Some results in noncommutative ring theory

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Abstract. In this paper we survey some results on the structure of noncommutative rings. We focus particularly on nil rings, Jacobson radical rings and rings with finite Gelfand–Kirillov dimension.

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1. Introduction

We present here a brief outline of results and examples related mainly to noncommutative nil rings. In this exposition rings are noncommutative and associative. A vector space R is called an algebra (or a K-algebra) if R is equipped with a binary operation

$$* : (R, R) \to R,$$

called multiplication, such that for any $a, b, c \in R$ and for any $\alpha \in K$, we have $(a+b)*c = a*c + b*c$, $a*(b+c) = a*b + a*c$, $(a*b)*c = a*(b*c)$, $\alpha(a*b) = (\alpha a)*b = a*(\alpha b)$.

It is known that simple artinian rings, commutative simple rings and simple right noetherian rings of characteristic zero have unity elements [35]. In this text, rings are usually without 1. In fact nil rings and Jacobson radical rings cannot have unity elements.

2. Nil rings

The most important question in this area is the Köthe Conjecture, first posed in 1930. Köthe conjectured that a ring R with no nonzero nil (two-sided) ideals would also have no nonzero nil one-sided ideals, [24], see also [15] and [27]. This conjecture is still open despite the attention of many noncommutative algebraists. It is a basic question concerning the structure of rings.

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The truth of the conjecture has been established for many classes of rings: typically, one proves that for a given class of rings, the sum of all nil one-sided ideals is nil. The most famous examples of such results are the proof of the conjecture in the case of algebras over uncountable fields by Amitsur, and the fact that nil ideals are nilpotent in the class of noetherian rings, proved by Levitski, see [27]. However, as indicated above, Köthe's conjecture is still open in the general case.

An element r in a ring R is said to be *nilpotent* if $r^n = 0$ for some n. A ring R is a *nil* ring if every element of R is nilpotent, and the ring R is *nilpotent* if $R^n = 0$ for some n. A more appropriate definition in the case of infinitely generated rings is the following. A ring R is *locally nilpotent* if every finitely generated subring of R is nilpotent. A thorough understanding of nil and nilpotent rings is important for an attempt to understanding general rings.

In addition, nil rings have some applications in group theory. The following famous theorem was proved in 1964 by Golod and Shafarevich. *For every field F there exists a finitely generated nil F-algebra R which is not nilpotent* ([20]). Recall that a group G is said to be torsion (or periodic) if every $g \in G$ has a finite order. Golod used the group $1 + R$, when F has positive characteristic, to get a counterexample to the General Burnside Problem: *Let G be a finitely generated torsion group. Is G necessarily finite*?

There are many open questions concerning nil rings. As mentioned before, the most important is now known as the *Köthe Conjecture* and was posed by Köthe in 1930: if a ring R has no nonzero nil ideals, does it follow that R has no nonzero nil one-sided ideals? Köthe himself conjectured that the answer would be in the affirmative ([24], [27], [37]).

There are many assertions equivalent to the Köthe Conjecture: For example, the following are equivalent to Köthe's conjecture:

1. The sum of two right nil ideals in any ring is nil.
2. (Krempa [26]) For every nil ring R the ring of 2 by 2 matrices over R is nil.
3. (Fisher, Krempa [18]) For every ring R, R^G is nil implies R is nil (G is the group of automorphisms of R, R^G the set of G-fixed elements).
4. (Ferrero, Puczylowski [17]) Every ring which is a sum of a nilpotent subring and a nil subring must be nil.
5. (Krempa [26]) For every nil ring R the polynomial ring $R[x]$ in one indeterminate over R is Jacobson radical.
6. (Smoktunowicz [44]) For every nil ring R the polynomial ring $R[x]$ in one indeterminate over R is not left primitive.
7. (Xu [49]) The left annihilators of a single element in every complement of a nil radical in a maximal left nil ideal satisfy a.c.c.

Recall that a ring R is Jacobson radical if for every $r \in R$ there is $r' \in R$ such that $r + r' + rr' = 0$. Every nil ring is Jacobson radical. The largest ideal in a ring R, which is Jacobson radical is called the *Jacobson radical* of R. The Jacobson radical of a ring R equals the intersection of all (right) primitive ideals of R (I is a primitive ideal in R if I/R is primitive). Recall that a ring R is (right) primitive if there is a maximal right ideal Q such that $Q + I = R$ for every nonzero ideal I in R and there is $b \in R$ such that $br - r \in Q$ for every $r \in R$ ([13]).

The Köthe Conjecture is said to hold for a ring R if the ideal generated by the nil left ideals of R is nil. Köthe's conjecture holds for the class of Noetherian rings (Levitzki, [27], [32]), Goldie rings (Levitzki, [32]), rings with right Krull dimension (Lenagan [29], [15]), monomial algebras (Beidar, Fong [6]), PI rings (Rasmyslow–Kemer–Braun [14], [34], [22], [12]), algebras over uncountable fields (Amitsur [27], [36]).

There are many related results, some are indicated in the following.

Theorem 2.1 (Levitzki; [32]). *Let R be a right Noetherian ring. Then every nil one-sided ideal of R is nilpotent.*

Theorem 2.2 (Lenagan [29]). *If R has right Krull dimension, then nil subrings of R are nilpotent.*

Theorem 2.3 (Gordon, Lenagan and Robson, Gordon and Robson; [15]). *If R has right Krull dimension, then the prime radical of R is nilpotent.*

The prime radical of R is a nil ideal and is equal to the intersection of all prime ideals in R.

Theorem 2.4 (Beidar, Fong [6]). *Let X be a nonempty set, $Z = \langle X \rangle$ the free monoid on X, Y an ideal of the monoid Z, and F a field. Then the Jacobson radical of the monomial algebra $F[Z/Y]$ is locally nilpotent.*

In the case of characteristic zero the result is due to Jaspers and Puczylowski, [21]. Earlier, Belov and Gateva-Ivanova [10] showed that the Jacobson radical of a finitely generated monomial algebra over a field is nil. However, it is not true that the Jacobson radical of a finitely generated monomial algebra is nilpotent, since it was shown by Zelmanov [50] that there is a finitely generated prime monomial algebra with a nonzero locally nilpotent ideal.

Theorem 2.5 (Razmyslov–Kemer–Braun [34], [22], [12]; [14]). *If R is a finitely generated PI-algebra over a field then the Jacobson radical of R is nilpotent.*

Razmyslov [34] proved this for rings satisfying all identities of matrices, Kemer [22] for algebras over fields of characteristic zero. Later Braun [12] proved the nilpotency of the radical in any finitely generated PI algebra over a commutative noetherian ring. Amitsur has previously shown that the Jacobson radical of a finitely generated PI algebra over a field is nil.

Another famous result is the Nagata–Higman Theorem:

Theorem 2.6 (Nagata–Higman; [19]). *If A is an associative algebra of characteristic p such that $a^n = 0$ for all $a \in A$ and $p > n$ or $p = 0$ then A is nilpotent.*

For interesting results related to Nagata–Higman's theorem see [19].

A theorem of Klein [23] asserts that if R is a nil ring of bounded index then $R[x]$ is a nil ring of bounded index.

In 1956 Amitsur [27] showed that if R is a nil algebra over an uncountable field, then the polynomial ring $R[x]$ in one indeterminate over R is also nil. The situation is completely different for countable fields, as was shown by the author in 2000.

Theorem 2.7 (Smoktunowicz [43]). *For every countable field K there is a nil K-algebra N such that the polynomial ring in one indeterminate over N is not nil.*

This answers a question of Amitsur. Another important theorem by Amitsur is the following.

Theorem 2.8 (Amitsur; [27]). *Let R be a ring. Then the Jacobson radical of the polynomial ring $R[x]$ is equal to $N[x]$ for some nil ideal N of R.*

In 1956 Amitsur conjectured that if R is a ring, and $R[x]$ has no nil ideals then it is semiprimitive (i.e. the Jacobson radical of $R[x]$ is zero). This assertion is true for many important classes of rings, as mentioned above. However, the following theorem shows that this conjecture does not hold in general: *There is a nil ring N such that the polynomial ring in one indeterminate over N is Jacobson radical but not nil* ([41]). For some generalizations of this theorem see [45]. This theorem is true in a more general setting: *For every natural number n, there is a nil ring N such that the polynomial ring in n commuting indeterminates over N is Jacobson radical but not nil*.

Recall that, as shown by Krempa in [26], Köthe's conjecture is equivalent to the assertion that polynomial rings over nil rings are Jacobson radical. However, homomorphic images of polynomial rings over nil rings with nonzero kernels are often Jacobson radical, as is shown by the next result.

Theorem 2.9 (Smoktunowicz [44]). *Let R be a nil ring and $R[x]$ the polynomial ring in one indeterminate over R. Let I be an ideal in $R[x]$ and M the ideal of R generated by coefficients of polynomials from I. Then $R[x]/I$ is Jacobson radical if and only if $R[x]/M[x]$ is Jacobson radical.*

The following are interesting open questions on nil rings.

Question 1 (Latyshev, [16], pp. 12). Let A be an associative algebra with a finite number of generators and relations. If A is a nil algebra must it be nilpotent?

Question 2 (Amitsur; [33]). Let A be an associative algebra with a finite number of generators and relations. Does it follow that the Jacobson radical of A nil?

3. Algebraic algebras

The most well-known question in this area is the Kurosh Problem ([15], [36]). *Let R be a finitely generated algebra over a field F such that R is algebraic over F. Is R finite dimensional over F?*

This problem has a negative solution in general. The famous construction of Golod and Shafarevich in the 1960s produced a finitely generated nil algebra which is not nilpotent ([20]). This was then used to construct a counterexample to the Burnside Conjecture, one of the biggest outstanding problems in group theory at that time. Zelmanov was later awarded the Fields Medal for his solution of the Restricted Burnside Problem [27].

However, the Kurosh Problem is still open for the key special case of a division ring:

Question 3 (Kurosh's problem for division rings [16], [36]). Let R be a finitely generated algebra over a field F such that R is algebraic over F and R is a division ring. Does it follow that R a finite dimensional vector space over its center?

Again, as with the nil ring problems, there are many partial results. The Kurosh Problem for division rings is still open in general, but it is answered affirmatively for F finite and for F having only finite algebraic field extensions, in particular, for F algebraically closed ([36]). By Levitzki's and Kaplanski's theorem, Kurosh's conjecture is also true if there is a bound on the degree of elements in R ([15]). It is unknown whether Kurosh's problem for division rings has a positive answer in the case of algebras over uncountable fields. Also the following question is still open: Is Kurosh's conjecture true for division rings with finite Gelfand–Kirillov dimension, and in particular for division rings with quadratic growth? There are obvious connections with problems in nil rings. A nil element is obviously algebraic, and, in the converse direction, it is possible to construct an associated graded algebra connected with an algebraic algebra in such a way that the positive part is graded nil, i.e., all homogeneous elements are nil. On the other hand, the Kurosh Problem has a negative solution for rings with finite Gelfand–Kirillov dimension ([30]), for simple rings ([42]), for primitive rings ([2]), for finitely generated primitive rings ([8]), and for finitely generated algebraic primitive rings ([9]). However, a natural question arising from the general Kurosh Problem remains open:

Question 4 (Small's question). Let R be a finitely generated simple algebra with 1 over a field F such that R is algebraic over F. Is R a finite dimensional vector space over its center?

Another open question on division rings, which has been around for years, is the following:

Question 5. Let K be a field and let R be a finitely generated algebra which is a division ring. Does it follow that R is a finitely generated vector space over K?

As far as I know this question is very much open even with various conditions, like e.g. Gelfand–Kirillov dimension 2. It has been shown by Small ([38]) that a division ring which is a homomorphic image of a graded noetherian ring (of course, by a non graded ideal) must be finite dimensional. There is a similar open question concerning rings:

Question 6 ([16], p. 20). Does there exist an infinite associative division ring which is finitely generated as a ring?

4. Algebras with finite Gelfand–Kirillov dimension

The Gelfand–Kirillov dimension measures the rate at which an algebra is generated by a generating set. The GK dimension is zero for algebras which are finite dimensional and an elementary counting argument shows that the next possible dimension is one. However, Borho and Kraft showed that any real number value greater than or equal to two is possible ([25]). Bergman's famous Gap Theorem establishes that there is no algebra with GK dimension strictly between one and two ([11], see also [25]). A theorem of Small and Warfield asserts that an affine prime algebra R over a field F of GK dimension 1 is a finite module over its center, and that its center is a finitely generated F-algebra of GK dimension 1 ([40], [25]). In the special case when R is a finitely generated domain over an algebraically closed field with GK dimension 1, it follows by Small–Warfield's and Tsen's theorem (see [15]) that R is in fact commutative ([47]). A theorem of Small, Stafford and Warfield shows that a finitely generated algebra with GK dimension 1 is close to being commutative in that it must satisfy a polynomial identity ([39], [25]).

The graded case has attracted interest in the last decade or so with the development of noncommutative algebraic geometry. Here progress is being made by studying algebras with restricted conditions, including conditions on the growth of the algebras. Low GK dimension examples are obviously of interest. Since the theory is developing by analogy with the classical projective case, one typically deals with graded algebras. Thus dimensions should be increased by one compared to the ungraded case. The first interesting case is to study graded domains of GK dimension two; that is, noncommutative projective curves. This was done in a famous paper in the *Inventiones Mathematicae* by Artin and Stafford about 10 years ago. In fact Artin and Stafford described in [3] the structure of finitely graded domains in terms of algebras related to automorphisms of elliptic curves. They were able to tell when such algebras are noetherian, primitive, PI, etc. In this paper they formulated the analogue of the Bergman Gap Theorem: there should be no graded (by natural numbers) domain with GK dimension strictly between two and three, and they were able to exclude the open interval $(2, 11/5)$. The author has recently established in [46] the truth of the full conjecture.

There are several connecting threads between the three areas mentioned above. As stated earlier, nil elements are algebraic, and graded nil algebras can be constructed from algebraic algebras as associated graded rings. The Golod–Shafarevich construction yields a nil but *not* nilpotent algebra which has exponential growth and so certainly infinite GK dimension.

In recent work with Lenagan, the author has constructed an example of a finitely generated nil but not nilpotent algebra that has finite GK dimension (≤ 20). The precise growth condition dividing nilpotent and nil but not nilpotent is tantalizing. Certainly, nil algebras with GK dimension 1 are easily seen to be nilpotent. It may be that the dividing line is of quadratic growth.

In this area the following question remains open and may be considered to be a test question for new methods. Is there a finitely generated nil algebra with quadratic growth which is not nilpotent? An F-algebra R has quadratic growth if there is a constant c and a generating subspace V of R such that $\dim_F(V + V^2 + \cdots + V_n) < cn^2$ for all $n > 0$. In particular GKdim $R \leq 2$.

In connection with this problem, a recent result of Bartholdi is pertinent. In 2004 Bartholdi proved the following result.

Theorem 4.1 (Bartholdi [4]). *Let K be an algebraic field extension of F_2. Then there exist a finitely generated graded K-algebra R such that all homogeneous elements of R are nil, but the algebra has a transcendental invertible element. In particular, R is graded nil but not nil. This algebra R has also a subalgebra isomorphic to the ring of 2×2 matrices over R.*

In more detail, Bartholdi showed that an affine 'recurrent transitive' algebra (without unit) constructed from Grigorchuk's group of intermediate growth is of quadratic growth. Moreover, assuming that the base field is an algebraic extension of F_2, the algebra is Jacobson radical and not nil. This algebra R was earlier studied by Ana Christina Vieira in [48], who showed that R is prime and for every non-zero two sided ideal I of R, R/I is finite-dimensional.

Another way to construct examples of finitely generated algebras was introduced by Markov and later extended by Beidar ([5]), Bell and Small ([7], [8], [32], [36]). The effect of Markov's result is to allow constructions first in infinitely generated algebras, thus simplifying the problem, and then, by using Markov's method, to bring the construction into a finitely generated algebra.

Theorem 4.2 (Markov [31]). *Let K be a field, and let R be a prime, countably generated K-algebra. Then there exists a prime K-algebra A generated by two elements x, y such that R is isomorphic to a right ideal of A, namely to xR .*

Recall that $T \subseteq R$ is a corner of an algebra R if T is a subalgebra of R and $TRT \subseteq T$. Markov's theorem was extended by Small who showed (around 1982, unpublished) that if K is a field and T is a prime, countably generated K algebra then there exists a finitely generated, prime K-algebra A such that T is a corner of A.

It is possible to apply this result in many situations. For example, in [8], Bell and Small applied the result to show that there is a finitely generated algebraic primitive algebra which is infinitely dimensional over its center. In 2003 Bell proved the following extension of Small's theorem. *Let K be a field, and let T be a prime, countably generated K-algebra of Gelfand–Kirillov dimension $\alpha < \infty$. Then there exists a finitely generated, prime K-algebra A of Gelfand–Kirillov dimension $\alpha + 2$ such that T is a corner of A* (see [7]).

Bell's theorem above is related to another question of Small: if R is a noetherian affine algebra with quadratic growth, does it follow that R is either primitive or PI? This is true in the graded case, as was shown by Artin and Stafford in 2000. According to Small, it is also true if every non-zero prime ideal in R is maximal.

An application by Bell of his theorem is the following example which is a counterexample to another question of Small. There is a prime, affine algebra with Gelfand–Kirillov dimension 2 which is not PI and not primitive. This algebra has a nonzero Jacobson radical. The following result of Lanski, Resco and Small assures that usually an affinization of a primitive ring is still primitive:

Theorem 4.3 (Lanski, Resco, Small [28]). *Let R be a prime ring. Then the following is true:*

1. *Let V be a right ideal of R. Then R is a primitive ring exactly when $V/(V \cap l(V))$ is a primitive ring, where $l(V) = \{r \in R : rV = 0\}$.*
2. *If R contains an idempotent e, then R is a primitive ring if and only if eRe is a primitive ring.*

5. Simple rings

A ring R (possibly without 1) is called simple if $R^2 \neq 0$ and R has no proper two-sided ideals. Levitzki, Jacobson, Kaplansky and others asked if there is a simple nil ring. An example of a simple ring which is a Jacobson radical ring (that is, $R = J(R)$ where $J(R)$ denotes the Jacobson radical of R) was found by Sasiada in 1961, see e.g. [15]; however, this ring is not nil. Note that the polynomial ring in one indeterminate over Sasiada's ring is left and right primitive ([44]). By Nakayama's lemma a simple Jacobson radical ring cannot be finitely generated. Since every nil ring is Jacobson radical, a simple nil ring also cannot be finitely generated. A few years ago examples of simple nil rings were constructed by the author ([15]).

Theorem 5.1 (Smoktunowicz [42]). *For every countable field K there is a simple nil algebra over K.*

Notice that all rings in that paper were graded by integers. The following natural question remains open.

Question 7. Is there a simple noncommutative nil algebra over an uncountable field?

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