

Galois deformations and arithmetic geometry of Shimura varieties

Kazuhiro Fujiwara*

Abstract. Shimura varieties are arithmetic quotients of locally symmetric spaces which are canonically defined over number fields. In this article, we discuss recent developments on the reciprocity law realized on cohomology groups of Shimura varieties which relate Galois representations and automorphic representations.

Focus is put on the control of ℓ -adic families of Galois representations by ℓ -adic families of automorphic representations. Arithmetic geometrical ideas and methods on Shimura varieties are used for this purpose. A geometrical realization of the Jacquet–Langlands correspondence is discussed as an example.

Mathematics Subject Classification (2000). 11F55, 11F80, 11G18.

Keywords. Galois representations, automorphic representations, Shimura variety.

1. Introduction

For a reductive group G over \mathbb{Q} and a homogeneous space X under $G(\mathbb{R})$ (we assume that the stabilizer K_∞ is a maximal compact subgroup modulo center), the associated modular variety is defined by

$$M_K(G, X) = G(\mathbb{Q}) \backslash G(\mathbb{A}_f) \times X / K.$$

Here $\mathbb{A}_f = \prod'_{p \text{ prime}} \mathbb{Q}_p$ (the restricted product) is the ring of finite adeles of \mathbb{Q} , and K is a compact open subgroup of $G(\mathbb{A}_f)$.

$M_K(G, X)$ has finitely many connected components, and any connected component is an arithmetic quotient of a Riemannian symmetric space. It is very important to view $\{M_K\}_{K \subset G(\mathbb{A}_f)}$ as a projective system as K varies, and hence as a tower of varieties. The projective limit

$$M(G, X) = \varprojlim_{K \subset G(\mathbb{A}_f)} M_K(G, X)$$

admits a right $G(\mathbb{A}_f)$ -action, which reveals a hidden symmetry of this tower. At each finite level, the symmetry gives rise to Hecke correspondences. For two compact open

*The author was partially supported by the 21st century COE program of JSPS. The author is grateful to the Research Institute for Mathematical Sciences of Kyoto University, and Columbia University for hospitality during the writing of this paper.

subgroups $K, K' \subset G(\mathbb{A}_f)$ and $g \in G(\mathbb{A}_f)$, g defines a correspondence $[K'gK]$: the first projection $M_{K \cap g^{-1}K'g} \rightarrow M_K$, and the second is the projection $M_{K \cap g^{-1}K'g} \rightarrow M_{g^{-1}K'g} \simeq M_{K'}$.

The Betti cohomology group $H_B^*(M_K(G, X), \mathbb{Q})$ admits the action of the Hecke algebra $H(G(\mathbb{A}_f), K)_{\mathbb{Q}}$, the convolution algebra formed by the compactly supported \mathbb{Q} -valued K -biinvariant functions on $G(\mathbb{A}_f)$. The characteristic function χ_{KgK} of a double coset KgK acts as correspondence $[Kg^{-1}K]$.

When all connected components of X are hermitian symmetric spaces, we call $M_K(G, X)$ a Shimura variety and denote it by $\text{Sh}_K(G, X)$. The arithmetic theory was begun by Shimura, and developed in his series of articles (cf. [44], [45], [47], [48]).

Shimura varieties have the following special features¹:

- They are quasi-projective algebraic varieties over \mathbb{C} (and smooth when K is small enough).
- They are canonically defined over the number field $E(G, X)$ (Shimura's theory of canonical models). $E(G, X)$ is called the reflex field. For a field extension E' of E , $\text{Sh}(G, X)_{E'}$ denotes this model as defined over E' .

Shimura established the existence of canonical models when they are moduli spaces of abelian varieties with PEL structure [48], and even in some cases which are not moduli spaces of motives (exotic models, see [47])². The functoriality of the canonical models is Shimura's answer to Hilbert's 12th problem (explicit construction of class fields).

In the case of Shimura varieties, $\text{Sh}(G, X)_E = \varprojlim_{K \subset G(\mathbb{A}_f)} \text{Sh}_K(G, X)_E$ is canonically defined over $E = E(G, X)$, which yields more symmetries than other modular varieties. The Galois group $G_E = \text{Gal}(\bar{E}/E)$ acts on the tower³, and hence the symmetry group is enlarged to $G_E \times G(\mathbb{A}_f)$. This symmetry acts on ℓ -adic étale cohomology groups $H_{\text{ét}}^*$ instead of the Betti cohomology H_B^* . The decomposition of the cohomology groups as Galois–Hecke bimodules is *the non-abelian reciprocity law* for Shimura varieties.

Aside from Shimura varieties, let us mention one more case which plays a very important role in the p -adic study of automorphic forms. When G is \mathbb{Q} -compact and X is a point, Hida noticed the arithmetic importance of $M(G, X)$ [21], though it is not a Shimura variety in general. These are called Hida varieties. Hida varieties are zero-dimensional, but the action of $G(\mathbb{A}_f)$ is quite non-trivial.

2. Non-abelian class field theory

We make the reciprocity law more explicit. Let $q(G)$ be the complex dimension of $\text{Sh}(G, X)$. The ℓ -adic intersection cohomology group $IH_{\text{ét}}^*(\text{Sh}_K(G, X)_{\bar{E}}) =$

¹We assume that (G, X) satisfies Deligne's axioms [11], [12] for non-connected Shimura varieties.

²The general solution is due to Borovoi and Milne.

³In the following, for a field F , the absolute Galois group $\text{Gal}(\bar{F}/F)$ of F is denoted by G_F .

$IH_{\text{ét}}^*(\overline{\text{Sh}(G, X)}_{\min, \bar{E}}, \overline{\mathbb{Q}_\ell})$ of the minimal compactification $\overline{\text{Sh}(G, X)}_{\min, E}$ is pure in each degree, and the most interesting part lies in the middle degree $IH_{\text{ét}}^{q(G)}$. In general, the decomposition of IH^* is understood by Arthur packets, by granting Arthur’s celebrated conjectures on local and global representations of reductive groups⁴. See [32] for a conjectural description in the general case.

Let us take a simple (but still deep) example. For a totally real number field F of degree g , let $I_F = \{\iota: F \hookrightarrow \mathbb{R}\}$ be the set of all real embeddings (regarded as the set of infinite places of F). One chooses a quaternion algebra D central over F which is split at one $\iota_0 \in I_F$ and ramifies at the other infinite places.

$G_D = \text{Res}_{F/\mathbb{Q}} D^\times$ is the Weil restriction of the multiplicative group of D , and $X_D = \mathbb{P}_{\mathbb{C}}^1 \setminus \mathbb{P}_{\mathbb{R}}^1 = \mathbb{C} \setminus \mathbb{R}$ is the Poincaré double half plane. The resulting $\text{Sh}(G_D, X_D)$ is a system of algebraic curves, namely the modular curve for $D = M_2(\mathbb{Q})$ and Shimura curves for the other cases, which has a canonical model over $F = E(G_D, X_D)$.

We fix a field isomorphism $\overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$. The decomposition of $IH_{\text{ét}}^1$ as a $G_F \times H(G(\mathbb{A}_f), K)$ -module is given by

$$IH_{\text{ét}}^1(\text{Sh}_K(G_D, X_D)_{\bar{F}}) = \bigoplus_{\pi \in \mathcal{A}_D} \rho_\pi \otimes \pi_f^K.$$

Here \mathcal{A}_D is the set of irreducible cuspidal representations $\pi = \pi_f \otimes \pi_\infty$ of $G_D(\mathbb{A})$ such that the infinite part π_∞ is cohomological for the trivial coefficient and the Jacquet–Langlands correspondent $JL(\pi)$ on $\text{GL}_2(\mathbb{A}_F)$ is cuspidal⁵. $\rho_\pi: G_F \rightarrow \text{GL}_2(\overline{\mathbb{Q}_\ell})$ is the ℓ -adic Galois representation attached to π which has the same L -function as π . The identity

$$L(s, \rho_\pi) = L(s, \pi, \text{st})$$

holds, where $L(s, \rho_\pi)$ is the Artin–Hasse–Weil L -function of ρ_π , and $L(s, \pi, \text{st})$ is the standard Hecke L -function of π . This identity (or rather identities between their local factors) is the *non-abelian reciprocity law* realized on $IH_{\text{ét}}^1$. The modular curve case is due to Eichler and Shimura. The Shimura curve case is due to Shimura [47], Ohta [35] and Carayol [5]. It fits into the general representation theoretical framework due to Langlands. The Langlands correspondence is a standard form of non-abelian class field theory which predicts a correspondence between Galois and automorphic representations.

In the following, we discuss the relationship between ℓ -adic families of Galois representations and ℓ -adic families of automorphic representations (with applications to the Langlands correspondence as Wiles did for elliptic curves over \mathbb{Q} in his fundamental work [57]). We show how arithmetic geometrical ideas and methods on Shimura varieties are effectively used for this purpose⁶.

⁴It is more natural (and sometimes more convenient because of vanishing theorems known in harmonic analysis) to introduce coefficient sheaves attached to representations of G .

⁵The finite part π_f of π is defined over $\overline{\mathbb{Q}_\ell}$ via identification $\overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$.

⁶Though the viewpoint of motives is extremely important, an emphasis is put on Galois representations in this article.

3. Galois deformations and nearly ordinary Hecke algebras for GL_2

In this section F denotes a totally real field, and I_F is as in §2. Fix an ℓ -adic field E_λ with integer ring o_λ and the maximal ideal λ .

By $I_{F,\ell}$ we mean the set of all field embeddings $F \hookrightarrow \overline{\mathbb{Q}}_\ell$. $I_{F,\ell}$ is canonically identified with $\coprod_{v|\ell} I_{F_v}$, where F_v is the local field at v , and I_{F_v} is the set of all continuous embeddings $F_v \hookrightarrow \overline{\mathbb{Q}}_\ell$. $I_{F,\ell}$ is identified with I_F by the isomorphism $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$ chosen in §2.

Let Σ be a set of finite places which contains all places dividing ℓ . Let G_Σ denote $\text{Gal}(F_\Sigma/F)$, where F_Σ is the maximal Galois extension of F which is unramified outside Σ .

For $G = GL_{2,F}$, Hida has produced a very big Hecke algebra in [22] by the method of cohomological ℓ -adic interpolation⁷. Let $\mathcal{X}_\Sigma^{\text{gl}}$ be the maximal pro- ℓ abelian quotient of G_Σ and $\mathcal{X}_\ell^{\text{loc}}$ be the pro- ℓ completion of $\prod_{v|\ell} o_v^\times$. A pair $((k_\iota)_{\iota \in I_F}, w)$ consisting of an integer vector $(k_\iota)_{\iota \in I_F}$ and an integer w is called discrete type if $k_\iota \equiv w \pmod 2$ and $k_\iota \geq 2$ for all $\iota \in I_F$. A continuous character $\chi: \mathcal{X}_\Sigma^{\text{gl}} \times \mathcal{X}_\ell^{\text{loc}} \rightarrow \overline{\mathbb{Q}}_\ell^\times$ is algebraic of type $((k_\iota)_{\iota \in I_F}, w)$ if the quotient $\chi / (\chi_{\text{cycle}}^{-w} \cdot \prod_{v|\ell} \prod_{\iota \in I_{F_v}} \chi_\iota^{k_\iota - 2})$ is of finite order. Here χ_{cycle} is the cyclotomic character, and χ_ι is the ℓ -adic character $o_v^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times$ defined via the embedding ι .

Then a nearly ordinary Hecke algebra T_Σ is a finite local $o_\lambda[[\mathcal{X}_\Sigma^{\text{gl}} \times \mathcal{X}_\ell^{\text{loc}}]]$ -algebra having the following properties:

- T_Σ is generated by the standard Hecke operators at all finite places.
- Take an ℓ -adic integer ring $o'_{\lambda'}$. For an o_λ -algebra homomorphism $f: T_\Sigma \rightarrow o'_{\lambda'}$ such that the induced o_λ -homomorphism $o_\lambda[[\mathcal{X}_\Sigma^{\text{gl}} \times \mathcal{X}_\ell^{\text{loc}}]] \rightarrow o'_{\lambda'}$ defines an algebraic character of discrete type $((k_\iota)_{\iota \in I_F}, w)$, there is a cuspidal $GL_2(\mathbb{A}_F)$ representation π which is defined by a holomorphic Hilbert cusp form of type $((k_\iota)_{\iota \in I_F}, w)$.
- Any cuspidal representations obtained by specializations as above are nearly ordinary, that is, the action of standard Hecke operators at $v|\ell$ are ℓ -adic units.
- T_Σ is the biggest o_λ -algebra with the above properties. T_Σ is the universal ring which connects all nearly ordinary cuspidal representations.

Assume that for some π appearing from T_Σ by a specialization, the mod λ -reduction $\bar{\rho}_\pi$ of ρ_π is absolutely irreducible. Then there is a Galois representation $\rho_{T_\Sigma}: G_\Sigma \rightarrow GL_2(T_\Sigma)$ which interpolates ρ_π 's for various π ℓ -adically (see [56] for the ordinary Hecke algebra: the method of pseudo-representation of Wiles). The local representation $\rho_{T_\Sigma}|_{G_{F_v}}$ for $v|\ell$ is *nearly ordinary*, that is, it has an expression

$$\rho_{T_\Sigma}|_{G_{F_v}} \simeq \begin{pmatrix} \chi_{1,v} & * \\ 0 & \chi_{2,v} \end{pmatrix}$$

⁷This method was found by Shimura in [46] in the late 1960s.

for some local characters $\chi_{i,v} : G_{F_v} \rightarrow T_\Sigma^\times$ ($i = 1, 2$) which are made explicit by the universal character of $\mathcal{X}_\ell^{\text{loc}}$. Also $\det \rho_{T_\Sigma}|_{\mathcal{X}_\Sigma^{\text{gl}}}$ is the universal character of $\mathcal{X}_\Sigma^{\text{gl}}$ twisted by χ_{cycle}^{-1} .

Next we introduce the deformation-theoretical viewpoint due to Mazur. Let R_Σ be the universal nearly ordinary deformation ring of $\bar{\rho}$. We put the nearly ordinary condition at $v|\ell$ and no restrictions at other $v \in \Sigma$. There is a canonical ring homomorphism $R_\Sigma \rightarrow T_\Sigma$ which is surjective (one recovers standard Hecke operators from ρ_{T_Σ}).

Theorem 3.1. *Assume the following conditions:*

1. ℓ is odd, and $\bar{\rho}|_{F(\zeta_\ell)}$ is absolutely irreducible⁸.
2. $\bar{\rho}|_{G_{F_v}}^{\text{ss}}$ is either indecomposable, or is a sum of distinct characters for $v|\ell$ (G_{F_v} -distinguished).

Then the nearly ordinary Hecke algebra T_Σ is a finite $\mathfrak{o}_\lambda[[\mathcal{X}_\Sigma^{\text{gl}} \times \mathcal{X}_\ell^{\text{loc}}]]$ -flat algebra of complete intersection and is isomorphic to the universal nearly ordinary deformation ring R_Σ .

When $F = \mathbb{Q}$, this is a fundamental result of Wiles [57] supplemented by his collaboration with Taylor [55]⁹. For general F , this result is due to the author and is a special case of [18]. For a precise construction of T_Σ at a finite level, see [18]. The above version of the theorem follows easily from it.

So Hida’s theory of nearly ordinary Hecke algebras for $\text{GL}_{2,F}$ is almost completed when the residual representation is absolutely irreducible.

We discuss some applications of the theorem. The first application is to the modularity of Galois representations¹⁰. The following theorem, which is a partial contribution to the modularity of elliptic curves over F (the Taniyama–Shimura conjecture) is an example:

Theorem 3.2. *Let E be an elliptic curve over F , and let $\rho_{E,\ell}$ be the associated Galois representation on the Tate module $T_\ell(E)$. Assume the following conditions:*

1. 3 splits completely in F .
2. $\rho_{E,3} \pmod{3}$ remains absolutely irreducible over $F(\zeta_3)$.
3. E is semi-stable at all $v|3$ and ordinary if it admits a good reduction.

Then there is a cuspidal representation π_E of infinity type $((2, \dots, 2), -2)$ such that $\rho_{E,\ell}$ is isomorphic to ρ_{π_E} over $\overline{\mathbb{Q}}_\ell$ for any ℓ . In particular $L(E, s) = L(s, \pi_E, \text{st})$ holds for the Hasse–Weil L -function $L(E, s)$ of E .

⁸There is an exceptional case when $\ell = 5$ and $[F(\zeta_5) : F] = 2$ which is not treated in [18]. This case will be discussed on another occasion.

⁹Also with a supplement by Diamond [13] to treat some exceptional local representations.

¹⁰One may apply the solvable base change technique to know the modularity, so only a weaker form of Theorem 3.1 is needed. Usually $[F : \mathbb{Q}]$ becomes even after a base change, so the Mazur principle in the even degree case [17] is indispensable for this approach.

One can use other division points (5-division points, for example) to establish similar modularity results.

By the solvable base change technique, one can also deduce a quite interesting result for $\mathrm{GL}_2, \mathbb{Q}$. See the work of Khare and Wintenberger [30], [28] for such directions. In particular Khare proves Serre's conjecture for mod ℓ level 1 forms of \mathbb{Q} .

The second application is in Iwasawa theory, especially the Selmer group for the symmetric square of GL_2 -representations [23]. Here we need the full form of Theorem 3.1. This theorem is also effective in solving classical problems. See [24] for such an example (Eichler's integral basis problem).

To establish Theorem 3.1, what we shall do is something like proving Weber's theorem (all abelian extensions of \mathbb{Q} are cyclotomic). In our case we already have some tower of Galois extensions which is explicitly described by automorphic forms via the reciprocity law, and we would like to know that it exhausts all extensions which satisfy reasonable conditions.

Let us list the main ingredients in the proof of Theorem 3.1, which are now standard. It consists of 4 steps:

- 1st step. *Level and weight optimization.* Define some minimal Hecke algebra T_{\min} . It is necessary to find a minimal Σ with more restrictive minimality conditions.
- 2nd step. *Compatibility of local and global Langlands correspondences.* Define the Galois representation $\rho_{T_{\min}} : G_F \rightarrow \mathrm{GL}_2(T_{\min})$, and determine the local behaviour of $\rho_{T_{\min}}$, namely local restrictions $\rho_{T_{\min}}|_{G_{F_v}}$.
- 3rd step. *$R = T$ in the minimal case.* Define the minimal deformation ring R_{\min} and show that $R_{\min} \simeq T_{\min}$.
- 4th step. *Reduction to the minimal case.* Show $R_{\Sigma} \simeq T_{\Sigma}$ by reducing it to the minimal case (raising the level).

The basic strategy for the third and fourth steps exists for general modular varieties $M(G, X)$, and will be discussed in the next two sections.

The first step applied to $\mathrm{GL}_2, \mathbb{Q}$ is Serre's ε -conjecture on the optimization of level and weights for modular $\mathrm{GL}_2(\overline{\mathbb{F}}_{\ell})$ -representations. This is proved in a series of works. See [40] for the related references. For general GL_2, F , outside ℓ , the optimization of the level is completely understood; see [26], [27], [17] for the ramified case and the Mazur principle, [36] for the Ribet type theorem [39]. These results are enough to prove Theorem 3.1 in the nearly ordinary case. However, optimization of level and weight at $v|\ell$ is still insufficient for further progress¹¹. The solvable base change technique in [50] is useful for modularity questions.

For the second step, one first constructs $\rho_{T_{\Sigma}}$ by an ℓ -adic interpolation from various ρ_{π} 's which appear from Shimura curves by Wiles' method of pseudo-representations [56].

¹¹A very naive version of Serre's ε -conjecture is false for $F \neq \mathbb{Q}$ for trivial reasons (one can not attain a conductor which is prime to ℓ in general). Diamond has formulated a refined conjecture and is making progress.

For $\mathrm{GL}_{2,F}$, $\rho_\pi|_{G_{F_v}}$ for $v \nmid \ell$ depends only on the v -component π_v of $\pi = \otimes_w \pi_w$ and is obtained by the local Langlands correspondence for GL_{2,F_v} [5], [52]. For $v|\ell$, the Weil–Deligne representation attached to the potentially stable representation $\rho_\pi|_{G_{F_v}}$ is compatible with π_v when ρ_π comes from Shimura curves [41], [42]. However, this information is too weak to determine $\rho_\pi|_{G_{F_v}}$. For example, information on the Hodge filtration is lacking. Breuil has formulated a p -adic version of the local Langlands correspondence [2].

For general Shimura varieties, this requires a detailed study of bad reductions. For n -dimensional representations as defined by Clozel [9], compatibility outside ℓ is established. (See [20] for the semi-simplification case, and see Taylor and Yoshida [54] for the general case).

For $v|\ell$, p -adic Hodge theory is the basic tool at present, especially for general Shimura varieties.

Remark 3.3. 1. One may allow finite flat deformations at $v|\ell$ when F_v is absolutely unramified [18]. Kisin has developed a local theory for finite flat deformation which allows ramification when the residue field at v is \mathbb{F}_ℓ , with an application to modularity theorems [31].

2. In the case of \mathbb{Q} , Ramakrishna and Khare [29] have found an interesting method without any reduction to the minimal case via use of special deformations, that is, by adding Steinberg type conditions at several auxiliary primes¹². Moreover, Ramakrishna has studied various Galois deformations over \mathbb{Q} , which may not be modular a priori, .

When $\bar{\rho}$ is absolutely reducible, T_Σ is used to prove Iwasawa’s main conjecture for class characters of totally real fields. Skinner and Wiles [49], [51] have obtained a modularity result even in the absolutely reducible case, by showing partial results on the relationship between the versal hull of R_Σ and T_Σ using many ideas and techniques including Taylor–Wiles systems.

4. Taylor–Wiles systems: the formalism

In [55], Taylor and Wiles have invented a marvelous argument to show a nice ring theoretical property of T_{\min} : T_{\min} is a complete intersection in the case of modular curves. This property may seem technical at first glance, but has the very deep implication that R_{\min} is T_{\min} in the case they considered. The original argument by Taylor and Wiles has been improved by now; Faltings suggested the direct use of deformation rings rather than Hecke algebras, and the further refinement we now describe is due to Diamond [14] and myself.

For a number field F , let $|F|_f$ be the set of finite places of F . q_v denotes the cardinality of the residue field $k(v)$ for $v \in |F|_f$.

¹²A freeness assertion is necessary for this approach. Using this assertion, one can compute congruence modules to come back to unrestricted deformations. Technical assumptions are made on $\bar{\rho}$ in [29].

Let k_λ be the residue field of o_λ .

Definition 4.1. Let H be a torus over F of relative dimension d , and let X be a set of finite subsets of $|F|_f$ that contains \emptyset . Let R be a complete noetherian local o_λ -algebra with the residue field k_λ , and M an R -module which is finitely generated as an o_λ -module. A Taylor–Wiles system $\{R_Q, M_Q\}_{Q \in X}$ for (R, M) consists of the following data:

- For $Q \in X$ and $v \in Q$, H is split at v , and $q_v \equiv 1 \pmod{\ell}$. We denote the ℓ -Sylow subgroup of $H(k(v))$ by Δ_v , and Δ_Q is defined as $\prod_{v \in Q} \Delta_v$ for $Q \in X$.
- For $Q \in X$, R_Q is a complete noetherian local $o_\lambda[\Delta_Q]$ -algebra with the residue field k_λ , and M_Q is an R_Q -module. For $Q = \emptyset$, $(R_\emptyset, M_\emptyset) = (R, M)$.
- There is a surjection of local o_λ -algebras

$$R_Q/I_Q R_Q \rightarrow R$$

for each non-empty $Q \in X$ ¹³. Here $I_Q \subset o_\lambda[\Delta_Q]$ denotes the augmentation ideal of $o_\lambda[\Delta_Q]$.

- The homomorphism $R_Q/I_Q R_Q \rightarrow \text{End}_{o_\lambda} M_Q/I_Q M_Q$ factors through R , and $M_Q/I_Q M_Q$ is isomorphic to M as an R -module.
- M_Q is free and of rank α as an $o_\lambda[\Delta_Q]$ -module for a fixed $\alpha \geq 1$.

In [55], the condition that R_Q is Gorenstein and M_Q is a free R_Q -module is required.

Taylor–Wiles systems are commutative ring theoretic versions of Kolyvagin’s Euler systems. The following theorem is the most important consequence of the existence of Taylor–Wiles systems.

Theorem 4.2 (Complete intersection and freeness criterion). *For a Taylor–Wiles system $\{R_Q, M_Q\}_{Q \in X}$ for (R, M) and a torus H of dimension d , assume the following conditions:*

1. For any $m \in \mathbb{N}$

$$v \in Q \Rightarrow q_v \equiv 1 \pmod{\ell^m}$$

holds for infinitely many $Q \in X$.

2. The cardinality r of Q is independent of $Q \in X$ for non-empty Q .
3. R_Q is generated by at most dr elements as a complete local o_λ -algebra for non-empty $Q \in X$.

¹³For deformation rings, $R_Q/I_Q R_Q \simeq R$ usually holds. The condition here is relaxed so that it applies to Hecke algebras directly.

Then one has:

- (complete intersection property) R is o_λ -flat and of relative complete intersection of dimension zero;
- (freeness) M is a free R -module.

In particular, M is a faithful R -module. If we denote the image of R in $\text{End}_{o_\lambda} M$ by T , we have $R \simeq T$ as a consequence of Theorem 4.2. So the complete intersection and the freeness criterion is also regarded as an *isomorphism criterion*. The Auslander–Buchsbaum formula for regular local rings plays a very important role in the proof of the theorem.

The reason why we have the complete intersection property is the following. When Q and N vary, a well chosen limit of $R_Q/m_{R_Q}^N$ tends to be a power series ring over o_λ in dr variables¹⁴. This is implied by condition (3) of 4.2. Usually $R = R_Q/I_Q R_Q$ holds, thus R is defined by $d \cdot \sharp Q = dr$ equations in R_Q , and hence it is a complete intersection in the limit.

If the complete intersection property and freeness both hold, then this pair of properties is inherited by descendants of (R, M) . This is formulated in the following way.

Definition 4.3. 1. An admissible quintet is a quintet $(R, T, \pi, M, \langle , \rangle)$, where R is a complete local o_λ -algebra, T is a finite flat o_λ -algebra, $\pi : R \rightarrow T$ is a surjective o_λ -algebra homomorphism, M is a faithful finitely generated T -module which is o_λ -free, and $\langle , \rangle : M \otimes_{o_\lambda} M \rightarrow o_\lambda$ is a perfect pairing which induces $M \simeq \text{Hom}_{o_\lambda}(M, o_\lambda)$ as a T -module.

2. An admissible quintet $(R, T, \pi, M, \langle , \rangle)$ is distinguished if R is a complete intersection and M is R -free (and hence π is an isomorphism).

3. By an admissible morphism from $(R', T', \pi', M', \langle , \rangle')$ to $(R, T, \pi, M, \langle , \rangle)$ we mean a triple (α, β, ξ) . Here $\alpha : R' \rightarrow R, \beta : T' \rightarrow T$ are surjective o_λ -algebra homomorphisms making the following diagram

$$\begin{array}{ccc}
 R' & \xrightarrow{\alpha} & R \\
 \pi' \downarrow & & \downarrow \pi \\
 T' & \xrightarrow{\beta} & T
 \end{array}$$

commutative, and $\xi : M \hookrightarrow M'$ is an injective T' -homomorphism onto an o_λ -direct summand. (Note that we do not assume the restriction of \langle , \rangle' to $\xi(M)$ is \langle , \rangle .)

A Taylor–Wiles system gives rise to a distinguished admissible quintet for a suitably chosen pairing on M under the conditions of Theorem 4.2.

¹⁴In applications, R_Q is a deformation ring. This implies that the deformation functor which defines R_Q behaves as if it were unobstructed for a well-chosen limit of Q .

Assume that (α, β, ξ) is an admissible morphism from $(R', T', \pi', M', \langle \cdot, \cdot \rangle')$ to $(R, T, \pi, M, \langle \cdot, \cdot \rangle)$. There is an abstract criterion for $(R', T', \pi', M', \langle \cdot, \cdot \rangle')$ to be a distinguished quintet if $(R, T, \pi, M, \langle \cdot, \cdot \rangle)$ is distinguished.

By duality we have

$$\hat{\xi}: M' \simeq \text{Hom}_{o_\lambda}(M', o_\lambda) \rightarrow \text{Hom}_{o_\lambda}(M, o_\lambda) \simeq M$$

such that

$$\langle \xi(x), y \rangle' = \langle x, \hat{\xi}(y) \rangle \quad \text{for all } x \in M, y \in M'.$$

We fix an o_λ -algebra homomorphism $f_T: T \rightarrow o_\lambda$. We define f_R (resp. $f_{R'}$) as $f_T \circ \pi$ (resp. $f \circ \beta \circ \pi'$).

Theorem 4.4 (abstract level raising formalism). *For an admissible morphism between admissible quintets $(R', T', \pi', M', \langle \cdot, \cdot \rangle') \rightarrow (R, T, \pi, M, \langle \cdot, \cdot \rangle)$, we assume the following conditions:*

1. $(R, T, \pi, M, \langle \cdot, \cdot \rangle)$ is distinguished.
2. T and T' are reduced, $M' \otimes_{o_\lambda} E_\lambda$ is $T' \otimes_{o_\lambda} E_\lambda$ -free, and its rank is the same as the rank of M over T .
3. $\hat{\xi} \circ \xi(M) = \Delta \cdot M$ holds for some non-zero divisor Δ in T .
4. An inequality

$$\begin{aligned} & \text{length}_{o_\lambda} \ker f_{R'} / (\ker f_{R'})^2 \\ & \leq \text{length}_{o_\lambda} \ker f_R / (\ker f_R)^2 + \text{length}_{o_\lambda} o_\lambda / f_T(\Delta) o_\lambda \end{aligned}$$

holds.

Then $(R', T', \pi', M', \langle \cdot, \cdot \rangle')$ is also distinguished, that is, $R' \simeq T'$, R is a complete intersection, and M' is T' -free.

A generalization of Wiles’ isomorphism criterion in [57] by Lenstra is used in the proof. Though the main two theorems in this section are a consequence of the general commutative algebra machinery, they are extremely deep. In the next section, we will see how modular varieties should yield the setup formulated in this section.

Remark 4.5. 1. The idea of an admissible quintet first appeared in the work of Ribet in [38]. $M/\hat{\xi} \circ \xi(M)$ is the congruence module.

2. The o_λ -direct summand property in Definition 4.3, (3) is an abstract form of what is known as “Ihara’s Lemma”.

3. $\ker f_R / (\ker f_R)^2$ is regarded as a Selmer group. When R is a complete intersection, its o_λ -length is computed and equal to $\text{length}_{o_\lambda} o_\lambda / \eta_R$. Here η_R is the ideal generated by the image of 1 under $o_\lambda \xrightarrow{\hat{f}_R} \text{Hom}_{o_\lambda}(R, o_\lambda) \simeq R \xrightarrow{f_R} o_\lambda$ as introduced by Wiles in [57]. Note that the finiteness of the Selmer group is a consequence of the reducedness of T (Taylor–Wiles systems do not give finiteness directly).

5. Taylor–Wiles systems: a strategy for the construction

Take a pair (G, X) which defines a modular variety, and take a finite dimensional representation $\nu: G \rightarrow \text{Aut}_{E_\lambda} V_{E_\lambda}$ defined over E_λ . For simplicity, we assume that G is quasi-split over \mathbb{Q} . ν defines a $G(\mathbb{A}_f)$ -equivariant local system \mathcal{F}_ν on $M(G, X)$ ¹⁵, which yields a family of local systems \mathcal{F}_ν^K on $M_K(G, X)$ at each level K . By choosing an o_λ -lattice in V_{E_λ} , \mathcal{F}_ν^K has an o_λ -structure $\mathcal{F}_{\nu, o_\lambda}^K$. We limit ourselves to the adelic action and the Hecke algebra action which respect the integral structure. Let Σ be a finite set of primes containing ℓ , and for a factorizable $K = \prod_q K_q$, K_Σ and K^Σ are defined by $K_\Sigma = \prod_{q \in \Sigma} K_q$, $K^\Sigma = \prod_{q \notin \Sigma} K_q$.

For simplicity, we only consider Hecke operators outside Σ ¹⁶. $H(G(\mathbb{A}_f^\Sigma), K^\Sigma)_{o_\lambda}$ is the Hecke algebra formed by o_λ -valued functions on the adèle group without components in Σ . Let us begin with a remark on homological algebras: $R\Gamma_B(M_K(G, X), \mathcal{F}_{\nu, o_\lambda}^K)$ belongs to $D^+(H(G(\mathbb{A}_f^\Sigma), K^\Sigma)_{o_\lambda})$, the derived category of $H(G(\mathbb{A}_f^\Sigma), K^\Sigma)_{o_\lambda}$ -complexes bounded from below. This is shown by taking the canonical flasque resolution of $\mathcal{F}_{\nu, o_\lambda}^K$.

Assume that K_q is maximal and hyperspecial for $q \notin \Sigma$. Then $H(G(\mathbb{A}_f^\Sigma), K^\Sigma)_{o_\lambda} = \otimes_{q \notin \Sigma} H(G(\mathbb{Q}_q), K_q)_{o_\lambda}$ is commutative. Let m_Σ denote a maximal ideal of $H(G(\mathbb{A}_f^\Sigma), K^\Sigma)_{o_\lambda}$. Throughout this section we make the following assumption:

Assumption 5.1 (Vanishing of cohomology for $\overline{\mathbb{F}}_\ell$ -coefficients). For any (sufficiently small) K ,

$$H_B^i(M(G, X)_K, \mathcal{F}_{\nu, o_\lambda}^K \otimes_{o_\lambda} k_\lambda)_{m_\Sigma} = 0$$

for $i \neq q(G)$.

This type of vanishing statement is known over E_λ when the weight of ν is sufficiently regular, but for torsion coefficients it is difficult to prove a vanishing statement¹⁷. Moreover, we need to have a vanishing claim *which holds uniformly* in K .

Let $M_\Sigma = H_B^{q(G)}(M(G, X)_K, \mathcal{F}_{\nu, o_\lambda}^K)_{m_\Sigma}$ be the localization of the middle dimensional cohomology group at m_Σ ¹⁸. By Assumption 5.1, it is o_λ -free.

T_Σ is defined as the image of $H(G(\mathbb{A}_f^\Sigma), K^\Sigma)_{o_\lambda}$ in M_Σ . We call it the ℓ -adic Hecke algebra.

When $M(G, X)$ is a Shimura variety $\text{Sh}(G, X)$ with reflex field E , we further assume, by using the étale cohomology, that a reciprocity law of the following form for $(\text{Sh}(G, X), \nu)$ holds:

$$M_\Sigma \otimes_{o_\lambda} \overline{\mathbb{Q}}_\ell \simeq \oplus_{\pi \in \Pi_\rho} V_\rho \otimes (\pi_f^K)^{a(\pi)}.$$

¹⁵For the center $Z(G)$ of G , the image of $Z(G)(\mathbb{Q}) \cap K$ by ν must be trivial for some K .

¹⁶We also need Hecke actions at Σ . At ℓ , one should use a semi-subgroup $G(\mathbb{Q}_\ell)_+$ of $G(\mathbb{Q}_\ell)$ because the full action of $G(\mathbb{Q}_\ell)$ does not preserve the lattice structure in general.

¹⁷In fact, there are non-zero torsion classes in general unless one puts restrictive conditions on m_Σ .

¹⁸Further localizations by elements in $H(G(\mathbb{A}_\Sigma), K_\Sigma)$ are necessary.

Here ${}^L G$ is the L -group of G , Π_ρ is a packet which corresponds to $\rho: G_E \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell)$ and is cohomological for v ¹⁹. V_ρ is a G_E -representation defined by a finite dimensional representation of ${}^L G$ determined by v . We try to construct a Taylor–Wiles system for (T_Σ, M_Σ) .

As is suggested in the introduction, a modular variety for a given level K does not admit a symmetry by a group. To have a group action, we must use an open subgroup of $G(\mathbb{A}_f)$ which is smaller than K . Take a \mathbb{Q} -parabolic subgroup P of G , and a quotient torus H of P which is disjoint from the center $Z(G)$, that is, the image of $Z(G)$ in H is trivial. It is very important to have a torus which is disjoint from the center: this gives us a *geometric direction*, which, in the case of Shimura varieties, cannot be obtained from abelian extensions of the reflex field. This feature is quite different from Euler systems.

For a prime q , let $K_{P,q} = \{g \in G(\mathbb{Z}_q), g \bmod q \in P(\mathbb{F}_q)\}$ be a parahoric subgroup at q defined by P , and let $K_{P,H,q} = \{g \in K_{P,q}, g \bmod q \in \ker(P(\mathbb{F}_q) \rightarrow H(\mathbb{F}_q)^\ell)\}$ be a subgroup depending on H . Here $H(\mathbb{F}_q)^\ell$ is the maximal subgroup whose order is prime to ℓ . Then $K_{P,q}/K_{P,H,q}$ is isomorphic to Δ_q , the ℓ -Sylow subgroup of $H(\mathbb{F}_q)$.

For a finite set of finite primes Q which is disjoint from Σ , one sets

$$K_{P,Q} = \prod_{q \in Q} K_{P,q} \cdot K^Q,$$

$$K_{P,H,Q} = \prod_{q \in Q} K_{P,H,q} \cdot K^Q,$$

and $\Delta_Q = \prod_{q \in Q} \Delta_q$. Since H is disjoint from the center, the natural covering

$$\pi_Q: \text{Sh}_{K_{P,H,Q}} \rightarrow \text{Sh}_{K_{P,Q}}$$

is an étale Galois covering with Galois group Δ_Q if K^Q is small enough to make group actions free. We view $R\Gamma_B(\text{Sh}_{K_{P,H,Q}} \mathcal{F}_{v,o_\lambda}^{K_{P,H,Q}})$ as an object in

$$D^+(H(G(\mathbb{A}_f^{\Sigma \cup Q}), K^{\Sigma \cup Q})_{o_\lambda}).$$

Then we make the following simple observation (*perfect complex argument*):

Lemma 5.2. *Let $\pi: X \rightarrow Y$ be an étale Galois covering between finite dimensional manifolds²⁰ with Galois group G . Let \mathcal{F} be a smooth Λ -sheaf on Y . Then $R\Gamma_B(X, \pi^* \mathcal{F})$ is represented by a perfect complex of $\Lambda[G]$ -modules, and*

$$R\Gamma_B(X, \pi^* \mathcal{F}) \otimes_{\Lambda[G]}^{\mathbb{L}} \Lambda[G]/I_G \simeq R\Gamma_B(Y, \mathcal{F})$$

holds in $D^b(\Lambda[G])$. Here I_G is the augmentation ideal of $\Lambda[G]$, and the map is induced by the trace map.

¹⁹We are ignoring the role of centralizer groups, endoscopy, and so on. See [32] for a precise conjectural description. Moreover, we only have information on the semi-simplification V_ρ^{ss} .

²⁰These manifolds must satisfy reasonable finiteness conditions on cohomology groups, which are true for the algebraic varieties or modular varieties that we are interested in.

By Lemma 5.2, we know that $R\Gamma_B(\mathrm{Sh}_{K_{P,H,Q}}, \mathcal{F}_{v,o_\lambda}^{K_{P,H,Q}})$ is, in $D^b(o_\lambda[\Delta_Q])$, a perfect complex of $o_\lambda[\Delta_Q]$ -modules, that is, it is quasi-isomorphic to a bounded complex of free $o_\lambda[\Delta_Q]$ -modules. We assume 5.1 holds and localize at the maximal ideal m_Q of $H(G(\mathbb{A}_f^{\Sigma \cup Q}), K^{\Sigma \cup Q})_{o_\lambda}$ below m_Σ . Then

$$M_Q = H_B^{q(G)}(\mathrm{Sh}_{K_{P,H,Q}}, \mathcal{F}_{v,o_\lambda}^{K_{P,H,Q}})_{m_Q}$$

is $o_\lambda[\Delta_Q]$ -free since it is the only non-zero cohomology of a perfect $o_\lambda[\Delta_Q]$ -complex. Then $M_Q \otimes_{o_\lambda[\Delta_Q]} o_\lambda[\Delta_Q]/I_{\Delta_Q} = M_{0,Q}$. Here $M_{0,Q} = H_B^{q(G)}(\mathrm{Sh}_{K_{P,Q}}, \mathcal{F}_{v,o_\lambda}^{K_{P,Q}})_{m_Q}$.

We define T_Q as the image of $H(G(\mathbb{A}_f^{\Sigma \cup Q}), K^{\Sigma \cup Q})_{o_\lambda}$ in $\mathrm{End}_{o_\lambda} M_Q$. Then the expectation is that $M_{0,Q}$ is a direct sum of M for a good choice of Q , and that (T_Q, M_Q) forms a Taylor–Wiles system for (T_Σ, M_Σ) as Q varies. Hence, a system for deformation rings will be obtained for appropriate choices of deformation functors.

To relate $M_{0,Q}$ to the original M_Σ , we already need information from the Galois parameter for the Langlands (or Arthur) packets for automorphic forms which contributes to $M_{0,Q}$, since we need to remove non-spherical components from $M_{0,Q}$. Some version of the compatibility of local and global parametrizations is needed.

This program for constructing a Taylor–Wiles system, and in particular for combining the perfect complex argument and the vanishing assumption 5.1 to obtain a system, was made explicit and carried out in two special cases in [18]. One case is when (G, X) defines a Shimura curve, that is, $G = G_D$ as in §3. The Hecke actions on H_B^0 and H_B^2 are easily determined for Shimura curves, and Assumption 5.1 is true if we localize at the maximal ideals of Hecke algebras which correspond to absolutely irreducible two dimensional representations. In another case, Hida varieties are used to treat some situations which do not arise from Shimura curves.

In general, the program is very effective when G defines Hida varieties, since the vanishing assumption is trivially true. To study non-abelian class field theory, especially the question of modularity, or deformations of absolutely irreducible representations, this case is quite useful, since one can apply the Jacquet–Langlands correspondence for inner forms (assuming that it is already known) to convert the problem to a compact inner form. For unitary groups, this approach was taken by Harris and Taylor, showing the $R = T$ theorem for the n -dimensional representations of CM fields constructed by Clozel [9] under several restrictive assumptions. Arithmetic geometrical difficulties are all encoded in the compatibility of local and global Langlands correspondences outside ℓ and in the description of local monodromy at ℓ .

Unfortunately, Hida varieties do not admit any natural Galois action. Since one needs to have finer information coming from geometry, it is still an important problem to construct Taylor–Wiles systems for higher dimensional Shimura varieties.

There are several possible solutions. One solution would be to establish Assumption 5.1, that is, a vanishing theorem for $\overline{\mathbb{F}}_\ell$ -coefficients. A partial solution will be given in the sequel for unitary groups with signature $(m, 1)$ by using a geometrical realization of the Jacquet–Langlands correspondence. As far as the author knows,

these unitary Shimura varieties are the only examples where the program is carried out for the middle dimensional cohomology in arbitrary higher dimensions.

For other approaches in the case of Siegel modular varieties using p -adic Hodge theory, see [34]. A Taylor–Wiles system for the Hilbert–Siegel case, especially $G\mathrm{Sp}_{4,\mathbb{Q}}$, has been studied by Tilouine.

Another possible solution would be to avoid the use of vanishing theorems. We have enough information about the alternating sum of cohomology groups, so in working with a suitable K_0 -group of virtual Galois–Hecke bimodules instead of Galois–Hecke bimodules, Taylor–Wiles systems might work for virtual representations.

Remark 5.3. The perfect complex argument is simple but very powerful in the cohomological study of congruences between automorphic forms. There are several other uses of the perfect complex argument.

One example would be the construction of nearly ordinary Hecke algebra for GL_2 , with an *exact control theorem*.

In particular for the nearly ordinary Hecke algebra T_Σ , there is a faithful T_Σ -module M_Σ with the following properties:

- M_Σ is free over $\Lambda = o_\lambda[[\mathcal{X}_\Sigma^{\mathrm{gl}} \times \mathcal{X}_\ell^{\mathrm{loc}}]]$.
- Take an algebraic character $\chi: \Lambda \rightarrow o'_{\lambda'}$ of discrete type $((k_i)_{i \in I_F}, w)$ such that $\tilde{\chi} = \chi / (\chi_{\mathrm{cycle}}^{-w} \cdot \prod_{v|\ell} \prod_{i \in I_{F_v}} \chi_i^{k_i-2})$ is of order prime to ℓ . $M_\Sigma \otimes_{\Lambda, \chi} o'_{\lambda'}$ is a lattice in the space of nearly ordinary forms of type $((k_i)_{i \in I_F}, w)$ with “nebencharacter” $\tilde{\chi}$. Moreover, when the degree $[F : \mathbb{Q}]$ is even²¹, it is *exactly* the image of $H_B^0(M_K, \mathcal{F}_{((k_i)_{i \in I_F}, w), o'_{\lambda'}}^K)$. Here M_K is a Hida variety associated to a definite quaternion algebra, and $\mathcal{F}_{((k_i)_{i \in I_F}, w), o'_{\lambda'}}^K$ is a smooth $o'_{\lambda'}$ -sheaf on M_K (cf. [17]) which realizes the reciprocity law for forms of type $((k_i)_{i \in I_F}, w)$ over $\overline{\mathbb{Q}_\ell}$.

Another example would be the level optimization for GL_2 in a special case, which gives an interpretation of Carayol’s lemma [7], [17].

6. Geometric Jacquet–Langlands correspondence

We use the strategy in Section 5 for some unitary Shimura varieties where the non-abelian reciprocity is established by Kottwitz [33], and we construct a Taylor–Wiles system for the middle dimensional cohomology group. To do this, we establish an explicit Jacquet–Langlands correspondence which preserves o_λ -lattice structures by arithmetic geometrical means, in particular by analyzing bad reductions of Shimura varieties. This is called a geometric Jacquet–Langlands correspondence. This first

²¹One uses Shimura curves when the degree is odd.

appeared in the level optimization problem mentioned in §3 in the case of modular curves. This also gives an arithmetic geometrical meaning of Hida varieties.

Let us give a simple example. Let $S_{D,K} = \text{Sh}(G_D, X_D)_K$ be a Shimura curve as in §2. For a finite place v of F , assume that D is split at v , and $v \nmid \ell$. Let \tilde{D} be a quaternion algebra which is definite at all infinite places, non-split at v , but has the same local invariants as D at the other finite places. \tilde{D} defines a Hida variety $M_{\tilde{D}}$, and we have an isomorphism $D^\times(\mathbb{A}_{F,f}^v) \simeq \tilde{D}^\times(\mathbb{A}_{F,f}^v)$ outside v .

Assume $K = \prod_u K_u$ is factorizable, and K_v is an Iwahori subgroup of GL_{2,F_v} . We set $\tilde{K}_v = o_{\tilde{D}_v}^\times$ for a maximal order $o_{\tilde{D}_v}$ of D_v , and we view $\tilde{K} = \tilde{K}_v \cdot \prod_{u \neq v} K_u$ as a subgroup of $\tilde{D}^\times(\mathbb{A}_{F,f})$.

Then we have the following geometric realization of the Jacquet–Langlands [25], and of the Shimizu [43] correspondence²²:

Proposition 6.1. *For $\Lambda = E_\lambda$, we have a (non-canonical) isomorphism as $H(D^\times(\mathbb{A}_{F,f}^{\ell,v}), K^{\ell,v})_{o_\lambda}$ -modules*

$$W_0 H_{\text{ét}}^1(S_{D,K}, \Lambda) \simeq H^0(M_{\tilde{D},\tilde{K}}, \Lambda)$$

modulo Eisenstein modules. W_0 is the weight 0 part of the weight filtration, and the decomposition with respect to the Hecke action gives the Jacquet–Langlands correspondence.

Here we view the Hida variety $M_{\tilde{D},\tilde{K}}$ as a variety over $\overline{k(v)}$, and we regard it as the set of supersingular points²³, that is, the set of points which correspond to formal o_v -modules of height 2. Since the special fiber at v of the arithmetic model of $S_{D,K}$ has two kinds of components which meet at supersingular points, the claim follows from the standard calculation of the weight filtration using the dual graph of the special fiber²⁴. There is also a variant for any finite o_λ -algebra Λ , which preserves o_λ -lattice structures.

So the Jacquet–Langlands correspondence is realized on the weight filtration of local monodromy action where the Shimura variety admits a bad reduction. This is our starting point²⁵.

6.1. Arithmetic model of unitary Shimura varieties. As in §2, F denotes a totally real field and $[F : \mathbb{Q}] = g$. I_F is the set of the embeddings $\iota : F \hookrightarrow \mathbb{R}$. Take an imaginary quadratic field E_0 over \mathbb{Q} , and let $E = E_0 \cdot F$ be the composite field. Let $\text{Gal}(E/F) = \langle \sigma \rangle$.

Let D be a central division algebra over E of dimension n^2 , and let $*$: $D \rightarrow D$ be a positive involution of the second kind, that is, an involution which induces σ on E .

²²There is also a version when D is ramified at v , which plays an essential role in [39], [36].

²³This identification is non-canonical.

²⁴The Hecke action on the set of the irreducible components is Eisenstein, and we are ignoring this part.

²⁵It will be desirable to have a motivic correspondence over a global field.

We require the following condition on ι at infinity: At one infinite place $\iota_0 \in I_F$, $*$ is equivalent to $g \mapsto J^t \bar{g} J^{-1}$ with $J = \begin{pmatrix} 1_{n-1} & 0 \\ 0 & -1 \end{pmatrix}$, and it is equivalent to the standard involution $g \mapsto {}^t \bar{g}$ for the other $\iota \neq \iota_0$.

Let $U(D) = \{g \in D^{\text{op}\times}, g \cdot g^* = 1_D\}$ be the unitary group. Let $GU(D) = \{g \in D^{\text{op}\times}, g \cdot g^* = \nu(g) \cdot 1_D, \nu(g) \in \mathbb{G}_{m,F}\}$ be the group of unitary similitudes, seen as a reductive group over F .

Let $G' = \text{Res}_{F/\mathbb{Q}} U(D)$ be the Weil restriction of the unitary group, and let G be the inverse image of $\mathbb{G}_{m,\mathbb{Q}}$ by $\text{Res}_{F/\mathbb{Q}} GU(D) \rightarrow \text{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}$.

$$1 \rightarrow G' \rightarrow G \xrightarrow{\nu} \mathbb{G}_{m,\mathbb{Q}} \rightarrow 1.$$

We fix an embedding $E_0 \hookrightarrow \mathbb{C}$. This defines a CM-type on E , and for each $\iota: F \hookrightarrow \mathbb{R}$, we have identifications

$$E \otimes_{F,\iota} \mathbb{R} \simeq \mathbb{C}, \quad D \otimes_{F,\iota} \mathbb{R} \simeq M_n(\mathbb{C}).$$

It follows that

$$G'_{\mathbb{R}} \simeq U(n-1, 1) \times U(n)^{g^{-1}}$$

by our choice.

Define a group homomorphism

$$h_0: \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}} \rightarrow G_{\mathbb{R}}$$

$$z \mapsto h_0(z) = \left(\left(\begin{matrix} z \cdot 1_{n-1} & 0 \\ 0 & \bar{z} \end{matrix} \right), z 1_n, \dots, z 1_n \right).$$

The associated symmetric space

$$X = G_{\infty}(\mathbb{R})/K_{\infty}$$

(K_{∞} is the centralizer of $h_0(\sqrt{-1})$) is a complex ball of dimension $n-1$.

For a compact open subgroup $K \subset G(\mathbb{A}_f)$, the corresponding Shimura variety $\text{Sh}_K(G, X)$ is compact by our assumption on D , and the reflex field $E(G, X)$ is E . We view $\text{Sh}_K(G, X)$ as a generalization of Shimura curves. The determination of the reciprocity law is due to Kottwitz [33], and the bad reductions have been studied especially by Rapoport and Zink [37], Harris, and Taylor [20].

The moduli interpretation of these varieties is given by Shimura. We need arithmetic models over the integers, so we consider it over $o_E \otimes_{\mathbb{Z}} \mathbb{Z}_p$ by fixing a prime p . The details are found in [20]. To avoid technical problems which arise from obstructions to the Hasse principle for G , we assume that n is odd in this exposition.

We fix a finite place w of the reflex field E of residual characteristic p . Assume p splits completely in E_0 , and let \wp be the place of E_0 below w . P_{\wp} is the set of places of E which divide \wp .

Since p is split in E_0 , the unitary group has a simpler form, and we have an isomorphism

$$G_{\mathbb{Q}_p} \simeq \mathbb{G}_{m,\mathbb{Q}_p} \times \prod_{u \in P_{\wp}} \text{Res}_{E_u/\mathbb{Q}_p} D_u^{\text{op}\times}.$$

In the following, we only consider compact open subgroups K_p of $G(\mathbb{Q}_p)$ which are of the form $\mathbb{Z}_p^\times \cdot \prod_{u \in P_\wp} K_u$ for $K_u \subset D^{\text{op}\times}(E_u)$. We assume moreover that D is split at w , and we identify the w -component with $H_w = D_w^{\text{op}\times} \simeq \text{GL}_{n, E_w}$.

Take a maximal order o_D of D which is stable under $*$. $o_D \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is identified with $\prod_{u \in P_\wp} o_{D_u} \times \prod_{u \in P_\wp} o_{D_{\sigma(u)}}$. For $u \in P_\wp$, let e_u denote the projector corresponding to o_{D_u} .

For an o_w -scheme U , let A be an abelian scheme over U of relative dimension gn^2 with o_D -multiplication. We assume that the o_D -action on the relative Lie algebra $\text{Lie}A/U$ satisfies the following conditions:

1. For $u \in P_\wp$ different from w , $e_u \text{Lie}A/U$ is zero.
2. $e_w \text{Lie}A/U$ is a locally free o_U -module of rank n , and the o_w -action on $e_w \text{Lie}A/S$ is the multiplication through $o_w \rightarrow \Gamma(U, \mathcal{O}_U)$.

By condition (1), for the p -divisible group $A[p^\infty]$, $e_u A[p^\infty]$ is an étale o_u -divisible module for $u \in P_\wp$ different from w . We denote the Tate module by $T_u(A)$.

We take a compact open subgroup $K = K_p \cdot K^p \subset G(\mathbb{A}_f)$. We assume that $K_w = \text{GL}_n(o_w)$ and that K^p is sufficiently small. We denote the product $\mathbb{Z}_p^\times \cdot \prod_{u \in P_\wp, u \neq w} K_u \cdot K^p$ by K^w . Then $K = K_w \cdot K^w$ by the definition. We view D as a left D^{op} -module and denote it by V ²⁶. The integral structure is given by $V_{\mathbb{Z}} = o_D$.

Definition 6.2. Let U be an o_w -scheme, let A be an abelian scheme over U with o_D -multiplication which satisfies the Lie algebra conditions described above, and let \bar{p} be a homogeneous polarization of A . Then (A, \bar{p}) is of type $(o_D, V_{\mathbb{Z}}; K)$ if the following rigidification structures are attached:

1. For any point $s \in U$, there is a polarization $\lambda \in \bar{p}$ of degree prime to p such that λ induces $*$ on D as its Rosati involution.
2. For any geometric point \bar{s} of characteristic p , a class $k \pmod{K^w}$ of o_D -linear symplectic similitudes

$$k: \prod_{u \in P_\wp, u \neq w} T_u(A)_{\bar{s}} \times T^p(A)_{\bar{s}} \simeq V_{\mathbb{Z}} \otimes \hat{\mathbb{Z}}^p.$$

Here $\pi_1(U, \bar{s})$, the étale fundamental group, is mapped to K^w .

By a standard argument, the moduli functor

$$U \mapsto \mathcal{S}h_K(U),$$

where $\mathcal{S}h_K(U)$ is the set of the isomorphism classes of polarized abelian schemes of type $(o_D, V_{\mathbb{Z}}; K)$ over U , is representable by a projective smooth scheme $\mathcal{S}h_K$ over o_w , which gives a model for $\text{Sh}_K(G, X)$.

²⁶With a symplectic form which we do not make precise here.

Let $\mathcal{A}^{\text{univ}}$ be the universal abelian scheme over \mathcal{H}_K . Then o_{D_w} acts on $e_w \mathcal{A}^{\text{univ}}[p^\infty]$, and is isomorphic to $M_n(o_w)$. By Morita equivalence, this gives an o_w -divisible module \mathcal{C}_w of height n on \mathcal{H}_K , so that $e_w \mathcal{A}^{\text{univ}}[p^\infty] \simeq \mathcal{C}_w^{\oplus n}$.

For general $K_w \subset H_w(E_w) \simeq \text{GL}_n(E_w)$, we have a flat arithmetic model \mathcal{H}_K over o_w by using the notion of level structures due to Drinfeld [15].

6.2. Inductive structure. We briefly describe an inductive structure of arithmetic models in characteristic p . The inductive structure was first studied by Boyer in the function field case [1]. In our unitary case it is due to Harris and Taylor in [20].

For $N \geq 1$, set $K_N = K_{N,w} \cdot K^w$, $K_{N,w} = \ker(H_w(o_w) \rightarrow H_w(o_w/m_w^N))$, $X_N = \mathcal{H}_{K_N}$, and $S = \text{Spec } o_w$. Let s (resp. η) denote the closed (resp. generic) point of S .

For $1 \leq h \leq n$, $X_{N,s}^{[h]}$ is the set of points in the special fiber $(X_N)_s$ of X_N where the étale (resp. connected) part of \mathcal{C}_w has height $n-h$ (resp. height h). $X_{N,s}^{[h]}$ is locally closed in $(X_N)_s$, and the canonical filtration on $X_{N,s}^{[h]}$

$$0 \rightarrow \mathcal{C}_w^{\text{conn}} \rightarrow \mathcal{C}_w \rightarrow \mathcal{C}_w^{\text{ét}} \rightarrow 0$$

has the property that $\mathcal{C}_w^{\text{ét}}$ is étale locally isomorphic to $(E_w/o_w)^{n-h}$.

We consider the formal completion $\mathcal{X}_N^{[h]} = \hat{X}_N|_{X_{N,s}^{[h]}}$ of X_N along $X_{N,s}^{[h]}$. Note that Hecke correspondences induce correspondences on $\mathcal{X}_N^{[h]}$.

Let P_h be the standard maximal parabolic subgroup of H_w which fixes the filtration $0 \subset o_w^h \subset o_w^n$.

Define $\mathcal{Y}_N^{[h]}$ to be the subspace of $\mathcal{X}_N^{[h]}$ where the canonical filtration of $\mathcal{C}_w[m_w^N]$ is compatible with the filtration $0 \subset (m_w^{-N}/o_w)^h \subset (m_w^{-N}/o_w)^n$ via the universal Drinfeld level structure $(m_w^{-N}/o_w)^n \rightarrow \mathcal{C}_w[m_w^N]$. $P_h(o_w/m_w^N)$ acts naturally on $\mathcal{Y}_N^{[h]}$.

We consider the pro-systems of formal schemes

$$\mathcal{X}_\infty^{[h]} = \varprojlim_N \mathcal{X}_N^{[h]} \quad \text{and} \quad \mathcal{Y}_\infty^{[h]} = \varprojlim_N \mathcal{Y}_N^{[h]}.$$

Note that $H_w(o_w)$ (resp. $P_h(o_w)$)-action on $\mathcal{X}_\infty^{[h]}$ (resp. $\mathcal{Y}_\infty^{[h]}$) is extended to $H_w(E_w)$ (resp. $P_h(E_w)$).

Then the canonical morphism²⁷

$$H_w(E_w) \wedge^{P_h(E_w)} \mathcal{Y}_\infty^{[h]} \rightarrow \mathcal{X}_\infty^{[h]}$$

is in fact an isomorphism by a method of Boyer ([1], [20]), that is,

$$\mathcal{X}_\infty^{[h]} \simeq H_w(E_w) \wedge^{P_h(E_w)} \mathcal{Y}_\infty^{[h]}.$$

²⁷For a group G and a right (resp. left) G -space X (resp. Y), $X \wedge^G Y$ denotes the contracted product, i.e., by viewing Y as a right G -space, $X \wedge^G Y = X \times Y/G$, where G acts diagonally on $X \times Y$.

Since the space itself is “parabolically induced from P_h ”, then for $\Lambda = \overline{\mathbb{F}}_\ell$, we have an isomorphism of admissible modules²⁸

$$\varinjlim_N H_c^i(X_{N,\bar{s}}^{[h]}, R^j \psi(\Lambda)) \simeq \text{Ind}_{P_h(E_w)}^{H_w(E_w)} \varinjlim_N H_c^i(Y_{N,\bar{s}}^{[h]}, R^j \psi(\Lambda)),$$

where $Y_N^{[h]}$ is the underlying scheme of $\mathcal{Y}_N^{[h]}$. In particular for $h < n$, this implies that the $H_v(E_w)$ -representations occurring in proper support cohomologies $\varinjlim_N H_c^i(X_{N,\bar{s}}^{[h]}, R^j \psi(\Lambda))$ are induced from admissible representations of $P_h(E_w)$. Admissibility follows from the finiteness of nearby cycle cohomologies.

6.3. Vanishing theorem. We take a supercuspidal representation in the sense of Vignéras

$$\bar{\pi}_w : H_w(E_w) \rightarrow \text{Aut } V_{\bar{\pi}_w}.$$

Here $V_{\bar{\pi}_w}$ is an $\overline{\mathbb{F}}_\ell$ -vector space, that is, $\bar{\pi}_w$ is admissible, irreducible and cannot be obtained by a non-trivial parabolic induction. We assume that $\bar{\pi}_w^{K_w}$ is non-zero. $\bar{\pi}_w$ corresponds to a finite dimensional irreducible representation of $H(H_w(E_w), K_w)_{\overline{\mathbb{F}}_\ell}$.

Theorem 6.3 (Vanishing theorem for $\overline{\mathbb{F}}_\ell$ -coefficients). *For a finite place w of E of residual characteristic p , assume that p is split in E_0 , D is split at w , and $p \neq \ell$. For a compact open subgroup $K = K_w \cdot K^w$, assume moreover that K_w is contained in an Iwahori subgroup and K^w is sufficiently small. Then for any o_λ -local system $\mathcal{F}_{v,o_\lambda}^K$ on Sh_K corresponding to a finite dimensional representation v of G , $\bar{\pi}_w$ does not appear as a subquotient of $H_B^i(\text{Sh}_K, \mathcal{F}_{v,o_\lambda}^K \otimes_{o_\lambda} \bar{k}_\lambda)$ unless $i = n - 1$.*

We give a sketch of the proof. We assume $\mathcal{F}_{v,o_\lambda}^K = o_\lambda$, since the general case is shown similarly. We set $\Lambda = \overline{\mathbb{F}}_\ell$. By the comparison theorem, the Betti cohomology is canonically isomorphic to the étale cohomology. We reduce the vanishing theorem to:

Lemma 6.4 (Localization principle). *For $X = \mathcal{S}h_K$, the kernel and the cokernel of the canonical map*

$$\begin{aligned} H_{\text{ét}}^i(X_{\bar{\eta}}, \Lambda) &\simeq H_{\text{ét}}^i(X_{\bar{s}}, R\psi(\Lambda)) \rightarrow H_{\text{ét}}^i(X_{\bar{s}}^{[n]}, R\psi(\Lambda)|_{X_{\bar{s}}^{[n]}}) \\ &= H_{\text{ét}}^0(X_{\bar{s}}^{[n]}, R^i \psi(\Lambda)|_{X_{\bar{s}}^{[n]}}) \end{aligned}$$

do not admit $\bar{\pi}_w$ as a subquotient, that is, the supercuspidal part of the global cohomology is isomorphic to nearby cycle fibers at “supersingular points”.

To show the lemma, we assume that $K_w = K_{N,w}$ for simplicity²⁹. For $\alpha \geq 1$, we denote $\mathcal{S}h_{K_\alpha}$ by X_α . Note that, for $\beta \geq \alpha \geq 1$, the $K_{\alpha,w}$ -invariants $H_{\text{ét}}^i(X_{\beta,\bar{\eta}}, \Lambda)^{K_{\alpha,w}}$

²⁸By using the regular base change theorem in étale cohomology to compare the algebraic and the formal situation.

²⁹The general case follows from the perfect complex argument. Even the assumption on K_w can be weakened.

are $H_{\text{ét}}^i(X_{\alpha, \bar{\eta}}, \Lambda)$ since $K_{\alpha, w}/K_{\beta, w}$ has order prime to ℓ . The same is true for the nearby cycle fibers for X_α and X_β as the group action is free on the generic fiber, so it suffices to prove that the kernel and the cokernel of the homomorphism

$$\varinjlim_{\alpha} H_{\text{ét}}^i(X_{\alpha, \bar{\eta}}, \Lambda) \rightarrow \varinjlim_{\alpha} H_{\text{ét}}^0(X_{\alpha, \bar{s}}^{[n]}, R^i \psi(\Lambda)|_{X_{\alpha, \bar{s}}^{[n]}})$$

do not admit $\bar{\pi}_w$ as a subquotient. Note that both sides are admissible $H_w(E_w)$ -representations. We work modulo the Serre subcategory of admissible representations generated by non-supercuspidal representations.

Then 6.4 follows from the result in 6.2, since the contribution of the proper support cohomologies of nearby cycle sheaves from $X_{N, s}^{[h]}$ for $h < n$ is parabolically induced from P_h .

We go back to the proof of 6.3. Since the nearby cycle is perverse,

$$R^i \psi(\Lambda)_{\bar{x}} = 0 \quad \text{for } i > n - 1,$$

so we get

$$H_{\text{ét}}^i(X_{\bar{\eta}}, \Lambda) = 0 \quad \text{for } i > n - 1$$

by the localization principle, modulo non-supercuspidal representations. By Poincaré duality on $X_{\bar{\eta}}$, we have the vanishing of the cohomologies of degree strictly less than $n - 1$, since the contragradient of a supercuspidal representation is again supercuspidal.

Remark 6.5. The method of this proof also gives a vanishing theorem for \mathbb{Q}_ℓ -sheaves. The result in this case is proved by Harris [19] by using Clozel’s purity lemma and the base change lift to $D^{\text{op}\times}$.

Note that $X_{\bar{s}}^{[n]}$ consists of a single isogeny class preserving homogeneous polarization when we assume n is odd for simplicity (see [37], chapter 6). It is classified by an inner form G_- of G which is compact modulo the center at all infinite places. $G_-(\mathbb{Q}_p)$ is $\mathbb{Q}_p^\times \cdot \tilde{D}_w^{\text{op}\times} \cdot \prod_{u \in P_\wp, u \neq w} D_u^{\text{op}\times}$ for a division algebra \tilde{D}_w over E_w of invariant $\frac{1}{n}$, and $G_-(\mathbb{A}_f^p) \simeq G(\mathbb{A}_f^p)$ holds, that is, G and G_- are locally isomorphic except at w and ι_0 as chosen in 6.1. Take a compact open subgroup $K_- = K_{-, w} \cdot K^w$ of $G_-(\mathbb{A}_f)$ with $K_{-, w} = o_{\tilde{D}_w}^{\text{op}\times}$ for a maximal order $o_{\tilde{D}_w}$ of \tilde{D}_w . Then the underlying space of $X_{\bar{s}}^{[n]}$ is regarded as a Hida variety $M_{K_-}(G_-, X_-)$ where X_- is a point. The key point for this identification is a theorem of Drinfeld that formal o_w -modules of dimension one and height n are unique over an algebraically closed field and that the endomorphism ring is isomorphic to $o_{\tilde{D}_w}$. So the canonical homomorphism

$$H_{\text{ét}}^{n-1}(X_{\bar{\eta}}, \Lambda) \rightarrow H_{\text{ét}}^0(X_{\bar{s}}^{[n]}, R^{n-1} \psi(\Lambda)|_{X_{\bar{s}}^{[n]}})$$

connects the middle dimensional cohomologies of Shimura and Hida varieties in characteristic p , though the sheaf $R^{n-1} \psi(\Lambda)|_{X_{\bar{s}}^{[n]}}$ may seem to be complicated at first glance. But the sheaf is described, for $\Lambda = \mathbb{Q}_\ell$, by Drinfeld’s reciprocity

law as formulated by Carayol [6] and proved by Harris and Taylor in [20]. This is the geometric Jacquet–Langlands correspondence. More precisely, *the geometric Jacquet–Langlands isomorphism* for G and G_- .

6.4. Reciprocity law and Taylor–Wiles systems for unitary Shimura varieties.

For an irreducible automorphic representation π of $U_D(\mathbb{A}_F)$, Clozel shows the existence of the base change lift $\text{BC}(\pi)$ to $U_D(\mathbb{A}_E) \simeq D^{\text{op}\times}(\mathbb{A}_E)$ [9]. For an automorphic representation π of G , $\text{BC}(\pi) = (\psi, \Pi_D)$ is defined as a representation of $\mathbb{A}_E^\times \times D^{\text{op}\times}(\mathbb{A}_E)$ [20].

$H_B^i(\text{Sh}_K(G, X), \overline{\mathbb{Q}}_\ell)$ is decomposed by the irreducible automorphic $G(\mathbb{A})$ -representations $\pi = \pi_f \otimes \pi_\infty$ such that π_∞ is cohomological for the trivial coefficient. We denote by $H_B^i(\text{Sh}_K(G, X), \overline{\mathbb{Q}}_\ell)[\text{cusp}]$ the part where Π_D corresponds to a cuspidal representation Π_{GL_n} of $\text{GL}_n(\mathbb{A}_E)$ by the Jacquet–Langlands correspondence³⁰.

We use a Galois parameter $\rho: G_E \rightarrow (\text{GL}_1 \times \text{GL}_n)(\overline{\mathbb{Q}}_\ell)$ attached to ψ and Π_{GL_n} ([9], [20]). We denote by P_ρ the set of the isomorphism classes of the $G(\mathbb{A}_f)$ -representation π'_f such that there is some π as above which has π'_f as the finite part and Galois parameter ρ .

Then the reciprocity law of Kottwitz, which is strengthened by Clozel, takes the form

$$H_{\text{ét}}^{n-1}(\text{Sh}_K, \overline{\mathbb{Q}}_\ell)[\text{cusp}] = \bigoplus_{\pi_f \in P_\rho} V_{\pi_f} \otimes \pi_f,$$

where $V_{\pi_f}^{\text{ss}}$ is described by ρ ³¹.

Once the reciprocity is established, we are ready to construct a Taylor–Wiles system along the lines in §5, since we have the desired vanishing theorem in §6. Assume that $\bar{\rho}: G_E \rightarrow (\text{GL}_1 \times \text{GL}_n)(\overline{\mathbb{F}}_\ell)$ is obtained from a Galois parameter which appear in the reciprocity law by mod ℓ -reduction. We assume that D is split at $w \nmid \ell$ and that $\bar{\rho}|_{G_{E_w}}$ defines an absolutely irreducible representation on the standard representation of GL_n . With these assumptions, the vanishing assumption 5.1 is satisfied by using Theorem 6.3. We take a finite set Q of finite places of E so that Q is disjoint from all ramifications, $q_u \equiv 1 \pmod{\ell}$, the Frobenius image $\bar{\rho}(\text{Fr}_u)$ at u is a regular semi-simple element for all $u \in Q$. The compatibility of local and global Langlands correspondences plays an important role here.

Then the identification of a deformation ring R of $\bar{\rho}$ and the Hecke algebra T , as well as the freeness of the middle dimensional cohomology group M localized at the maximal ideal of the Hecke algebra, follows in the minimal case by the standard machinery in §4 under restrictive assumptions to define an appropriate deformation problem and to control the tangent space³².

³⁰Many technical conditions are omitted here. For example, at any place u of E , D_u is either split or a division algebra to be able to apply the global Jacquet–Langlands correspondence to $D^{\text{op}\times}(\mathbb{A}_E)$.

³¹It is conjectured that V_{π_f} is an irreducible G_E -representation of dimension n . It seems likely that the assertion about the dimension is a consequence of the recent progress on the fundamental lemma by Laumon and Ngo.

³²The basic assumptions are that: $\bar{\rho}$ has a large image, D is split at all places dividing ℓ and where $\bar{\rho}$ is ramified, and there is some π giving $\bar{\rho}$ such that π is minimally ramified and spherical at places dividing ℓ . More technical conditions are needed.

As is already mentioned in §5, Harris and Taylor already used Hida varieties associated to unitary groups which are compact at infinity to prove $R = T$ ³³. Thus our approach here does not give new information on the deformation ring, but it gives us hope that a further investigation might be possible for the middle dimensional cohomology of Shimura varieties along the lines sketched in §5.

7. Concluding remarks

Finally, the author would like to mention an application of the geometric Jacquet–Langlands correspondence to the theory of cohomological ℓ -adic automorphic forms of Emerton [16] and Urban which is closely related to the Coleman–Mazur construction of ℓ -adic families of automorphic forms for $\mathrm{GL}_{2,\mathbb{Q}}$ [10]. Assume $G_{\mathbb{Q}_\ell}$ is split with a split maximal torus $T_{\mathbb{Q}_\ell}$. We fix $K^\ell \subset G(\mathbb{A}_f^\ell)$. Consider the projective limit

$$\mathrm{Sh}_{K^\ell}(G, X) = \varprojlim_{K_\ell \subset G(\mathbb{Q}_\ell)} \mathrm{Sh}_{K_\ell \cdot K^\ell}(G, X).$$

By the vanishing theorem, under the supercuspidality assumption mod ℓ at a finite place $w \nmid \ell$, the ℓ -adic continuous complex $R\Gamma_{\mathrm{cont}}(\mathrm{Sh}_{K^\ell}(G, X), E_\lambda)$ reduces to one module H concentrated at the middle dimension. The Jacquet module of the space of locally analytic vectors in H is seen as the set of the global sections of a coherent sheaf \mathcal{E}_{G, K^ℓ} on the ℓ -adic rigid analytic torus \mathcal{T}_ℓ whose $\widehat{\mathbb{Q}_\ell}$ -valued point is $\mathrm{Hom}_{\mathrm{cont}}(T_{\mathbb{Q}_\ell}(\mathbb{Q}_\ell)/\overline{Z(F)} \cap K^\ell, \widehat{\mathbb{Q}_\ell}^\times)$ (the weight space).

On the other hand, for the compact twist G_- in 6.3, the cohomological construction using Hida varieties gives a coherent sheaf $\mathcal{E}_{G_-, K^\ell}$ on \mathcal{T}_ℓ . The geometric Jacquet–Langlands isomorphism in 6.3 suggests that \mathcal{E}_{G, K^ℓ} is described explicitly by $\mathcal{E}_{G_-, K^\ell}$, where $K^{\ell, w}$ is identified with $K_-^{\ell, w}$, preserving Hecke actions outside w . So the geometric Jacquet–Langlands correspondence, which preserves \mathbb{Z}_ℓ -lattice structures (or even structures over \mathbb{Z}_ℓ/ℓ^n), realizes a correspondence between cohomological ℓ -adic automorphic forms on G and G_- . The existence of such a correspondence for GL_2 and the Coleman–Mazur construction was questioned by Buzzard ([3] for the construction on the definite quaternion side), and answered by Chenevier [8].

The author would like to close this article with the following speculation: the functoriality principle for reductive groups should be true even for cohomological ℓ -adic automorphic forms.

³³After this paper was written, Taylor [53] has shown a potential automorphy for a large class of Galois representations by developing the techniques discussed in this article. As a consequence, he proved the Sato–Tate conjecture for elliptic curves quite generally.

References

- [1] Boyer, P., Mauvaise réduction des variétés de Drinfeld et correspondance de Langlands locale. *Invent. Math.* **138** (3) (1999), 573–629.
- [2] Breuil, C., Sur quelques représentations modulaires et p -adiques de $GL_2(\mathbf{Q}_p)$. I. *Compositio Math.* **138** (2) (2003), 165–188.
- [3] Buzzard, K.: On p -adic families of automorphic forms. *Modular curves and abelian varieties*, Progr. Math. 224, Birkhäuser, Basel 2004, 23–44.
- [4] Carayol, H., Sur la mauvaise réduction des courbes de Shimura. *Compositio Math.* **59** (1986), 151–230.
- [5] Carayol, H., Sur les représentations p -adiques associées aux formes modulaires de Hilbert. *Ann. Sci. École Norm. Sup. (4)* **19** (1986), 409–468.
- [6] Carayol, H., Nonabelian Lubin-Tate theory. *Automorphic forms, Shimura varieties, and L -functions*, Vol. II, Perspect. Math. 11, Academic Press, Boston, MA, 1990, 15–39.
- [7] Carayol, H., Sur les représentations galoisiennes mod ℓ attachées aux formes modulaires. *Duke Math. J.* **59** (1989), 785–801.
- [8] Chenevier, G.: Une correspondance de Jacquet-Langlands p -adique. *Duke Math. J.* **126** (1) (2005), 161–194.
- [9] Clozel, L., Représentations galoisiennes associées aux représentations automorphes auto-duales de $GL(n)$. *Inst. Hautes Études Sci. Publ. Math.* **73** (1991), 97–145.
- [10] Coleman, R., Mazur, B., The eigencurve. *Galois representations in arithmetic algebraic geometry* (Durham, 1996), London Math. Soc. Lecture Note Ser. 254, Cambridge University Press, Cambridge 1998, 1–113.
- [11] Deligne, P., Travaux de Shimura. In *Séminaire Bourbaki*, 23^{ème} année (1970/71), Exp. No. 389, Lecture Notes in Math. 244, Springer-Verlag, Berlin 1971, 123–165.
- [12] Deligne, P., Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques. In *Automorphic forms, representations and L -functions* (Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math. 33, Amer. Math. Soc., Providence, R.I., 1979, 247–289.
- [13] Diamond, F., On deformation rings and Hecke rings. *Ann. of Math. (2)* **144** (1) (1996), 137–166.
- [14] Diamond, F., The Taylor-Wiles construction and multiplicity one. *Invent. Math.* **128** (2) (1997), 379–391.
- [15] Drinfeld, V. G., Elliptic modules. *Mat. Sb. (N.S.) Ser. 94* **136** (1974), 594–627; English transl. *Math. USSR Sb.* **23** (1974), 561–592.
- [16] Emerton, M., On the interpolation of eigenvalues attached to automorphic Hecke eigenforms. *Invent. Math.* **164** (2006), 1–84
- [17] Fujiwara, K., Level optimization in the totally real case. Preprint, arXiv:math.NT/0602586.
- [18] Fujiwara, K., Deformation rings and Hecke algebras in the totally real case. Preprint, arXiv:math.NT/0602606.
- [19] Harris, M., Supercuspidal representations in the cohomology of Drinfeld upper half spaces; elaboration of Carayol’s program. *Invent. Math.* **129** (1) (1997), 75–119.
- [20] Harris, M., Taylor, R., *The geometry and cohomology of some simple Shimura varieties*. Ann. of Math. Stud. 151, Princeton University Press, Princeton, NJ, 2001.

- [21] Hida, H., On p -adic Hecke algebras for GL_2 over totally real fields. *Ann. of Math.* **128** (1988), 295–384.
- [22] Hida, H., Nearly ordinary Hecke algebras and Galois representations of several variables. In *Algebraic analysis, geometry, and number theory* (Baltimore, MD, 1988), Johns Hopkins University Press, Baltimore, MD, 1989, 115–134.
- [23] Hida, H., Adjoint Selmer groups as Iwasawa modules. In *Proceedings of the Conference on p -adic Aspects of the Theory of Automorphic Representations* (Jerusalem, 1998); *Israel J. Math.* **120** (Part B) (2000), 361–427.
- [24] Hida, H., The integral basis problem of Eichler. *Internat. Math. Res. Notices* **2005** (34) (2005), 2101–2122.
- [25] Jacquet, H., Langlands, R. P., *Automorphic forms on GL_2* . Lecture Notes in Math. 114, Springer-Verlag, Berlin 1970.
- [26] Jarvis, F., Mazur’s principle for totally real fields of odd degree. *Compositio Math.* **116** (1999), 39–79.
- [27] Jarvis, F., Level lowering for modular mod l representations over totally real fields. *Math. Ann.* **313** (1999), 141–160.
- [28] Khare, C., On Serre’s modularity conjecture for 2-dimensional mod p representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ unramified outside p . Preprint, arXiv:math.NT/0504080.
- [29] Khare, C., Ramakrishna, R., Finiteness of Selmer groups and deformation rings. *Invent. Math.* **154** (1) (2003), 179–198.
- [30] Khare, C., Wintenberger, J. P., On Serre’s reciprocity conjecture for 2-dimensional mod p representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Preprint, arXiv:math.NT/0412076.
- [31] Kisin, M., Modularity of potentially Barsotti-Tate representations and moduli of finite flat group schemes. Preprint.
- [32] Kottwitz, R. E., Shimura varieties and λ -adic representations. In *Automorphic forms, Shimura varieties, and L -functions*, Vol. I, *Perspect. Math.* 10, Academic Press, Boston, MA, 1990, 161–209.
- [33] Kottwitz, R. E., On the λ -adic representations associated to some simple Shimura varieties. *Invent. Math.* **108** (3) (1992), 653–665.
- [34] Mokrane, A., Tilouine, J., Cohomology of Siegel varieties with p -adic integral coefficients and applications. Cohomology of Siegel varieties. *Astérisque* **280** (2002), 1–95.
- [35] Ohta, M., On the zeta function of an abelian scheme over the Shimura curve. *Japan J. Math.* **9** (1983), 1–26.
- [36] Rajaei, A., On the levels of mod l Hilbert modular forms. *J. Reine Angew. Math.* **537**, (2001), 33–65.
- [37] Rapoport, M., Zink, Th., *Period spaces for p -divisible groups*. Ann. of Math. Stud. 141, Princeton University Press, Princeton, NJ, 1996.
- [38] Ribet, K. A., Congruence relations between modular forms. In *Proceedings of the International Congress of Mathematicians* (Warszawa, 1983), Vol. 1, PWN, Warsaw 1984, 503–514.
- [39] Ribet, K. A., On modular representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ arising from modular forms. *Invent. Math.* **100** (1990), 431–476.
- [40] Ribet, K. A., Report on mod ℓ representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. In *Motives* (Seattle, WA, 1991), Proc. Sympos. Pure Math. 55, Amer. Math. Soc., Providence, RI, 1994, 639–676.

- [41] Saito, T., Modular forms and p -adic Hodge theory. *Invent. Math.* **129** (3) (1997), 607–620.
- [42] Saito, T., Hilbert modular forms and p -adic Hodge theory. Preprint, 1999.
- [43] Shimizu, H., On zeta functions of quaternion algebras. *Ann. of Math. (2)* **81** (1965), 166–193.
- [44] Shimura, G., Correspondences modulaires et les fonctions zeta de courbes. *J. Math. Soc. Japan* **10** (1958), 1–28.
- [45] Shimura, G., Number fields and zeta functions associated with discontinuous groups and algebraic varieties. In *Proceedings of the International Congress of Mathematicians* (Moscow, 1966), Izdat. “Mir”, Moscow 1968, 290–299.
- [46] Shimura, G., An ℓ -adic method in the theory of automorphic forms. In *Collected papers*, Vol. II: 1967–1977, Springer-Verlag, New York 2002, 237–272.
- [47] Shimura, G., Construction of class fields and zeta functions of algebraic curves. *Ann. of Math.* **85** (1967), 58–159.
- [48] Shimura, G., On canonical models of arithmetic quotient of bounded symmetric domains. *Ann. of Math.* **91** (1970), 144–222.
- [49] Skinner, C. M., Wiles, A. J., Residually reducible representations and modular forms. *Inst. Hautes Études Sci. Publ. Math.* **89** (1999), 5–126.
- [50] Skinner, C. M., Wiles, A. J., Base change and a problem of Serre. *Duke Math. J.* **107** (1) (2001), 15–25.
- [51] Skinner, C. M., Wiles, A. J., Nearly ordinary deformations of irreducible residual representations. *Ann. Fac. Sci. Toulouse Math. (6)* **10** (1) (2001), 185–215.
- [52] Taylor, R., On Galois representations associated to Hilbert modular forms. *Invent. Math.* **98** (1989), 265–280.
- [53] Taylor, R., Automorphy for some ℓ -adic lifts of automorphic mod ℓ Galois representations, II. Preprint.
- [54] Taylor, R., Yoshida, T., Compatibility of local and global Langlands correspondences. *J. Amer. Math. Soc.*, to appear.
- [55] Taylor, R., Wiles, A. J., Ring theoretic properties of certain Hecke algebras. *Ann. of Math.* **141** (3) (1995), 553–572.
- [56] Wiles, A. J., On ordinary λ -adic representations associated to Hilbert modular forms. *Invent. Math.* **94** (1988), 529–573.
- [57] Wiles, A. J., Modular elliptic curves and Fermat’s last theorem. *Ann. of Math.* **141** (1995), 443–551.

Graduate School of Mathematics, Nagoya University, Nagoya, Aichi, Japan

E-mail: fujiwara@math.nagoya-u.ac.jp