\kappa > 4$, some modifications are necessary, but (2) remains valid – even for $\kappa \geq 8$ when the curves become plane-filling.

Now back to the SLE-based advances of the recent past. Many of these concern or were motivated by (nonrigorous) results in the Statistical Physics literature about two-dimensional critical phenomena. Critical points of physical systems typically happen at very specific values of physical parameters, such as where the vapor pressure curve in a liquid/gas system ends. Critical systems have many remarkable properties, such as random fluctuations that normally are observable only on microscopic scales manifesting themselves macroscopically. A related feature is that many quantities at or approaching the critical point have power law behavior, with the non-integer powers, known as *critical exponents* (as well as other macroscopic features, such as the scaling limits we will discuss later), believed to satisfy "universality", i.e., microscopically distinct models in the same spatial dimension should have the same exponents at their respective critical points. Two-dimensional critical systems turn out to have an additional remarkable property, which is at the heart of both the SLE approach and its predecessors in the physics literature – that is conformal invariance on the macroscopic scale.

As in the case of Brownian intersection exponents, many of the SLE-based results in two dimensions were rigorous proofs of exponent values that had been derived earlier by nonrigorous arguments – primarily those of what is known in the Physics literature as "Conformal Field Theory", which dates back to the work of Polyakov and collaborators in the 1970s and 1980s [31], [4], [5] – see also [10], [30], [9]. Other results were brand new. I'll discuss a few of these in more detail, but, as noted before, what is particularly exciting is that the SLE-based approach is not solely a rigorization of what already had existed in the physics literature but also a conceptually quite complementary approach to that of Conformal Field Theory. Werner in particular has emphasized the need to understand that complementary relationship; this has led, e.g., to a focus on the "restriction property", as in his paper about the conformally invariant measure on self-avoiding loops [39]. That paper is one example of a burgeoning interest in extending the original SLE focus on random curves to the case of random *loops*, but still with conformal invariance properties, both in the specific case of percolation scaling limits [6], [7] and in the more general contexts of Brownian "loop soups" [27], [37] and Conformal Loop Ensembles as currently being studied by Scott Sheffield and Werner.

Next are some more examples of the results obtained in the last six years or so.

4. Brownian frontier

Let $W(t)$ be a planar Brownian motion. The complement in the plane of the curve segment $W([0, t])$ is a countable union of open sets, one of which is infinite; the boundary of that infinite component is called the Brownian frontier. As a consequence of deep relations that planar Brownian motion and its intersection exponents have with SLE_6 and its exponents (see [19], [23]), Lawler, Schramm and Werner obtained the following, proving a celebrated conjecture of Mandelbrot [29].

Theorem 2 ([21]). *The Hausdorff dimension of the planar Brownian frontier is* $4/3$.

5. Loop-erased walks

A different set of results are stated somewhat informally in the next theorem. They concern loop-erased random walks and related random objects on lattices. Unlike the percolation case discussed next, these results about continuum scaling limits, in which the lattice scale shrinks to zero, are not restricted to a particular lattice.

Theorem 3 ([24]). *Let D be* (*say*) *a Jordan domain in the plane; then the scaling limits of loop-erased random walk, the uniformly random spanning tree and the related lattice-filling curve in D are, respectively,* (*radial*) SLE_2*, a continuum "*SLE_2*-based tree" and the plane-filling* (*chordal*) SLE_8.

Scaling limits of lattice models are among the most interesting and often the most difficult results. To do them well requires the successful combination of concepts and techniques from three different areas: conformal geometry (as in the classical Löwner evolutions where the driving function is nonrandom), stochastic analysis (since for SLE the driving function is a Brownian motion), and the probability theory of lattice models (e.g., random walks or percolation or Ising models or ...). The work of Werner combines all three ingredients admirably well.

6. Percolation

Before closing, let me discuss one more example which demonstrates how these three areas can interact – scaling limits of two-dimensional critical percolation. The physics community knew (nonrigorously) the exponent values and even some geometric information in the form of specific formulas for scaling limits of crossing probabilities between boundary segments of domains. These formulas were derived by Cardy [9] following Aizenman's conjecture that they should be conformally invariant – see [15]. But there was little understanding of the scaling limit geometry of objects like cluster "interfaces" – see Figure 1.

In [33], Schramm argued that the limit of one particular interface, the "exploration path," should be SLE_6. Smirnov, for the triangular lattice, then proved [34] that (A) the crossing probabilities do converge to the conformally invariant Cardy formulas, sketched an argument as to how that could lead to (B) convergence of the whole exploration path to SLE_6 and argued further that one should be able to extend these results to (C) a "full scaling limit" for the family of "interface loops" of all clusters. In [35], Smirnov and Werner then proved certain percolation exponents, using exploration path convergence (B), while in [22], Lawler, Schramm and Werner combined the full scaling limit (C) with percolation arguments to prove another exponent value that is stated below.

Convergence in (B) and (C) can be proved by using a considerable amount of lattice percolation machinery [6], [7], [8] – including results of Kesten, Sidoravicius and Zhang [14] about six-fold crossings of annuli and of Aizenman, Duplantier and Aharony [3] about narrow "fjords." Then the percolation exponent results of Werner and coauthors apply and provide another excellent example of how the combination of the three ingredients mentioned above work together – e.g., the next theorem proves a prediction of den Nijs and Nienhuis [10], [30].

Theorem 4 ([22]). *In critical site percolation on the triangular lattice,*

$$\textbf{Prob}\,[\text{cluster of origin has diameter} \geq R] = R^{-5/48+o(1)} \quad \text{as } R \to \infty\,. \tag{3}$$

7. Conclusion

I close with some comments about continuum models of Probability Theory and their relation to other areas of Mathematics which are exemplified by the work of Wendelin Werner. Traditionally, a major focus of Probability Theory, and especially so in France, has been on continuum objects such as Brownian Motion and Stochastic Calculus, with SLE and related processes as the latest continuum objects in the pantheon. Those of us raised in a different setting, such as Statistical Mechanics, sometimes regard lattice models as more "real" or "physical." But this is a narrow view. It is only the continuum models which possess extra properties, like conformal invariance in the two-dimensional setting, that relate Probability Theory to other well-developed areas of Mathematics. Such relations and interactions have become of increasing importance in recent years and will continue to do so. Even if one is primarily interested in the original lattice models, it is quite clear that their properties, such as critical exponents and critical universality, cannot be understood without a deep analysis of the continuum models that arise in the scaling limit. Thanks to the work of Wendelin Werner, his collaborators, and others, one might say that now we are all "continuistas."

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