

Invariants of singularities of pairs

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Abstract. Let X be a smooth complex variety and Y be a closed subvariety of X , or more generally, a closed subscheme of X . We are interested in invariants attached to the singularities of the pair (X, Y) . We discuss various methods to construct such invariants, coming from the theory of multiplier ideals, D -modules, the geometry of the space of arcs and characteristic p techniques. We present several applications of these invariants to algebra, higher dimensional birational geometry and to singularities.

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1. Introduction

Let X be a smooth complex variety of dimension n and Y be a closed subvariety of X (or more generally, a closed subscheme of X). We are interested in studying the singularities of the pair (X, Y) . The general setup is to assume only that X is normal and \mathbb{Q} -Gorenstein, as in [32]. However, several of the approaches we will discuss become particularly transparent if we assume, as we do, the smoothness of the ambient variety. The following are some examples of pairs.

Examples 1.1. (i) Let $X = \mathbb{C}^n$ and let Y be a hypersurface defined by an equation $f(x_1, \dots, x_n) = 0$. For instance f can be the Fermat hypersurface $x_1^m + x_2^m + \dots + x_n^m = 0$, which has an isolated singularity of multiplicity m at the origin.

(ii) If X is a smooth projective variety and L is a line bundle on X , then we take Y to be the base locus of the complete linear system $|L|$, i.e. $Y = \bigcap_{D \in |L|} D$.

(iii) Let X be a smooth affine variety with coordinate ring R . If $I \subseteq R$ is an ideal, then we take Y to be the closed subscheme defined by I .

In what follows we present various invariants attached to such pairs and we discuss some of their applications. Our main point is that the same invariants that play an important role in higher dimensional algebraic geometry arise also in several other approaches to singularities.

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2. Multiplier ideals

Multiplier ideals were first introduced by J. Kohn for solving certain partial differential equations. Siu and Nadel introduced them to complex geometry. We discuss below these ideals in the context of algebraic geometry.

Let X be a smooth complex affine variety and Y be a closed subscheme of X . Suppose that the ideal of Y is generated by f_1, \dots, f_m , and let λ be a positive real number. We define the multiplier ideal of (X, Y) of coefficient λ as follows:

$$\mathcal{J}(X, \lambda \cdot Y) = \left\{ g \in \mathcal{O}_X \mid \frac{|g|^2}{\left(\sum_{i=1}^m |f_i|^2\right)^\lambda} \text{ is locally integrable} \right\}.$$

Example 2.1. Let $X = \mathbb{C}^n$ and let Y be the closed subscheme of X defined by $f = x_1^{a_1} \dots x_n^{a_n}$. Then

$$\mathcal{J}(X, \lambda \cdot Y) = (x_1^{\lfloor \lambda a_1 \rfloor} \dots x_n^{\lfloor \lambda a_n \rfloor}),$$

where $\lfloor \alpha \rfloor$ denotes the integer part of α . Equivalently, if H_i is the divisor defined by $x_i = 0$, then g is in $\mathcal{J}(X, \lambda \cdot Y)$ if and only if

$$\text{ord}_{H_i} g \geq \lfloor \lambda a_i \rfloor,$$

for $i = 1, \dots, n$.

We can use a log resolution of singularities and the above example to give in general a more geometric description of the multiplier ideals of (X, Y) . By Hironaka's Theorem there is a log resolution of singularities of the pair (X, Y) , i.e. a proper birational morphism

$$\mu: X' \longrightarrow X$$

with the following properties. The variety X' is smooth, $\mu^{-1}(Y)$ is a divisor, and the union of $\mu^{-1}(Y)$ and the exceptional locus of μ has simple normal crossings. The relative canonical divisor $K_{X'/X}$ is locally defined by the determinant of the Jacobian $J(\mu)$ of μ , hence it is supported on the exceptional locus of μ . We write $\mu^{-1}(Y) = \sum_{i=1}^N a_i E_i$ and $K_{X'/X} = \sum_{i=1}^N k_i E_i$, where the E_i are distinct smooth irreducible divisors in X' such that $\sum_{i=1}^N E_i$ has only simple normal crossing singularities.

Suppose that x_1, \dots, x_n are local coordinates in X and y_1, y_2, \dots, y_n are local coordinates for an open set in X' . Note that

$$\mu^* dx_1 \dots dx_n d\bar{x}_1 \dots d\bar{x}_n = |\det(J(\mu))|^2 dy_1 \dots dy_n d\bar{y}_1 \dots d\bar{y}_n. \quad (1)$$

The local integrability of a function g on X can be expressed as a local integrability condition on X' via the change of variable formula. This reduces us to a monomial situation, similar to that in Example 2.1. One deduces that $g \in \mathcal{J}(X, \lambda \cdot Y)$ if and only if

$$\text{ord}_{E_i} g \geq \lfloor \lambda a_i \rfloor - k_i$$

for every i . Equivalently, if we put $\lfloor \lambda \mu^{-1}(Y) \rfloor = \sum_i \lfloor \lambda a_i E_i \rfloor$, then

$$\mathcal{J}(X, \lambda \cdot Y) = \mu_* \mathcal{O}_{X'}(K_{X'/X} - \lfloor \lambda \mu^{-1}(Y) \rfloor). \tag{2}$$

We refer to [34] for details.

Note that because of the original definition, it follows that this expression for $\mathcal{J}(X, \lambda \cdot Y)$ is independent of the choice of a resolution of singularities. On the other hand, the formula (2) applies also when X is non necessarily affine. Note also that this formula implies that if $\lambda_1 \geq \lambda_2$, then

$$\mathcal{J}(X, \lambda_1 \cdot Y) \subseteq \mathcal{J}(X, \lambda_2 \cdot Y).$$

If λ is small enough, then $\lambda a_i < k_i + 1$ for $i = 1, \dots, N$. This implies that

$$\text{ord}_{E_i} 1 \geq \lfloor \lambda a_i \rfloor - k_i,$$

hence $\mathcal{J}(X, \lambda \cdot Y) = \mathcal{O}_X$. This leads us to the definition of the log canonical threshold of the pair (X, Y) : this is the smallest λ such that $\mathcal{J}(X, \lambda \cdot Y) \neq \mathcal{O}_X$, i.e.

$$c = \text{lc}(X, Y) = \min_i \left\{ \frac{k_i + 1}{a_i} \right\}.$$

We may regard $\frac{1}{c}$ as a refined version of multiplicity. In general a singularity with a smaller log canonical threshold tends to be more complex.

The first appearance of the log canonical threshold was in the work of Arnold, Gusein-Zade and Varchenko (see [2] and [48]), in connection with the behavior of certain integrals over vanishing cycles. In the last decade this invariant has enjoyed renewed interest due to its applications to birational geometry. The following is probably the most interesting open problem about log canonical thresholds.

Conjecture 2.2 (Shokurov). For every n , the set

$$\{\text{lc}(X, Y) \mid \dim(X) = n, Y \subset X\}$$

satisfies the Ascending Chain Condition: it contains no strictly increasing sequences.

We can consider also higher jumping numbers. In general, we say that λ is a jumping number of (X, Y) , if

$$\mathcal{J}(X, \lambda \cdot Y) \subsetneq \mathcal{J}(X, (\lambda - \varepsilon) \cdot Y)$$

for all $\varepsilon > 0$. If λa_i is not an integer, then $\lfloor \lambda a_i \rfloor = \lfloor (\lambda - \varepsilon) a_i \rfloor$ for sufficiently small positive ε . We see that a necessary condition for λ to be a jumping number is that λa_i is an integer for some i . In particular, if λ is a jumping number, then it is rational and has a bounded denominator.

The following theorem gives a periodicity property of the jumping numbers.

Theorem 2.3. (i) If $Y = D$ is a hypersurface in X , then

$$\mathcal{J}(X, \lambda \cdot D) \cdot \mathcal{O}_X(-D) = \mathcal{J}(X, (\lambda + 1) \cdot D).$$

(ii) (Ein and Lazarsfeld [16]) For every Y defined by the ideal I_Y , if $\lambda \geq \dim X - 1$, then

$$\mathcal{J}(X, \lambda \cdot Y) \cdot I_Y = \mathcal{J}(X, (\lambda + 1) \cdot Y).$$

Corollary 2.4. If $\lambda > \dim X - 1$, then λ is a jumping number for (X, Y) if and only if so is $(\lambda + 1)$.

We conclude that the set of jumping numbers of the pair (X, Y) is a discrete subset of \mathbb{Q} and it is eventually periodic with period one.

Example 2.5. If Y is a smooth subvariety of X of codimension e , then the set of jumping numbers of the pair (X, Y) is $\{e, e + 1, \dots\}$. In particular $\text{lc}(X, Y) = e$.

Example 2.6. (Howald) Let $X = \mathbb{C}^n$ and let Y be the closed subscheme defined by a monomial ideal \mathfrak{a} . If $a = (a_1, a_2, \dots, a_n) \in \mathbb{N}^n$, we denote the monomial $x_1^{a_1} \dots x_n^{a_n}$ by x^a . Consider the Newton polyhedron $P_{\mathfrak{a}}$ associated with \mathfrak{a} : this is the convex hull of those $a \in \mathbb{N}^n$ such that $x^a \in \mathfrak{a}$. Using toric geometry Howald showed in [27] that

$$\mathcal{J}(X, \lambda \cdot Y) = (x^a \mid a + e \in \text{Int}(\lambda \cdot P_{\mathfrak{a}})),$$

where $e = (1, \dots, 1)$. In particular, the log canonical threshold c of (X, Y) is characterized by the fact $c \cdot e$ lies on the boundary of $P_{\mathfrak{a}}$.

For example, suppose that \mathfrak{a} is the ideal $(x_1^{a_1}, \dots, x_n^{a_n})$. In this case, the boundary of $P_{\mathfrak{a}}$ is

$$\{u = (u_1, \dots, u_n) \in \mathbb{R}_+^n \mid \sum_{i=1}^n \frac{u_i}{a_i} = 1\}.$$

Therefore $\text{lc}(X, Y) = \sum_i \frac{1}{a_i}$.

Example 2.7. Suppose that $X = \mathbb{C}^2$ and Y is the plane cuspidal curve defined by $x^3 + y^5 = 0$. Then the set of jumping numbers for Y is periodic with period 1. The jumping numbers in $(0, 1]$ are $\{\frac{8}{15}, \frac{11}{15}, \frac{13}{15}, \frac{14}{15}, 1\}$.

One reason that multiplier ideals have been very powerful in studying questions in higher dimensional algebraic geometry is that they appear naturally in a Kodaira type vanishing theorem. The following statement is the algebraic version of a result due to Nadel. In our context, it can be deduced from the Kawamata–Viehweg Vanishing Theorem (see [34]).

Theorem 2.8. Let X be a smooth projective variety and Y a closed subscheme of X defined by the ideal I_Y . If A is a line bundle such that $I_Y \otimes A$ is globally generated, and if L is a line bundle such that $L - A$ is big and nef, then for every $i > 0$

$$H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(X, Y)) = 0.$$

3. Applications of multiplier ideals

One of the most important applications of multiplier ideals is the following theorem of Siu (see [44] and [45]) on the deformation invariance of plurigenera.

Theorem 3.1. *Let $f : X \rightarrow T$ be a smooth projective morphism of relative dimension n between two smooth irreducible varieties. If we denote by X_t the fiber $f^{-1}(t)$ for each $t \in T$, then for every fixed $m > 0$, the dimension of the cohomology group $H^0(X_t, (\Omega_{X_t}^n)^{\otimes m})$ is independent of t .*

The techniques involved in the proof of this theorem have been recently applied by Siu, Hacon and McKernan to study one of the outstanding problems in higher dimensional algebraic geometry, the finite generation of the canonical ring (see, for example, [23]).

In a different direction, there are applications of multiplier ideals to singularities of theta divisors on abelian varieties. Let (X, Θ) be a principally polarized abelian variety, that is, Θ is an ample divisor on an abelian variety X such that $\dim H^0(X, \mathcal{O}_X(\Theta)) = 1$. The following result is due to Ein and Lazarsfeld [17].

Theorem 3.2. *Let (X, Θ) be a principally polarized abelian variety. If Θ is irreducible, then Θ has at most rational singularities.*

Corollary 3.3. *Let (X, Θ) be a principally polarized abelian variety of dimension g , with Θ irreducible. If*

$$\Sigma_k(\Theta) = \{x \in X \mid \text{mult}_x(\Theta) \geq k\},$$

then for every $k \geq 2$ we have $\text{codim}(\Sigma_k(\Theta), X) \geq k + 1$. In particular, Θ is a normal variety and $\text{mult}_x(\Theta) \leq g - 1$ for every singular point x on Θ .

Remark 3.4. The fact that Θ is normal was first conjectured by Arbarello, De Concini and Beauville. When X is the Jacobian of a curve, the fact that Θ has only rational singularities was proved by Kempf. Note also that in this case, a classical theorem of Riemann expresses the multiplicity of Θ at a point in term of the dimension of the corresponding linear system on the curve. It was Kollár who first observed in [31] that one can use vanishing theorems to study the singularities of the theta divisor: he showed that for every principally polarized abelian variety (X, Θ) , we have $\text{lc}(X, \Theta) = 1$. Theorem 3.2 above is a strengthening of Kollár's result.

Multiplier ideals have been applied in several other directions: to Fujita's problem on adjoint linear systems [3], to Effective Nullstellensatz [16], to Effective Artin-Rees Theorem [20]. Building on work of Tsuji, recently Hacon and McKernan and independently, Takayama have used multiplier ideals to prove a very interesting result on boundedness of pluricanonical maps for varieties of general type (see [24] and [47]). We end this section with an application to commutative algebra due to Ein, Lazarsfeld and Smith [19].

Let X be a smooth n -dimensional variety and $Y \subseteq X$ defined by the reduced sheaf of ideals \mathfrak{a} . The m^{th} symbolic power of \mathfrak{a} is the sheaf $\mathfrak{a}^{(m)}$ of functions on X that vanish with multiplicity at least m at the generic point of every irreducible component of Y . If Y is smooth, then the symbolic powers of \mathfrak{a} agree with the usual powers, but in general they are very different.

Theorem 3.5. *If X is a smooth n -dimensional variety and if \mathfrak{a} is a reduced sheaf of ideals, then $\mathfrak{a}^{(mn)} \subseteq \mathfrak{a}^m$ for every m .*

4. Bounds on log canonical thresholds and birational rigidity

In this section we compare the log canonical threshold with the classical Samuel multiplicity. We give then an application of the inequality between these two invariants to a classical question on birational rigidity. Let X be a smooth complex variety and $x \in X$ a point. Denote by R the local ring of X at x , and by \mathfrak{m} its maximal ideal. The following result was proved by de Fernex, Ein and Mustață in [10].

Theorem 4.1. *Let \mathfrak{a} be an ideal in R that defines a subscheme Y supported at x . Let c be the log canonical threshold of (X, Y) , $l(R/\mathfrak{a})$ be the length of R/\mathfrak{a} and $e(\mathfrak{a})$ be the Samuel multiplicity of R along \mathfrak{a} . If $n = \dim R$, then we have the following inequalities.*

$$(i) \quad l(R/\mathfrak{a}) \geq \frac{n^n}{n! \cdot c^n}.$$

$$(ii) \quad e(\mathfrak{a}) \geq \frac{n^n}{c^n}. \text{ Furthermore, this is an equality if and only if the integral closure of } \mathfrak{a} \text{ is equal to } \mathfrak{m}^k \text{ for some } k.$$

The first assertion in (ii) above can be easily deduced from (i). The proof of (i) proceeds by reduction to the monomial case, via a Gröbner deformation. When \mathfrak{a} is monomial, the inequality follows by a combinatorial argument from the explicit description of the invariants.

Example 4.2. Suppose that $\mathfrak{a} = (x_1^{a_1}, \dots, x_n^{a_n})$. In this case $e(\mathfrak{a}) = \prod_{i=1}^n a_i$ and $lc(\mathfrak{a}) = \sum_{i=1}^n \frac{1}{a_i}$. The inequality in Theorem 4.1(ii) becomes

$$\prod_{i=1}^n a_i \geq \frac{n^n}{\left(\sum_{i=1}^n \frac{1}{a_i}\right)^n}.$$

This is equivalent to

$$\left(\frac{1}{n} \sum_{i=1}^n \frac{1}{a_i}\right)^n \geq \prod_{i=1}^n \frac{1}{a_i},$$

which is just the classical inequality between the arithmetic and the geometric mean.

Remark 4.3. When X is a surface, the inequality in (ii) above was first proved by Corti [9].

Theorem 4.1 is used in [11] to study the behavior of the log canonical threshold under a generic projection. More generally, one proves the following

Theorem 4.4. *Let $f: X \rightarrow Y$ be a smooth proper morphism of relative dimension k between two smooth complex varieties. If V is a locally complete intersection of codimension k in X such that $f|_V$ is finite, then*

$$\text{lc}(Y, f_*(V)) \leq \frac{\text{lc}(X, V)^k}{k^k}.$$

Using Theorem 4.4 and some beautiful geometric ideas of Pukhlikov [42], one gives in [11] a simple uniform proof for the following result.

Theorem 4.5. *If X is a smooth hypersurface of degree N in $\mathbb{C}\mathbb{P}^N$, with $4 \leq N \leq 12$, then X is birationally superrigid. In particular, every birational automorphism of X is biregular.*

Remark 4.6. Consider the group $\text{Aut}_{\mathbb{C}}(\mathbb{C}(X))$, the automorphism group of the field of rational functions of X . This is naturally isomorphic to $\text{Bir}_{\mathbb{C}}(X)$, the group of birational automorphisms of X . If X is birationally superrigid, then $\text{Bir}_{\mathbb{C}}(X) \simeq \text{Aut}_{\mathbb{C}}(X)$, the automorphism group of X . When X is a hypersurface of degree N in \mathbb{P}^N , X has no nonzero vector fields and therefore $\text{Aut}_{\mathbb{C}}(X)$ is a finite group. This shows that X is not a rational variety: if $\mathbb{C}(X)$ is purely transcendental, then $\text{Aut}_{\mathbb{C}}(X)$ will contain a subgroup isomorphic to the general linear group GL_n .

Remark 4.7. When $N = 4$, it is a classical theorem of Iskovskikh and Manin that X is birationally rigid [28]. They used this to show that the function field of a suitable quartic threefold provides a counterexample to the classical Lüroth problem: $\mathbb{C}(X)$ is a \mathbb{C} -subfield of the purely transcendental field $\mathbb{C}(x_1, x_2, x_3)$ but $\mathbb{C}(X)$ is not purely transcendental. For $N = 5$, Theorem 4.5 is a result of Pukhlikov [40]. The cases $N = 6, 7$ and 8 were first proved by Cheltsov [8]. We mention also that Pukhlikov [41] has shown that a generic hypersurface of degree N in \mathbb{P}^N is superrigid for every $N \geq 4$.

5. Bernstein–Sato polynomials

Let $f \in \mathbb{C}[x_1, x_2, \dots, x_n]$ be a nonzero polynomial. We denote by A_n the Weyl algebra of differential operators on \mathbb{A}^n , that is

$$A_n = \mathbb{C}[x_1, \dots, x_n, \partial_{x_1}, \dots, \partial_{x_n}].$$

Let s be another variable and consider the following functional equation:

$$b(s)f^s = P(s, x, \partial_x) \bullet f^{s+1}, \tag{3}$$

where $b(s) \in \mathbb{C}[s]$ and $P \in A_n[s]$. Here f^s is considered a formal symbol, and the action \bullet of P is defined via $\partial_{x_i} \bullet f^s = s f^{-1} \frac{\partial f}{\partial x_i} f^s$. On the other hand, if we let $s = m$ for an integer m , then (3) has the obvious meaning.

It is easy to see that the set of polynomials $b(s)$ for which there is P satisfying (3) is an ideal in the polynomial ring $\mathbb{C}[s]$. It was proved by Bernstein in [4] using the theory of holonomic D -modules that this ideal is nonzero. Its monic generator is denoted by $b_f(s)$ and is called the Bernstein–Sato polynomial of f . It is an interesting and subtle invariant of the singularities of the hypersurface defined by f .

Examples 5.1. (1) Making $s = 1$ in (3) we see that $b_f(-1) f^{-1}$ lies in $\mathbb{C}[x_1, \dots, x_n]$. If f is nonconstant, it follows that -1 is a root of b_f .

(2) If $f = x$, then $b_f(s) = (s + 1)$. Indeed, we have

$$(s + 1) f^s = \partial_x \bullet f^{s+1}.$$

More generally, if f defines a nonsingular hypersurface, then $b_f(s) = (s + 1)$.

(3) If $f = x_1^2 + \dots + x_n^2$, then $b_f(s) = (s + 1) \left(s + \frac{n}{2}\right)$ and

$$b_f(s) f^s = \frac{1}{4} (\partial_{x_1}^2 + \dots + \partial_{x_n}^2) \bullet f^{s+1}.$$

(4) If $f = x^2 + y^3$, then $b_f(s) = \left(s + \frac{5}{6}\right) (s + 1) \left(s + \frac{7}{6}\right)$ and

$$b_f(s) f^s = \left(\frac{1}{27} \partial_y^3 + \frac{1}{6} y \partial_x^2 \partial_y + \frac{1}{8} x \partial_x^3 + \frac{3}{8} \partial_x^2 \right) \bullet f^{s+1}.$$

Computing Bernstein–Sato polynomials in general is quite subtle (see [49]). On the other hand, there has been a lot of recent progress in algorithmic computation using Gröbner bases in the Weyl algebra (see [43]).

We describe now the connection between the roots of the Bernstein–Sato polynomial of f and the jumping numbers of the hypersurface Y defined in \mathbb{A}^n by f . An important theorem of Kashiwara [30] asserts that all the roots of $b_f(s)$ are negative rational numbers. Building on Kashiwara’s work, Lichtin made this more explicit in [35], describing a connection between the roots of $b_f(s)$ and a log resolution of the pair (\mathbb{A}^n, Y) . This says that if $\mu: X' \rightarrow \mathbb{A}^n$ is a log resolution of (X, Y) , then every root of $b_f(s)$ is of the form $-\frac{k_i+m}{a_i}$ for some i and some positive integer m (we use the notation introduced in §2). In particular, we see that every root of $b_f(s)$ is rational, and no larger than $-\text{lc}(\mathbb{A}^n, Y)$. However, we stress that unlike in the case of multiplier ideals, there is no explicit description of the Bernstein–Sato polynomial in terms of a log resolution.

On the other hand, the following result of Ein, Lazarsfeld, Smith and Varolin [20] shows that in a suitable range, all jumping numbers give roots of the Bernstein–Sato polynomial.

Theorem 5.2. *If $\lambda \in (0, 1]$ is a jumping number of (\mathbb{A}^n, Y) , then $-\lambda$ is a root of the Bernstein–Sato polynomial $b_f(s)$.*

The proof of this theorem uses the functional equation (3) and integration by parts. The case when $\lambda = \text{lc}(\mathbb{A}^n, Y)$ was proved by Kollár in [32]. Note that in conjunction with the above mentioned result of Lichtin, this gives the following

Corollary 5.3. *The largest root of $b_f(s)$ is $-\text{lc}(\mathbb{A}^n, Y)$.*

A different point of view on the connection between multiplier ideals and Bernstein–Sato polynomials was given by Budur and Saito. In fact, they show how to recover the multiplier ideals from a filtration that appears in D -module theory, the V -filtration. We present now their result.

Let t be a new variable, and let A_{n+1} denote the Weyl algebra corresponding to the affine space \mathbb{A}^{n+1} , with coordinates x_1, \dots, x_n, t . We consider the module B_f that is the first local cohomology module of \mathbb{A}^{n+1} along the embedding of \mathbb{A}^n as the graph of f , i.e.

$$B_f = \mathbb{C}[x_1, \dots, x_n, t]_{f-t} / \mathbb{C}[x_1, \dots, x_n, t].$$

Let δ be the class of $\frac{1}{f-t}$ in B_f (δ is the “delta-function associated to the graph of f ”).

Note that B_f has a natural structure of left module over A_{n+1} . Since $\partial_t^m \delta$ is the class of $\frac{m!}{(f-t)^{m+1}}$ in B_f , we see that B_f is free over $\mathbb{C}[x_1, \dots, x_n]$, with basis given by $\{\partial_t^j \delta \mid j \geq 0\}$.

The V -filtration is a decreasing filtration on B_f by left A_n -submodules V^α indexed by $\alpha \in \mathbb{Q}$, with the following properties:

- (i) $\bigcup_\alpha V^\alpha = B_f$.
- (ii) The filtration is semicontinuous and discrete in the following sense: there is a positive integer ℓ such that for every integer m and every $\alpha \in (\frac{m-1}{\ell}, \frac{m}{\ell}]$ we have $V^\alpha = V^{m/\ell}$.
- (iii) We have $t \cdot V^\alpha \subseteq V^{\alpha+1}$ for every α , with equality if $\alpha > 0$.
- (iv) We have $\partial_t \cdot V^\alpha \subseteq V^{\alpha-1}$ for every α .
- (v) For every α , if we put $V^{>\alpha} := \bigcup_{\beta > \alpha} V^\beta$, then $(\partial_t t - \alpha)$ is nilpotent on $V^\alpha / V^{>\alpha}$.

The key property is (v) above. One can think of the V -filtration as an attempt to diagonalize the operator $\partial_t t$ on B_f . It is not hard to show that if a filtration as above exists, then it is unique. Malgrange [36] proved the existence of the V -filtration using the existence of the Bernstein–Sato polynomial and the rationality of its roots. To explain the role played by $b_f(s)$ in the construction of the V -filtration, we mention that the equation (3) in the definition of b_f is equivalent with the following equality in B_f :

$$b(-\partial_t t) \cdot \delta = P(-\partial_t t, x, \partial_x) \cdot t \delta. \tag{4}$$

The following result of Budur and Saito [5] shows that the multiplier ideals can be obtained as a piece of the V -filtration. We consider $\mathbb{C}[x_1, \dots, x_n]$ embedded in B_f by $h \rightarrow h\delta$.

Theorem 5.4. *If Y is the hypersurface defined by f , then for every $\lambda > 0$ we have $\mathcal{F}(\mathbb{A}^n, \lambda \cdot Y) = V^{\lambda+\varepsilon} \cap \mathbb{C}[x_1, \dots, x_n]$, where $0 < \varepsilon \ll 1$.*

The assertion in Theorem 5.2 can be deduced from this statement. The proof of Theorem 5.4 involves two steps. First, one describes the V -filtration in the case when f defines a divisor with simple normal crossings: $f = x_1^{a_1} \dots x_n^{a_n}$. In this case, let us put $\mathcal{F}'(\mathbb{A}^n, \alpha \cdot Y) := \mathcal{F}(\mathbb{A}^n, (\alpha - \varepsilon) \cdot Y)$ for $0 < \varepsilon \ll 1$ (with the convention $\mathcal{F}'(\mathbb{A}^n, \alpha \cdot Y) = \mathbb{C}[x_1, \dots, x_n]$ if $\alpha \leq 0$). If we take V^α to be generated over A_n by $\mathcal{F}'(\mathbb{A}^n, (\alpha + j) \cdot Y) \partial_t^j \delta$, where j varies over the nonnegative integers, then one can check that these V^α satisfy the properties in the definition of the V -filtration. In particular, this easily implies the statement of Theorem 5.4 in this case. The hard part of the proof uses Saito's theory of mixed Hodge modules to deduce the general case of the theorem by relating the V -filtrations of f and of a log resolution.

We mention that Kashiwara constructed in [29] a V -filtration associated to several polynomials. Budur, Mustață and Saito used this in [7] to introduce and study the Bernstein–Sato polynomial associated to a subscheme not necessarily of codimension one, and to generalize Theorems 5.2 and 5.4 to this setting.

6. Spaces of arcs and contact loci

Let X be a smooth n -dimensional complex variety. Given $m \geq 0$, we denote by

$$X_m = \text{Hom}(\text{Spec } \mathbb{C}[t]/(t^{m+1}), X)$$

the space of m^{th} order jets on X . This carries a natural scheme structure. Similarly we define the space of formal arcs on X as

$$X_\infty = \text{Hom}(\text{Spec } \mathbb{C}[[t]], X).$$

These constructions are functorial, hence to every morphism $\mu: X' \rightarrow X$ we associate corresponding morphisms μ_m and μ_∞ . Thanks to the work of Kontsevich, Denef, Loeser and others on motivic integration, in recent years these spaces have been very useful in constructing invariants of singular algebraic varieties. In what follows we describe some applications of these spaces to the study of singularities.

We have natural projection maps induced by truncation $X_{m+1} \rightarrow X_m$. Since X is smooth, this is locally trivial in the Zariski topology, with fiber \mathbb{A}^n . We similarly have projection maps $X_\infty \rightarrow X_m$. A subset C of X_∞ is called a *cylinder* if it is the inverse image of a constructible set S in some X_m . Moreover, C is called locally closed (closed, irreducible) if S is so. If C is a closed cylinder that is the inverse image of $S \subset X_m$, its codimension in X_∞ is equal to the codimension of S in X_m .

Consider a nonzero ideal sheaf $\mathfrak{a} \subseteq \mathcal{O}_X$ defining a subscheme $Y \subset X$. Given a finite jet or an arc γ on X , the order of vanishing of \mathfrak{a} – or the order of contact of the

corresponding scheme Y – along γ is defined in the natural way. Specifically, pulling \mathfrak{a} back via γ yields an ideal (t^e) in $\mathbb{C}[t]/(t^{m+1})$ or $\mathbb{C}[[t]]$, and one sets

$$\text{ord}_\gamma(\mathfrak{a}) = \text{ord}_\gamma(Y) = e.$$

(Take $\text{ord}_\gamma(\mathfrak{a}) = m + 1$ when \mathfrak{a} pulls back to the zero ideal in $\mathbb{C}[t]/(t^{m+1})$ and $\text{ord}_\gamma(\mathfrak{a}) = \infty$ when it pulls back to the zero ideal in $\mathbb{C}[[t]]$.) For a fixed integer $p \geq 0$, we define the *contact locus*

$$\text{Cont}^p(Y) = \text{Cont}^p(\mathfrak{a}) =_{\text{def}} \{\gamma \in X_\infty \mid \text{ord}_\gamma(\mathfrak{a}) = p\}.$$

Note that this is a locally closed cylinder: for $m \geq p$, it is the inverse image of

$$\text{Cont}^p(Y)_m = \text{Cont}^p(\mathfrak{a})_m =_{\text{def}} \{\gamma \in X_m \mid \text{ord}_\gamma(\mathfrak{a}) = p\}, \tag{5}$$

which is locally closed in X_m . A subset of X_∞ is called an *irreducible closed contact subvariety* if it is the closure of an irreducible component of $\text{Cont}^p(Y)$ for some p and Y .

Suppose now that W is an arbitrary irreducible closed cylinder in X_∞ . We can naturally associate a valuation of the function field of X to W as follows. If f is a nonzero rational function of X , we put

$$\text{val}_W(f) = \text{ord}_\gamma(f) \quad \text{for a general } \gamma \in W.$$

This valuation is not identically zero if and only if W does not dominate X .

If $\mu: X' \rightarrow X$ is a proper birational morphism, with X' smooth, and if E is an irreducible divisor on X' , then we define a valuation by

$$\text{val}_E(f) = \text{the vanishing order of } f \circ \mu \text{ along } E.$$

A valuation on the function field of X is called a *divisorial valuation* (with center on X) if it is of the form $m \cdot \text{val}_E$ for some positive integer m and some divisor E as above.

A key invariant associated to a divisorial valuation v is its *log discrepancy*. If E is a divisor as above, we put $k_E = \text{val}_E(\det(J(\mu)))$, where $J(\mu)$ is the Jacobian matrix of μ . Equivalently, k_E is the coefficient of E in the relative canonical divisor $K_{X'/X}$. Note that k_E depends only on val_E (it does not depend on the model X'). Given an arbitrary divisorial valuation $m \cdot \text{val}_E$, we define its log discrepancy as $m(k_E + 1)$.

Consider a divisor E on X' as above. If $C_m(E)$ is the closure of $\mu_\infty(\text{Cont}^m(E))$, then it is not hard to see that $C_m(E)$ is an irreducible closed contact subvariety of X_∞ such that $\text{val}_{C_m(E)} = m \cdot \text{val}_E$. The following result of Ein, Lazarsfeld and Mustaa [18] describes in general the connection between cylinders and divisorial valuations.

Theorem 6.1. *Let X be a smooth variety.*

- (i) *If W is an irreducible, closed cylinder in X_∞ that does not dominate X , then the valuation val_W is divisorial.*

- (ii) For every divisorial valuation $m \cdot \text{val}_E$, there is a unique maximal irreducible closed cylinder W such that $\text{val}_W = m \cdot \text{val}_E$: this is $W = C_m(E)$.
- (iii) The map that sends $m \cdot \text{val}_E$ to $C_m(E)$ gives a bijection between divisorial valuations of $\mathbb{C}(X)$ with center on X and the set of irreducible closed contact subvarieties of X_∞ .

The applicability of this result to the study of singularities is due to the following description of log discrepancy of a divisorial valuation in terms of the codimension of a certain set of arcs.

Theorem 6.2. *Given a divisorial valuation $v = m \cdot \text{val}_E$ with center on X , if $C_m(E)$ is its associated irreducible closed contact subvariety in X_∞ , then the log discrepancy of v is equal to $\text{codim}(C_m(E), X_\infty)$.*

Combining the statements of the above theorems, we deduce a lower bound for the codimension of an arbitrary cylinder in terms of the log discrepancy of the corresponding divisor.

Corollary 6.3. *If W is a closed, irreducible cylinder in X_∞ that does not dominate X , then $\text{codim}(W, X_\infty)$ is bounded below by the log discrepancy of val_W .*

Remark 6.4. The above two theorems also hold for singular varieties after some minor modifications using Nash's blow-up and Mather's canonical class.

The key ingredient in the proof of the above theorems is the following result due to Kontsevich, Denef and Loeser (see [14]). It constitutes the geometric content of the so-called Change of Variable Theorem in motivic integration. Suppose that $\mu: X' \rightarrow X$ is a proper, birational morphism of smooth varieties and let $K_{X'/X}$ be the relative canonical divisor.

Theorem 6.5. *Given integers $e \geq 0$ and $m \geq e$, consider the contact locus*

$$\text{Cont}^e(K_{X'/X})_m = \{\gamma' \in X'_m \mid \text{ord}_{\gamma'}(K_{X'/X}) = e\}.$$

If $m \geq 2e$, then $\text{Cont}^e(K_{X'/X})_m$ is a union of fibres of $\mu_m: X'_m \rightarrow X_m$, each of which is isomorphic to an affine space \mathbb{A}^e . Moreover, if

$$\gamma', \gamma'' \in \text{Cont}^e(K_{X'/X})_m$$

lie in the same fibre of μ_m , then they have the same image in X'_{m-e} .

As an application of Theorems 6.1 and 6.2, one gives in [18] a simple proof of the following result of Mustață [37] describing the log canonical threshold in terms of the geometry of the space of jets.

Theorem 6.6. *Let X be a smooth complex variety and Y be a closed subscheme of X defined by the nonzero ideal sheaf I_Y . Let X_m and Y_m be the spaces of m^{th} order jets of X and Y , respectively. If $c = \text{lc}(X, Y)$, then*

- (i) For every m we have $\text{codim}(Y_m, X_m) \geq c \cdot (m + 1)$. More generally, if $W \subset X_\infty$ is an irreducible closed cylinder that does not dominate X , then $\text{codim}(W, X_\infty) \geq c \cdot \text{val}_W(I_Y)$.
- (ii) If m is sufficiently divisible, then $\text{codim}(Y_m, X_m) = c \cdot (m + 1)$.
- (iii) We have $c = \lim_{m \rightarrow \infty} \frac{\text{codim}(Y_m, X_m)}{m+1}$.

The above results relating divisorial valuations with the space of arcs can be used to study more subtle invariants of singularities of pairs. Let Y be a closed subscheme of the smooth variety X , and let λ be a positive real number. We associate a numerical invariant to the pair $(X, \lambda \cdot Y)$ and to an arbitrary nonempty closed subset $B \subseteq X$, as follows.

Consider a divisorial valuation of the form val_E with center $c_X(E)$ in X (the center is the image of E in X). The *log discrepancy* of the pair $(X, \lambda \cdot Y)$ along E is

$$a(E, X, \lambda \cdot Y) = k_E + 1 - \lambda \cdot \text{val}_E(I_Y),$$

where I_Y is the ideal of Y in X . Note that if $Y = \emptyset$, we recover the log discrepancy of val_E . The idea is to measure the singularities of the pair $(X, \lambda \cdot Y)$ using the log discrepancies along divisors with center contained in B .

Definition 6.7. Let $B \subset X$ be a nonempty closed subset. The minimal log discrepancy of $(X, \lambda \cdot Y)$ over B is defined by

$$\text{mld}(B; X, \lambda \cdot Y) := \inf_{c_X(E) \subseteq B} \{a(E; X, \lambda \cdot Y)\}. \tag{6}$$

Remark 6.8. One can show that $\text{mld}(B; X, \lambda \cdot Y)$ is either $-\infty$ or a nonnegative real number. In fact, $\text{mld}(B; X, \lambda \cdot Y) \neq -\infty$ if and only if there is an open neighborhood U of B such that $\text{lc}(U, U \cap Y) \geq \lambda$. An important fact about minimal log discrepancies is that they can be computed using a log resolution of $(X, B \cup Y)$, see [1].

The following theorem of Ein, Mustařă and Yasuda [22] gives a description of minimal log-discrepancies in term of the geometry of the space of arcs.

Theorem 6.9. Let B be a nonempty, proper closed subset of X , and let $\pi : X_\infty \rightarrow X$ be the projection map. For every proper closed subscheme Y of X and for every λ and $\tau \in \mathbb{R}_+$ we have $\text{mld}(B; X, \lambda \cdot Y) \geq \tau$ if and only if for every irreducible closed cylinder $W \subseteq \pi^{-1}(B)$ we have

$$\text{codim}(W, X_\infty) \geq \lambda \cdot \text{val}_W(I_Y) + \tau.$$

The above theorem can be applied to study the behavior of singularities of pairs under restriction to a divisor. This is useful whenever one wants to do induction on dimension. Suppose that D is a smooth divisor on X . We want to relate the singularities of $(X, \lambda \cdot Y)$ with those of $(D, \lambda \cdot Y|_D)$. The adjunction formula suggests that the precise relation should be between $(X, D + \lambda \cdot Y)$ and $(D, \lambda \cdot Y|_D)$. The precise formula is the content of the following theorem from [22].

Theorem 6.10. *Let D be a smooth divisor on the smooth variety X and let B be a nonempty proper closed subset of D . If Y is a closed subscheme of X such that $D \not\subseteq Y$, and if $\lambda \in \mathbb{R}_+$, then*

$$\text{mld}(B; X, D + \lambda \cdot Y) = \text{mld}(B; D, \lambda \cdot Y|_D).$$

Remark 6.11. The notion of minimal log discrepancy plays an important role in the Minimal Model Program. It can be defined under weak assumptions on the singularities of X : one requires only that X is normal and \mathbb{Q} -Gorenstein. Kollár and Shokurov have conjectured the statement of Theorem 6.10 with the assumption that X and D are only normal and \mathbb{Q} -Gorenstein. It is easy to see that the inequality “ \leq ” holds in general, and the opposite inequality is known as Inversion of Adjunction (see [32] and [33] for a discussion of this conjecture and related topics). Theorem 6.10 has been generalized in [21] to the case when both X and D are normal locally complete intersections.

The interpretation of minimal log discrepancies in terms of spaces of arcs gives also the following semicontinuity statement. This was conjectured for an arbitrary (normal and \mathbb{Q} -Gorenstein) variety X by Ambro and Shokurov, see [1]. The statement below, due to Ein, Mustață and Yasuda [22] has been generalized to the case of a normal locally complete intersection variety in [21].

Theorem 6.12. *If X is a smooth variety and if Y is a closed subscheme of X , then for every $\lambda \in \mathbb{R}_+$, the function on X defined by $x \rightarrow \text{mld}(x; X, \lambda \cdot Y)$ is lower semicontinuous.*

We end with a result that translates properties of the minimal log discrepancy over the singular locus of a locally complete intersection variety into geometric properties of its spaces of jets.

Theorem 6.13. *Let X be a normal locally complete intersection variety of dimension n .*

- (i) *X_m has pure dimension $n(m + 1)$ for every m (and in this case X_m is also a locally complete intersection) if and only if $\text{mld}(X_{\text{sing}}; X, \emptyset) \geq 0$ (this says that X has log canonical singularities).*
- (ii) *X_m is irreducible for every m (and in this case it is also reduced) if and only if $\text{mld}(X_{\text{sing}}; X, \emptyset) \geq 1$ (this says that X has canonical singularities).*
- (iii) *X_m is normal for every m if and only if $\text{mld}(X_{\text{sing}}; X, \emptyset) > 1$ (this says that X has terminal singularities).*
- (iv) *In general, we have $\text{codim}((X_m)_{\text{sing}}, X_m) \geq \text{mld}(X_{\text{sing}}; X, \emptyset)$ for every m .*

Remark 6.14. The description in (ii) above was first proved in [38]. Note that since X is in particular Gorenstein, it is known that X has canonical singularities if and only

if it has rational singularities. All the statements in the above theorem were obtained in [22] and [21] combining the description of minimal log discrepancies in terms of spaces of arcs and Inversion of Adjunction.

7. Invariants in positive characteristic

Several invariants have been recently introduced in positive characteristic using the Frobenius morphism, invariants whose behavior is formally very similar to the ones we have discussed in characteristic zero. Moreover, there are interesting results and conjectures involving the comparison between the two sets of invariants via reduction mod p .

As in the case of singularities of pairs (X, Y) in characteristic zero, one can develop the theory under very mild assumptions on the ambient variety X (in fact, the positive characteristic theory does not even need the assumption that X is \mathbb{Q} -Gorenstein). For this one needs to use the full power of the theory of tight closure of Hochster and Huneke [26]. However, the definitions become particularly transparent if we assume X nonsingular. Therefore, in accord with the setup in the previous sections, we will make this assumption. The theory we present here is due to Hara and Yoshida [25] building on previous work of Hara, Smith, Takagi and Watanabe.

We work in the local setting with a regular local ring (R, \mathfrak{m}, k) of characteristic $p > 0$. Let $n = \dim(R)$ and let E be the top local cohomology module of R , $E = H_{\mathfrak{m}}^n(R)$. If x_1, \dots, x_n generate \mathfrak{m} , then

$$E \simeq R_{x_1 \dots x_n} / \sum_{i=1}^n R_{x_1 \dots \widehat{x}_i \dots x_n}. \tag{7}$$

The Frobenius morphism on R induces a Frobenius morphism F_E on E that via the isomorphism (7) takes the class of $u/(x_1 \dots x_n)^d$ to the class of $u^p/(x_1 \dots x_n)^{pd}$.

We want to study the singularities of the pair (X, Y) , where $X = \text{Spec}(R)$ and Y is defined by a nonzero ideal \mathfrak{a} . For every $r \geq 0$ and every $e \geq 1$, we put

$$Z_{r,e} := \ker(\mathfrak{a}^r F_E^e) = \{w \in E \mid h F_E^e(w) = 0 \text{ for all } h \in \mathfrak{a}^r\}.$$

Given a nonnegative real number λ , the *test ideal* of the pair $(X, \lambda \cdot Y)$ is

$$\tau(X, \lambda \cdot Y) := \text{Ann}_R \left(\bigcap_{e \geq 1} Z_{\lceil \lambda p^e \rceil, e} \right).$$

Here $\lceil \alpha \rceil$ denotes the smallest integer that is $\geq \alpha$.

As Hara and Yoshida show in [25], the test ideals $\tau(X, \lambda \cdot Y)$ enjoy formal properties similar to those of the multiplier ideals $\mathcal{J}(X, \lambda \cdot Y)$ in characteristic zero. In particular, we can consider the jumping numbers for the test ideals: these are the λ such that $\tau(X, \lambda \cdot Y) \subsetneq \tau(X, (\lambda - \varepsilon) \cdot Y)$ for every positive ε .

The set of jumping numbers for the test ideals are also eventually periodic with period one. However, two basic properties that for multiplier ideals follow simply from the description in terms of a log resolution are not known for test ideals: it is not known whether every jumping number for the test ideals is rational, and whether in every bounded interval there are only finitely many such jumping numbers. We want to stress that the problem does *not* come from the fact that we do not know, in general, whether such resolutions exist. Even when we have such resolutions, the invariants in characteristic p do not depend simply on the numerical data of the resolution (see Example 7.4 below for the case of the cusp).

There is a more direct description of the set of jumping numbers given by Mustața, Takagi and Watanabe in [39]. Suppose that J is a proper ideal of R containing \mathfrak{a} in its radical. For every $e \geq 1$, define $v^J(p^e)$ to be the largest r such that \mathfrak{a}^r is not contained in the e^{th} Frobenius power of J

$$J^{[p^e]} := (u^{p^e} \mid u \in J).$$

It is easy to see that $v^J(p^e)/p^e \leq v^J(p^{e+1})/p^{e+1}$, and the F -threshold of \mathfrak{a} with respect to J is defined by

$$c^J(\mathfrak{a}) := \sup_e \frac{v^J(p^e)}{p^e}.$$

It is shown in [39] that the set of F -thresholds of \mathfrak{a} (with respect to various J) is precisely the set of jumping numbers for the test ideals of (X, Y) . Note that the smallest F -threshold is obtained for $J = \mathfrak{m}$: this is an analogue of the log canonical threshold that was introduced and studied by Takagi and Watanabe in [46].

There are several interesting results and questions relating the invariants in characteristic zero and those obtained via reduction mod p . To keep the notation simple we will work in the following setup. Suppose that \mathfrak{a} is an ideal in $A[x_1, \dots, x_n]$, where A is the localization of \mathbb{Z} at some integer. Let Y be the subscheme of $X = \mathbb{A}_A^n$ defined by \mathfrak{a} . If p is a prime that is large enough, then by reducing mod p and localizing at (x_1, \dots, x_n) we get a closed subscheme Y_p in $X_p = \text{Spec } \mathbb{F}_p[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$ defined by the ideal \mathfrak{a}_p .

The multiplier ideals of the pair (X, Y) (more precisely, of its extension to \mathbb{C}) can be computed by a log resolution defined over \mathbb{Q} . After suitably localizing A we may assume that the multiplier ideals are defined over A , too. The following results relate the reduction mod p of the multiplier ideals with the test ideals. They are due to Hara and Yoshida [25], based on previous work of Hara, Smith, Takagi and Watanabe.

Theorem 7.1. *With the above notation, if $p \gg 0$, then for every λ we have*

$$\tau(X_p, \lambda \cdot Y_p) \subseteq \mathcal{J}(X, \lambda \cdot Y)_p.$$

Note that since our primes are large enough, the log resolution over \mathbb{Q} induces by reduction mod p log resolutions for (X_p, Y_p) . The proof of Theorem 7.1 is based on the use of local duality for the reduction mod p of the log resolution. The proof of the

next result is more involved, using the approach of Deligne and Illusie to the positive characteristic proof of the Kodaira Vanishing Theorem.

Theorem 7.2. *With the above notation, for every λ and for every $p \gg 0$ (depending on λ) we have*

$$\tau(X_p, \lambda \cdot Y_p) = \mathcal{J}(X, \lambda \cdot Y)_p.$$

Remark 7.3. We reinterpret the above statements in terms of jumping numbers. For simplicity, we restrict ourselves to the smallest such number: given \mathfrak{a} as above and $p \gg 0$, we want to compare the log canonical threshold c of the pair (X, Y) in some small neighborhood of the origin, with the F -pure threshold $c_p = c^m(\mathfrak{a}_p)$. Theorem 7.1 implies that for all $p \gg 0$ we have $c \geq c_p$, while Theorem 7.2 implies that $\lim_{p \rightarrow \infty} c_p = c$.

Example 7.4. Let \mathfrak{a} be generated by $f = x^2 + y^3$, whose log canonical threshold is $\frac{5}{6}$. Let $p > 3$ be a prime. One can show that if $p \equiv 1 \pmod{3}$, then the largest r such that f^r does not lie in (x^{p^e}, y^{p^e}) is given by $v(p^e) = \frac{5}{6}(p^e - 1)$ for every $e \geq 1$, so that $c_p = \frac{5}{6}$. On the other hand, if $p \equiv 2 \pmod{3}$, then $v(p) = \frac{5p-7}{6}$, while $v(p^e) = \frac{5p^e - p^{e-1} - 6}{6}$ for $e \geq 2$. Therefore in this case $c_p = \frac{5}{6} - \frac{1}{6p}$.

Conjecture 7.5. For every ideal \mathfrak{a} in $A[x_1, \dots, x_n]$ there are infinitely many primes p for which the F -pure threshold c_p is equal to the log canonical threshold c .

For a discussion of this conjecture we refer to [39]. We end by mentioning a connection between the positive characteristic invariants and the Bernstein–Sato polynomial. Suppose that $f \in A[x_1, \dots, x_n]$ is as above. We know that the Bernstein–Sato polynomial $b_f(s)$ has rational roots, and in fact, one can show that one can find an equation (3) as in the definition of $b_f(s)$ with P having rational coefficients. Therefore, after suitably localizing A , we may assume that both b_f and P have coefficients in A and that (3) holds over A . It follows that if p is a large enough prime, we get a similar equation over \mathbb{F}_p .

Consider now an ideal J in the ring $\mathbb{F}_p[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$, such that f_p lies in the radical of J . Let us apply (3) with $s = v^J(p^e)$, the largest integer such that f_p^r is not in $J^{[p^e]}$. Since the ideal $J^{[p^e]}$ is a module over the ring $\mathbb{F}_p[x, \partial_x]$, we deduce that $b_f(v^J(p^e)) \equiv 0 \pmod{p}$. Therefore the functions v^J give roots of $b_f \pmod{p}$. Sometimes one can use this observation to find actual roots of b_f .

Example 7.6. Let $f = x^2 + y^3$. We have described in Example 7.4 the function $v = v^J$ when J is the maximal ideal. If $p \equiv 1 \pmod{3}$, then $v(p^e) = \frac{5}{6}(p^e - 1)$. The above discussion implies that p divides $b_f(-5/6)$. Since there are infinitely many such primes, we deduce that $-5/6$ is a root of b_f . Similarly, if $p \equiv 2 \pmod{3}$, then it follows from the formula for $v(p)$ that $-7/6$ is a root of b_f , and from the formula for $v(p^e)$, with $e \geq 2$ that -1 is a root of b_f . Therefore we have obtained all roots of the Bernstein–Sato polynomial of f by this method.

A similar picture can be seen in other examples, though at the moment there is no general result in this direction. In [6] this approach was used to describe all the roots of the Bernstein–Sato polynomial of a monomial ideal. It would be very interesting to find a more conceptual framework that would explain the connection between the Bernstein polynomial and the invariants in positive characteristic.

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