Rational curves and rational points

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Abstract. We survey some recent results proving the existence of rational points over one dimensional function fields and finite fields on varieties containing many rational curves. We also consider conjectural extensions of these theorems to higher dimensional function fields.

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1. Introduction

In the past five years there has been substantial progress giving geometric interpretations and generalizations of classical theorems about C_1 fields. In this lecture I would like to focus on some results about the existence of rational points on certain classes of varieties over function fields of curves and over finite fields. The geometry of rational curves play a central role in this work.

Recent results of de Jong and Starr give hope that in the future we will see a similar development for function fields of higher dimensional varieties, although so far we do not have even a precise conjectural formulation. The program that they and others have proposed raises many interesting questions, most of which are still relatively unexplored.

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2. Classical results

All the results and conjectures that I want to consider have their origins in two theorems proven in the 1930s which guarantee that over certain fields, hypersurfaces of low degree in projective space possess rational points. To make this precise, we make a definition.

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Definition 2.1. A field K is said to be C_1 if every homogeneous polynomial $f \in K[x_0, \dots, x_n]$ has a nontrivial zero provided that $\deg(f) \leq n$.

The first two known classes of C_1 fields were provided by the following two well-known theorems.

Theorem 2.2 (Tsen). *If k is an algebraically closed field and C is an irreducible curve over k with function field K, then K is a C_1 field.*

Theorem 2.3 (Chevalley). *Finite fields are C_1.*

There are two natural ways to search for generalizations of these results: we can try to extend the class of fields considered, or extend the class of varieties. There are substantial generalizations in both directions. We will start by considering the second question, namely, for which types of varieties can we guarantee that they will possess points over finite fields or over one dimensional function fields? At the expense of making less than optimal statements, we will always restrict our attention to smooth projective varieties. In this context, there are many natural generalizations of the class of hypersurfaces above. The two most obvious ones for an algebraic geometer before 1992 were the following.

Remark 2.4. If X is a smooth hypersurface in $\mathbb{P}^n$ of degree d, then the following conditions are equivalent.

- $d \leq n$.
- X is Fano, that is, the canonical line bundle K_X has ample dual.
- $h^i(X, \mathcal{O}_X) = 0$ for all $i > 0$.

These naturally gives rise to four questions, do Tsen's or Chevalley's theorems generalize to either of these classes of varieties? The answer to one of these questions has been known for some time.

Theorem 2.5 (Katz). *If X is a projective variety over $\mathbb{F}_q$ such that $h^i(X, \mathcal{O}_X) = 0$ for all $i > 0$, then X has an $\mathbb{F}_q$-point.*

The other three have all been (nearly) settled in the new millennium, in large part due to a fundamental shift in viewpoint coming from a third geometric condition – *rational connectivity* – which is also equivalent for smooth hypersurfaces to the inequality $d \leq n$.

3. Rationally connected varieties

The basic theory of rational connectivity was developed independently by Campana [C] and Kollar, Miyaoka, and Mori [KMM]. We will give a quick review of some of the main points. Throughout, we work over a ground field k. To simplify our discussion we will restrict our attention to smooth projective varieties, and we will take k to be an uncountable algebraically closed field. A careful reader will notice that this last hypothesis will not be satisfied in many of the applications we refer to later. To avoid this, one must either make more careful definitions, or first base change to such a field. For a good treatment of the foundations of the theory of rationally connected varieties, we refer to [K] or [D].

Given a projective variety X, by a *rational curve* in X, we mean the image of a morphism $f : \mathbb{P}^1 \to X$.

Definition 3.1. A smooth projective variety X is *rationally connected* (RC) if two general points of X can be connected by a rational curve.

If $k = \mathbb{C}$ this notion is equivalent to many other conditions guaranteeing the existence of an abundance of rational curves in X. Since these other conditions can fail to be equivalent to rational connectedness over other fields (or for singular varieties), they have their own names.

Definition 3.2. X is *rationally chain connected* (RCC) if any two points of X can be joined by a chain of rational curves.

Definition 3.3. X is *separably rationally connected* (SRC) if there exists a morphism $f : \mathbb{P}^1 \to X$ such that $f^*(TX)$ is an ample vector bundle.

While this last definition appears quite different from the previous two, it is straightforward to verify that such a morphism deforms so freely that it guarantees the existence of a rational curve through two general points. Combining this with the elementary fact that every irreducible component of a degeneration of a rational curve is rational, we obtain the implications

$$\text{SRC} \implies \text{RC} \implies \text{RCC}.$$

In characteristic zero, by more delicate arguments, one can reverse these implications. The equivalence of these notions is quite convenient, because it is easy to see that the property of being rationally chain connected is closed in families, while that of being separably rationally connected is open. As a result, we find that in characteristic zero, rational connectivity is a deformation invariant property. This is one reason why the notion is particularly well suited to the classification theory of higher dimensional varieties.

The following beautiful theorem, which even over the complex numbers can be proven at this time only by making use of results in characteristic p, was established in [C] and [KMM].

Theorem 3.4. *Fano varieties are rationally chain connected.*

As a result, one can try to address questions about Fano varieties by studying the geometry of rational curves. Indeed, rational connectivity seems to be a more useful notion for questions about the existence of rational points.

4. Rational points on rationally connected varieties

For both finite fields and function fields, the study of rational points often takes as its starting point a more geometric interpretation of the notion of rational point. The very different methods used in the two cases reflect the difference in this geometric interpretation. We will start with the case of function fields.

If we let K be the function field of a smooth curve C defined over an algebraically closed field k, then given a projective variety X_K over K, we can always find a C model for X_K. That is, we can find a projective variety X over k together with a morphism $\pi : X \to C$ such that the base change to Spec K is X_K. A rational point of X_K then corresponds exactly to a section of π. Thus the geometry of rational points over K is intimately connected to the geometry of curves in higher dimensional varieties. From this point of view, then, the notion of rational connectivity is particularly appealing, since it is also directly connected with the same type of geometric objects. Moreover, it is not difficult to see that the variety X_K is geometrically rationally connected if and only if a general fiber of the morphism π is rationally connected.

Thus, the following theorem, which is proven in characteristic zero in [GHS] and in positive characteristic in [dJS] is a natural generalization of Tsen's theorem.

Theorem 4.1. *If C is a smooth curve over an algebraically closed field, and $\pi : X \to C$ is a proper morphism whose general fiber is separably rationally connected, then π admits a section.*

Combining with Theorem 3.4 we get a partial answer to one of the questions raised in Section 2.

Corollary 4.2. *If K is the function field of a curve defined over an algebraically closed field of characteristic zero, then every Fano variety over K has a rational point.*

To my knowledge the analogous question in positive characteristic remains open, since Fano varieties are not known to be separably rationally connected.

The proof of Theorem 4.1 is based on the geometry of Kontsevich's space of stable maps, $\overline{M}_g(X)$. This space, which compactifies the space of smooth curves in X by allowing them to degenerate to morphisms from nodal curves, was introduced into mathematics in order to study ideas coming from string theory. It is extremely useful in this context, however, because it has two advantages over the traditional compactifications of the space of curves in X. Its deformation theory is extremely simple and it has an obvious functorial property – given a morphism of varieties $\pi : X \to Y$, we get an induced morphism $\overline{M}(\pi) \colon \overline{M}_g(X) \to \overline{M}_g(Y)$. In the setting of Theorem 4.1, the point is that via degeneration methods, one can produce a curve B in X such that B corresponds to a smooth point of $\overline{M}_g(X)$ at which the differential of $\overline{M}(\pi)$ is surjective. As this last statement involves only the tangent spaces to the moduli spaces, it can be verified using elementary deformation theory, but nonetheless it is very powerful, since it guarantees that $\overline{M}(\pi)$ dominates at least one irreducible component of $\overline{M}_g(C)$. Thus, the proof is completed by finding a suitable degeneration of a morphism of curves, which can be done by elementary techniques.

In [GHMS], by making a careful study of the maps $\overline{M}(\pi)$ for general morphisms of varieties π it is shown that in a certain sense the property of rational connectivity is universal for the problem of finding rational points over function fields of curves. In other words, we cannot really hope to find a larger class of varieties which will always possess a point over function fields than the class given by Theorem 4.1.

In particular, there exist varieties X over $K(C)$ such that $h^i(X, \mathcal{O}_X) = 0$ for all $i > 0$ but without a rational point. We remark that while the methods of [GHMS] prove the existence of such a variety, they are not effective and give little insight into how to actually write one down. In [Laf] G. Lafon produces an explicit Enriques surface over $\mathbb{Q}(t)$ with no rational point even over $\mathbb{C}((t))$.

Turning our attention to finite fields, the theory has a rather different flavor. The geometric interpretation of an $\mathbb{F}_q$-point of a variety X is that it is a fixed point of the action of the Frobenius morphism. This point of view is very useful in connection with cohomological conditions, since one can use Lefschetz type theorems that relate fixed points to cohomological data. In particular, Theorem 2.5 was proven by establishing the analogue for the Frobenius morphism of the holomorphic Lefschetz fixed point theorem – that the number of points of a smooth projective variety over $\mathbb{F}_q$ is congruent modulo p to the alternating sum of the traces of Frobenius on the cohomology groups $H^i(X, \mathcal{O}_X)$. Since the trace of Frobenius on H^0 is obviously 1 the existence result follows immediately, as well as the sharper statement that the number of $\mathbb{F}_q$-points on X is congruent to 1 modulo p.

In [E], H. Esnault shows how it is possible to relate rational curves to this circle of ideas. She observes that an immediate consequence of rational chain connectivity is that the Chow group of zero cycles is $\mathbb{Z}$. A method introduced by Bloch (cf. [B]) can then be used to control the eigenvalues of the action of Frobenius on the étale (or crystalline) cohomology of X. This is done by thinking of the diagonal $\Delta \subset X \times X$ as a zero cycle in $X_{K(X)}$ and using the triviality of the Chow group to move it via rational equivalence to $[\{p\} \times X] \cup \Gamma$ where Γ is a cycle whose projection to the second factor is not dominant. Interpreting this as the decomposition of the identity map on the cohomology of X into the projection onto H^0 plus the projection onto the rest, together with formal (but deep) properties of étale cohomology then yield that the eigenvalues of the action of Frobenius on $H^i_{et}(X, \mathbb{Q}_l)$ are divisible by q for all $i > 0$. Now using the Lefschetz–Verdier trace formula, she concludes the following theorem.

Theorem 4.3. *If X is a smooth projective variety over $\mathbb{F}_q$ with $Ch_0(X \times \overline{K(X)} = \mathbb{Z}$, then X has a rational point.*

This has as immediate corollaries:

Corollary 4.4. *Every smooth, projective, rationally chain connected variety over a finite field has a rational point.*

Corollary 4.5. *Fano varieties over finite fields have rational points.*

As before, one actually gets a sharper statement – the number of rational points is congruent to 1 mod q.

5. Higher rational connectivity

We would now like to consider some more speculative material about the existence of points on varieties over more complicated fields. We will describe (in very rough terms) a program of de Jong and Starr, and following them, we will consider here only fields of the form $K(X)$ for X a complex variety. We encourage the reader to consider analogous questions over other classes of fields, where it is easy to make similar speculations.

Long before any of the work described here, Serge Lang gave an entirely different generalization of Tsen's theorem. Following him, we first generalize the definition of C_1.

Definition 5.1. A field K is called C_r if every homogeneous polynomial f in $K[x_0, \dots, x_n]$ has a nontrivial zero provided $d^r \leq n$.

Then Lang proves the following theorem.

Theorem 5.2. *If X is a variety of dimension r over an algebraically closed field, then the function field $K(X)$ is C_r.*

It is natural to ask whether there is a common generalization of Theorem 4.1 and Theorem 5.2. In other words, can we single out a class of abstract varieties which generalizes the class of degree d hypersurfaces in $\mathbb{P}^n$ with $d^r \leq n$ such that they will always have a rational point over function fields of dimension r? So far there are few positive results, but there is an analogy which is extremely tantalizing and which takes as its starting point the observation that rational connectivity is just the usual topological notion of path connectedness with the interval replaced by $\mathbb{P}^1$. If we systematically translate between topology and algebraic geometry using fiber bundle as the analogue of family and the interval as the analogue of $\mathbb{P}^1$, then Theorem 4.1 translates into the elementary topological fact that over a one dimensional manifold, any fiber bundle with connected fiber admits a section.

This topological fact has a straightforward generalization to higher dimensional bases. Namely, if $\phi\colon X \to M$ is a fibration with M an r-dimensional manifold and fiber F such that $\pi_i(F) = 0$ for all $i < r$, then ϕ admits a section. This suggests the hope that we could try to define some notion of higher rational connectedness and prove a theorem that r-rationally connected varieties over r dimensional function fields have rational points.

Unfortunately, such a theorem cannot hold. Certainly any generalization of the class of low degree hypersurfaces will include projective space itself (the case $d = 1$.)

It is well known, however, that over higher dimensional bases, there are families all of whose fibers are isomorphic to $\mathbb{P}^n$ but which do not admit a section. The obstruction to the existence of a section lies in the Brauer group of the base of the family. While this might dash one's hope, de Jong and Starr have taken the optimistic view that at least in dimension 2, the Brauer group might be the only obstruction.

In particular, the Brauer class associated to a family of projective spaces admits a generalization to this situation. Given a morphism $\phi\colon X \to B$, one gets a sequence

$$\mathrm{Pic}(X_K) \to \mathrm{Pic}(X_{\overline{K}})^G \to \mathrm{Br}(K).$$

If X has a K-point, the rightmost map in this sequence vanishes, thus the nonvanishing of this map is an obstruction to the existence of a K-point. We will refer to this as the Brauer obstruction. We remark that this obstruction will always vanish for the projection to B of a hypersurface of dimension at least 3 in $\mathbb{P}^n \times B$. Now we can state the

Metaconjecture. There exists a notion of rationally simply connected such that for any morphism $\pi\colon X \to B$ with B a complex surface and with rationally simply connected general fiber, ϕ admits a rational section if and only if the Brauer obstruction vanishes. Moreover, for smooth hypersurfaces, this notion should agree with the condition that $d^2 \leq n$.

There is a natural guess for how to go about formulating what it means for a variety X to be rationally simply connected. For a topological space, simple connectivity means that the space of loops is path connected, or equivalently that the space of paths between any two fixed points is path connected. By again replacing paths with rational curves, we would arrive at a provisional definition that the space of rational curves joining two general points of X should be rationally connected. Unfortunately, this is impossible, since the space of rational curves joining two points will not even be connected – a rational curve has a discrete invariant, its homology class, which is invariant under deformation. The most we can ask for is for the space of rational curves *of fixed topological type* joining two general points to be rationally connected. Finally, it seems more reasonable to ask for this only in sufficiently positive homology classes since rational connectedness itself is really only a condition on the high degree curves. There are various possible meanings of "sufficiently positive" and the optimal choice is not clear. At least if $\mathrm{Pic}(X) = \mathbb{Z}$ (which will be the case for smooth hypersurfaces), the condition is unambiguous.

Unfortunately, this does not seem to be enough, since de Jong and Starr are able to prove that smooth hypersurfaces of degree d in $\mathbb{P}^n$ satisfy this constraint provided that $d^2 \leq n + 1$ which in this case is too good of a theorem, since Lang's result is sharp. To repair this, they propose that there is an additional condition which might be seen as an analogue of the condition appearing in the definition of separable rational connectivity, namely the existence of a curve in the space of curves satisfying certain positivity properties. This condition is satisfied for hypersurfaces of the desired degree range. We refer the reader to their preprints for the precise conditions they use.

They also propose a strategy for establishing that families of such varieties over surfaces possess a rational section provided that the Brauer obstruction vanishes. To date, they are able to carry out this program only under extremely restrictive hypotheses, which in particular rule out almost all families of hypersurfaces of degree greater than 2. Nonetheless, the work they have done lends tremendous credibility to the belief that there should be theorems in this direction, providing a geometric explanation for (at least the $r = 2$ case of) Lang's theorem. Moreover, their existing results apply to families of Grassmannians and give a new proof of de Jong's period-index theorem.

At this point, one might be inclined to believe that we would at least have a reasonably clear idea of how to generalize this type of conjecture to higher rational connectivity, but there are further issues to confront in order to do this. Since rational connectivity is a property depending only on the birational equivalence class of a variety, in defining rationally simply connected, we did not have to choose a particular compactification of the space of rational curves joining two general points. However, under any definition currently considered, the property of being rationally simply connected is not invariant under birational modification (and should not be), so to generalize these ideas to higher dimension, one needs either to fix a choice of compactification or find a different framework in which to discuss these questions. This direction is wide open and seems likely to provide interesting – and difficult – questions for some time to come.

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