Rigidity of rational homogeneous spaces

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Abstract. Rigidity questions on rational homogeneous spaces arise naturally as higher dimensional generalizations of Riemann's uniformization theorem in one complex variable. We will give an overview of some results obtained in this area by the study of minimal rational curves and geometric structures defined by the their tangent directions.

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1. Introduction

Riemann's uniformization theorem in one complex variable says that the three basic Riemann surfaces, namely, the Riemann sphere $\mathbb{P}_1$, the complex plane $\mathbb{C}$ and the unit disc Δ, exhaust all simply connected Riemann surfaces. Finding an analog of this result in higher dimensions has been one of the main themes of research in complex geometry. A number of approaches from different view-points have been developed for this. Our approach here is to regard Riemann's uniformization theorem as a characterization of the three basic Riemann surfaces. From this approach, the uniformization problem in several complex variables is to find suitable conditions which characterize some basic classes of complex manifolds generalizing the three basic Riemann surfaces. The most natural higher-dimensional analogs of the three basic Riemann surfaces are Hermitian symmetric spaces. Thus the uniformization problem in several complex variables leads to the study of rigidity of Hermitian symmetric spaces.

A nice survey of results on the rigidity of Hermitian symmetric spaces obtained up to early 1990s was given in Siu's article 'Uniformization in Several Complex Variables' [28]. As one can see in [28], the methods employed are quite different depending on whether the Hermitian symmetric spaces are of compact type, of Euclidean type, or of non-compact type. The methods used for compact type have close connection with algebraic geometry, while those for Euclidean or non-compact types are closer to differential geometry. Since we have little expertise for the cases of Euclidean type or non-compact type, we will restrict our discussion to the case of compact type. From the view-point of algebraic geometry, it is more natural to consider a larger class of complex manifolds, the rational homogeneous spaces, which include Hermitian symmetric spaces of compact type. We will discuss rigidity problems of

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rational homogeneous spaces, especially those related to the problems considered in Siu's survey. Our aim is to report the progress on these problems made after 1990.

As explained in Siu's survey, there are two approaches to the uniformization in several complex variables, one via topological conditions and the other via curvature conditions. The approach by topological conditions asks under what additional condition a complex manifold diffeomorphic to a rational homogeneous space is biholomorphic to it. The natural additional condition one can consider is the Kähler condition, the Moishezon condition or the deformation condition. Regarding the first two conditions, little new development was made after [28]. On the other hand, there has been much progress under the deformation condition. This development will be discussed in Section 3. The approach by curvature conditions is to characterize rational homogeneous spaces as manifolds with positive curvature in a suitable sense. There are various ways to impose curvature conditions, but all reasonable assumptions contain the positivity of the anti-canonical bundle. Complex manifolds with positive anti-canonical bundles are called Fano manifolds. Rational homogeneous spaces are just homogeneous Fano manifolds. Thus the question is to characterize homogeneous spaces among Fano manifolds by certain curvature properties of the tangent bundle. Typical examples are the Hartshorne conjecture, the Frankel conjecture and the generalized Frankel conjecture whose solutions by Mori [26], Siu-Yau [29], and Mok [23] were surveyed in [28]. In Section 4, we will discuss the Campana–Peternell conjecture, which generalizes these three results in the context of rational homogeneous spaces.

The results we will discuss on the rigidity problem both under the deformation condition and under the curvature condition depend on the study of minimal rational curves of Fano manifolds. This study originated from Mori's solution of the Hartshorne conjecture [26] which concerns the projective space. To handle problems on general rational homogeneous spaces other than the projective space, it is important to study the tangent directions of minimal rational curves, more precisely, the variety of minimal rational tangents (see Section 2 for definition). We will start with a discussion of this concept and related matters, before we discuss specific rigidity problems in Section 3 and Section 4. The methods of the variety of minimal rational tangents in the study of Fano manifolds have many aspects and applications other than those related to rigidity problems of rational homogeneous spaces. Here we will concentrate only on those aspects directly related to rational homogeneous spaces. For the other aspects, we refer the reader to [11], [14] and [18].

Some comments on the notation. For a vector space V, its projectivization $\mathbb{P}V$ is the set of 1-dimensional subspaces of V. For a complex manifold X and $x \in X$, the holomorphic tangent space of X at x will be denoted by $T_x(X)$ and the holomorphic tangent bundle of X will be denoted by $T(X)$.

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2. Geometric structures arising from minimal rational curves

To handle the rigidity questions, we need to show that a given complex manifold with certain additional conditions is a rational homogeneous space. Thus a common problem in these questions is how to recognize rational homogeneous spaces. Since we are interested in algebro-geometric conditions, we should be able to recognize them in terms of algebro-geometric data. Some special cases, like the projective spaces or the hyperquadrics, can be handled by the properties of certain linear systems. However such approaches are hard to generalize to other rational homogeneous spaces.

To get a hint on this problem, we should look at previous results in uniformization problems concerning rational homogeneous spaces more general than projective spaces or hyperquadrics. Very few results of this type are known. In Siu's survey [28], we found only one result of this type, namely, Mok's solution [23] of the generalized Frankel conjecture, which we will discuss again in Section 4. In Mok's work, after a deformation of the metric by the heat equation, one is given a compact complex manifold X with a Kähler metric g of positive Ricci curvature and of nonnegative holomorphic bisectional curvature, and one has to show that X is a symmetric space. For this, Mok constructed a distinguished subvariety $\mathcal{C}$ of the projectivized tangent bundle $\mathbb{P}T(X)$ which is a proper subvariety unless X is a projective space, and showed that $\mathcal{C}$ is invariant under the holonomy action of the Riemannian metric g. This implies that X is symmetric by Berger's theorem on Riemannian holonomy.

How should we formulate an algebro-geometric analog of Mok's argument? Luckily, Mok's construction of the distinguished subvariety $\mathcal{C} \subset \mathbb{P}T(X)$ is essentially algebraic. His construction can be generalized to arbitrary Fano manifolds as follows.

Let X be a Fano manifold. By Mori [26] X is covered by rational curves. Let us fix an irreducible component $\mathcal{K}$ of the space of rational curves whose members sweep out an open subset of X and have minimal degree with respect to the anti-canonical bundle. Members of $\mathcal{K}$ are called minimal rational curves. Whenever we mention minimal rational curves below, we tacitly assume that a choice of $\mathcal{K}$ is made. For a general point $x \in X$, let $\mathcal{K}_x$ be the normalization of the subvariety of $\mathcal{K}$ parametrizing members of $\mathcal{K}$ passing through x. In [26], it is proved that $\mathcal{K}_x$ is a projective manifold. [20] showed that each member of $\mathcal{K}_x$ is immersed at x. Thus there exists a morphism

$$\tau_x : \mathcal{K}_x \longrightarrow \mathbb{P}T_x(X),$$

called the tangent morphism, assigning to each curve its tangent direction at x. Its image $\tau_x(\mathcal{K}_x)$ is called the variety of minimal rational tangents at x and denoted by $\mathcal{C}_x$. Let $\mathcal{C}$ be the closure of the union of $\{\mathcal{C}_x,\ \text{general } x \in X\}$ in $\mathbb{P}T(X)$. $\mathcal{C}$ is called the variety of minimal rational tangents. The variety $\mathcal{C}$ constructed in [23] on a Fano manifold X with nonnegative holomorphic bisectional curvature coincides with the variety of minimal rational tangents on X. It is helpful to look at some examples.

Example 1. Let X be the complex projective space $\mathbb{P}_n$. Minimal rational curves on X are just lines on the projective space. For any point $x \in X$ and any tangent direction

$\alpha \in \mathbb{P}T_x(X)$, there exists a line through x in the direction of α. Thus $\mathcal{C}_x = \mathbb{P}T_x(X)$ and $\mathcal{C} = \mathbb{P}T(X)$.

Example 2. Let X be the n-dimensional hyperquadric $\boldsymbol{Q}_n$ in $\mathbb{P}_{n+1}$. Minimal rational curves on X are just lines of $\mathbb{P}_{n+1}$ which lie on $\boldsymbol{Q}_n$. For a given point $x \in X$, the tangent directions to lines through x lying on $\boldsymbol{Q}_n$ defines a hyperquadric $\mathcal{C}_x \cong \boldsymbol{Q}_{n-2}$ in the projective space $\mathbb{P}T_x(X) \cong \mathbb{P}_{n-1}$.

Example 3. Let X be the Grassmannian $\mathbb{G}(u, v)$ of u-dimensional subspaces of a $(u+v)$-dimensional complex vector space V. There are two universal quotient bundles $\mathcal{U}$ of rank u and $\mathcal{V}$ of rank v such that $T(X) \cong \mathcal{U} \otimes \mathcal{V}$. There is a natural embedding of X into $\mathbb{P}\Lambda^u V$ called the Plücker embedding. Minimal rational curves on X are just lines of $\mathbb{P}\Lambda^u V$ lying on X. It is easy to check that for each $x \in X$, the variety of minimal rational tangents at x, $\mathcal{C}_x \subset \mathbb{P}T_x(X)$, is isomorphic to the set of pure tensors in $\mathbb{P}T_x(X) \cong \mathbb{P}(\mathcal{U}_x \otimes \mathcal{V}_x)$. In other words, $\mathcal{C}_x$ is isomorphic to the Segre embedding of $\mathbb{P}_{u-1} \times \mathbb{P}_{v-1}$.

Now we confront the second part of Mok's argument, which is more subtle: how to use the variety of minimal rational tangents to characterize rational homogeneous spaces. What should be the algebro-geometric analog of Riemannian holonomy and Berger's theorem? Since it is hard to answer this directly, let us approach from a more general setting. Riemannian holonomy is one special part of the general theory of geometric structures in differential geometry. In a broad sense, we can say that a geometric structure is given on a manifold X, if some extra conditions are imposed on the tangent bundle $T(X)$ or its associated fiber bundles. The existence of a distinguished subvariety $\mathcal{C}$ of the projectivized tangent bundle $\mathbb{P}T(X)$ of a complex manifold X can be regarded as a geometric structure on X. For example, the hyperquadric in Example 2 has the structure given by the hypersurface $\mathcal{C} \subset \mathbb{P}T(X)$ which defines a nondegenerate quadratic form on $T_x(X)$ up to a scalar. Such a structure is called a conformal structure. The Grassmannian in Example 3 has the variety of minimal rational tangents isomorphic to the Segre embedding. Such a geometric structure has been studied by differential geometers in connection with twistor theory.

Now that we have a geometric structure arising from minimal rational curves on any Fano manifold, how can we use them to characterize rational homogeneous spaces? A natural approach is by the equivalence problem for such geometric structures, which is the basic problem of E. Cartan's approach to differential geometry. The equivalence problem for the geometric structure is usually a local question. One may wonder whether such a local differential geometric study will be useful in dealing with problems formulated in algebraic geometry. More precisely, let us assume that there exists an analytic open subset $U \subset X$ such that the structure $\mathcal{C}|_U$ is equivalent to that of an analytic open subset of a rational homogeneous space S. What algebro-geometric consequence would this give? For example, can we say that X is biholomorphic to S? The answer is plainly no. A counterexample can be given just by setting X to be the blow-up of S along a subvariety. This example is certainly uninteresting. There are a number of ways to avoid these inessential problems. One simple way is just to restrict our discussion to complex manifolds of second Betti number 1. This may look like an over-simplification, but there is still a lot to say even under this restriction. From now on let us assume that our complex manifolds have $b_2 = 1$. In this case, we have the following result which shows that the approach via local equivalence of geometric structures is promising.

Theorem 2.1. *Let X be a Fano manifold of $b_2 = 1$ and $\mathcal{C} \subset \mathbb{P}T(X)$ be the variety of minimal rational tangents associated to a family of minimal rational curves. Let S be a rational homogeneous space of $b_2 = 1$ different from $\mathbb{P}_n$ and $\mathcal{C}' \subset \mathbb{P}T(S)$ be the variety of minimal rational tangents. Suppose there exist connected analytic open sets $U \subset X$ and $U' \subset S$ with a biholomorphic map $\varphi\colon U \to U'$ such that its differential $d\varphi\colon \mathbb{P}T(U) \to \mathbb{P}T(U')$ sends $\mathcal{C}|_U$ onto $\mathcal{C}'|_{U'}$. Then φ can be extended to a biholomorphic map $X \cong S$.*

This is a special case of [15], where a similar result was proved for a large class of Fano manifolds including rational homogeneous spaces. The main idea of proof in [15] is the extension of the biholomorphic map φ outside U by an analytic continuation 'along the minimal rational curves'. This extension is possible over the whole manifold X because X is rationally connected by minimal rational curves from $b_2 = 1$. One difficulty of the problem is to prove the univalence of the analytic continuation, where the simply connectedness of X is used crucially.

By Theorem 2.1, the problem of recognizing rational homogeneous spaces of $b_2 = 1$ is reduced to the local equivalence question for the geometric structure defined by the variety of minimal rational tangents. So the question is how to show the existence of the local equivalence map φ. One necessary condition is that for each general point $x \in X$, the variety of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ must be isomorphic to the variety of minimal rational tangents $\mathcal{C}'_s \subset \mathbb{P}T_s(S)$ at a base point $s \in S$. In fact, it is expected that this is a sufficient condition:

Conjecture 2.2. Let S be a rational homogeneous space of $b_2 = 1$ and $\mathcal{C}'_s \subset \mathbb{P}T_s(S)$ be the variety of minimal rational tangents at a base point $s \in S$. Let X be a Fano manifold of $b_2 = 1$ and $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ be the variety of minimal rational tangents at a general point $x \in X$ associated to a family of minimal rational curves. Suppose that $\mathcal{C}'_s \subset \mathbb{P}T_s(S)$ and $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ are isomorphic as projective subvarieties. Then X is biholomorphic to S.

This conjecture is known to be true for a number of cases. When S is the projective space, it follows from the following result proved in [4]:

Theorem 2.3. *Let X be a Fano manifold whose variety of minimal rational tangents at a general point is the whole $\mathbb{P}T_x(X)$. Then X is the projective space.*

The proof in [4] is by a careful study of the singularity of minimal rational curves. When S is the hyperquadric of dimension ≥ 3, Conjecture 2.2 follows from Miyaoka's result [22]. In fact, Miyaoka proved the following stronger result.

Theorem 2.4. *Let X be a Fano manifold of $b_2 = 1$ whose variety of minimal rational tangents at a general point x is a hypersurface in $\mathbb{P}T_x(X)$. Then X is the hyperquadric.*

One special feature of Miyaoka's proof is that it requires a study of rational curves which are not minimal. It would be very interesting if one can prove Theorem 2.4 using only minimal rational curves. Conjecture 2.2 for hyperquadrics can be proved using only minimal rational curves as we will see shortly.

Methods used in [4] or [22] for projective spaces and hyperquadrics are rather special, unrelated to ideas of E. Cartan's equivalence problems. These methods cannot be applied to other rational homogeneous spaces to solve Conjecture 2.2. For general rational homogeneous spaces, the most promising approach seems to be via the techniques used in the equivalence problem for the geometric structure. The assumption in Conjecture 2.2 means that a geometric structure modelled on that of S is given on a Zariski open subset of X. By Theorem 2.1, the question is whether this structure is isomorphic to the standard one on S. There is a procedure in Cartan's theory to solve such problems. When a geometric structure is given, Cartan's procedure gives certain curvature tensors on some principal bundle whose vanishing will guarantee that the structure is locally isomorphic to the flat model of the given geometric structure. In fact, certain geometric structures modelled on rational homogeneous spaces have been studied in differential geometry ever since Cartan and the curvature tensors have been computed (e.g. [7], [30]). However, the model geometric structures on rational homogeneous spaces studied in these works do not necessarily agree with our geometric structure arising from the variety of minimal rational tangents. Recall that for each rational homogeneous space S of $b_2 = 1$, there is an associated simple root of the Lie algebra of automorphisms of S. When the rational homogeneous space S is associated to a long simple root, the two geometric structures coincide, as proved in [16]. These include Hermitian symmetric spaces and homogeneous contact manifolds, the latter meaning rational homogeneous spaces of $b_2 = 1$ where the isotropy representation on the tangent space has an irreducible subspace of codimension 1.

Thus at least for S associated to a long simple root, we have certain computational tools to check the validity of Conjecture 2.2: there are certain curvature tensors defined on the Zariski open subset of X at each point of which the variety of minimal rational tangents is isomorphic to the variety of minimal rational tangents at a point of S and the main problem is to show the vanishing of these curvature tensors. In some cases, these curvature tensors vanish for a purely local algebraic reason as proved in [31]. But in many cases, including the cases of Hermitian symmetric spaces and homogeneous contact manifolds, proving the vanishing of these curvature tensors requires a delicate geometric argument using minimal rational curves. One approach is to proceed in the following two steps. The first step is to prove the vanishing of the curvature assuming that the geometric structure modelled on S is defined on the whole X, in other words, that $\mathcal{C}_x$ is isomorphic to $\mathcal{C}'_s$ for each $x \in X$. Under this assumption, one studies the behavior of the curvature tensor along minimal rational curves to conclude the vanishing. This has been done for Hermitian symmetric spaces in [12] and for homogeneous contact manifolds in [8]. The second step is to prove that the geometric structure modelled on S which is defined on a Zariski open subset of X extends to the whole X. This extension can be proved by looking at the local projective geometric invariants of the variety of minimal rational tangents $\mathcal{C}_x$ as x varies along a minimal rational curve, an idea introduced in [24]. In this manner, Conjecture 2.2 is proved for Hermitian symmetric spaces and homogeneous contact manifolds in [25]. It may be possible to extend these arguments further to prove Conjecture 2.2 when the rational homogeneous space is associated to a long simple root. However, Conjecture 2.2 for S associated to a short simple root looks much harder, and remains a challenge for future research.

Now that we know Conjecture 2.2 can be proved in many cases, how can we use this in the rigidity problems? The main question is how to get the assumption in Conjecture 2.2 from the conditions imposed on X in various rigidity questions. This depends on the individual rigidity problem and is often a difficult problem. We will see two examples below.

3. Deformation rigidity of rational homogeneous spaces

It is expected that the following generalization of Conjecture 3.5 in [28] holds. We will call it the deformation rigidity problem for rational homogeneous spaces.

Conjecture 3.1. Let $\pi : M \to \Delta$ be a smooth family of Fano manifolds such that the fiber M_t for each $t \neq 0$ is biholomorphic to a rational homogeneous space S. Then the central fiber M_0 is also biholomorphic to S.

More precisely speaking, this is the global deformation rigidity problem. It is well-known that the local deformation rigidity of rational homogeneous spaces follows from the vanishing $H^1(S, T(S)) = 0$, which is a consequence of Borel–Weil–Bott theorem.

Note that the Fano condition is necessary. In fact, the rational homogeneous space $S = \mathbb{P}_1 \times \mathbb{P}_1$ can be deformed to a non-Fano Hirzebruch surface. In this example, the automorphism group of S is not simple. One can also give examples whose automorphism groups are simple. For example, $S = \mathbb{P}T(\mathbb{P}_{2m+1})$ can be deformed to a non-Fano manifold of the form $\mathbb{P}(D \oplus L)$ where D is the null-correlation bundle and L is a line bundle on the odd-dimensional projective space $\mathbb{P}_{2m+1}$. In all these examples, $b_2(S) > 1$.

When $b_2(S) = 1$, the Fano condition is equivalent to the assumption that the central fiber is Kähler. In this case, it is expected that the following stronger rigidity holds.

Conjecture 3.2. Let $\pi : M \to \Delta$ be a smooth family of compact complex manifolds such that the fiber M_t for each $t \neq 0$ is biholomorphic to a rational homogeneous space S of $b_2 = 1$. Then the central fiber M_0 is also biholomorphic to S.

This is a generalization of Conjecture 3.4 in [28]. Conjecture 3.2 is proved for the projective space in [27] and for the hyperquadric in [9]. In these works, the main problem is to prove the existence of certain rational curves on M_0, which can play the role of minimal rational curves of Fano manifolds. Thus the nature of the problem is quite different from what we have discussed in Section 2. At the moment, it looks very hard to generalize the methods of [9] and [27] to other homogeneous spaces. For the other cases, the weaker conjecture, Conjecture 3.1, is already very interesting.

Regarding Conjecture 3.1, little work has been done when $b_2(S) > 1$. Conjecture 3.1 when $b_2(S) = 1$ has been proved in a series of works [10], [13] [16], [17], [19]. For the rest of the section, we will sketch the main ideas in this proof.

Taking the approach to rigidity problems discussed in Section 2, we see that one central problem is the following. Let $x \in M_0$ be a general point. Is the variety of minimal rational tangents at x, $\mathcal{C}_x \subset \mathbb{P}T_x(M_0)$, isomorphic to that of S?

To handle this question, we proceed as follows. Let us take a section $\{x_t, t \in \Delta\}$ of π such that $x_0 = x$. We compare the family of tangent morphisms

$$\tau_{x_t}\colon \mathcal{K}_{x_t} \to \mathcal{C}_{x_t} \subset \mathbb{P}T_{x_t}(M_t)$$

with the model

$$\tau_s\colon \mathcal{K}_s \to \mathcal{C}_s \subset \mathbb{P}T_s(S).$$

By the assumption τ_{x_t} is isomorphic to τ_s when $t \neq 0$. In particular, $\{\mathcal{K}_{x_t}, t \in \Delta\}$ is a smooth family of projective manifolds such that $\mathcal{K}_{x_t}$ is biholomorphic to $\mathcal{K}_s$ for $t \neq 0$. The first question to ask is whether $\mathcal{K}_{x_0}$ is biholomorphic to $\mathcal{K}_s$. This itself is a deformation rigidity problem. For many cases of S, $\mathcal{K}_s$ itself is a rational homogeneous space. In general, $\mathcal{K}_s$ is a variety very close to a rational homogeneous space. This indicates the possibility of using an induction on the dimension to solve Conjecture 3.1. Indeed, using additional information coming from the fact that $\mathcal{K}_{x_0}$ is the space of minimal rational curves through x_0, we can carry out this induction argument to show that $\mathcal{K}_{x_0}$ is biholomorphic to $\mathcal{K}_s$.

Now the problem is reduced to the study of the linear system defining τ_{x_t}. For the model S, τ_s is defined by a complete linear system when we view it as a morphism into the linear span of $\mathcal{C}_s$. Thus to show that τ_{x_0} is isomorphic to τ_s, it suffices to show that the linear span of $\mathcal{C}_{x_0}$ has the same dimension as that of $\mathcal{C}_s$. This leads to the study of the linear span of the variety of minimal rational tangents.

To crystalize the essential idea here, it is better to deal with a general Fano manifold X of $b_2 = 1$. Assume that we have a choice of the variety of minimal rational tangents $\mathcal{C} \subset \mathbb{P}T(X)$ such that for a general $x \in X$, $\mathcal{C}_x$ is irreducible. Let $D_x \subset T_x(X)$ be the linear span of $\mathcal{C}_x$. As x varies, D_x defines a Pfaffian system D on a Zariski open set of X. From the topological restriction $b_2(X) = 1$, one can show that D cannot be integrable. An essential property of D, which is one of the key points of [13], is that the Frobenius bracket at a general point

$$[\,,\,]\colon \Lambda^2 D_x \to T_x(X)/D_x$$

annihilates planes in D_x corresponding to tangent lines to the variety of minimal rational tangents $\mathcal{C}_x$. This follows from deformation theoretic properties of minimal rational curves.

Applying this general property of D to the variety of minimal rational tangents on M_0, one can show that if the linear span of $\mathcal{C}_{x_0}$ has dimension different from that of $\mathcal{C}_s$, then the Pfaffian system generated by $\mathcal{C}_{x_0}$ must be integrable, a contradiction to $b_2(M_0) = 1$. Thus we conclude that τ_{x_0} is isomorphic to τ_s.

From the discussion in Section 2, this already shows Conjecture 3.1 for Hermitian symmetric spaces and homogeneous contact manifolds. Also, when S is associated to a long simple root, we know, from Section 2, that there are some curvature tensors defined in a neighborhood of x_0 whose vanishing will guarantee that M_0 is biholomorphic to S. Since M_t has the same geometric structure modelled on S, the curvature tensor at x_0 is just the limit of the curvature tensor at x_t. As M_t, $t \neq 0$, is biholomorphic to S, the curvature tensor at x_t, $t \neq 0$, vanishes. By the continuity, the curvature tensor also vanishes at x_0. This completes the proof of Conjecture 3.1 when S is associated to a long simple root.

When S is associated to a short root, this argument does not work. In a sense, the difficulty lies in local differential geometry: the geometric structure given by the variety of minimal rational tangents has only part of the information needed for the geometric structure modelled on S studied by differential geometers. Thus we have only a 'crude' geometric structure, and in terms of this crude structure, S is not quite flat. One can still hope that there are certain structure constants involved and the constancy of the structure constants for $t \neq 0$ would imply that they remain unchanged at $t = 0$. Unfortunately, this idea can be worked out only in one special case [17].

For the final handling of Conjecture 3.1 for S associated to a short root, we do not deal with the equivalence problem directly. But instead we use the variety of minimal rational tangents to control the automorphism group of M_0. In a sense, instead of showing directly that the geometric structure defined by the variety of minimal rational tangents on M_0 is locally equivalent to that of S, we study the local automorphisms of the geometric structure to show that the group of automorphisms of M_0 is isomorphic to that of S. This suffices to conclude that M_0 is biholomorphic to S, in the setting of the deformation problem.

The study of the local automorphisms of a geometric structure leads to the theory of prolongations of linear Lie algebras (cf. [3], [21]). This theory is essentially algebraic and many works have been done in the context of the theory of filtered Lie algebras. Unfortunately, many results in this area requires that the linear Lie algebra involved is reductive, while the linear Lie algebra of the infinitesimal automorphisms of the variety of minimal rational tangents of a rational homogeneous space associated to a short root is not reductive. Thus from the view-point of Lie algebras, the variety of minimal rational tangents $\mathcal{C}_s$ is not really a nice object. But from the view-point of projective algebraic geometry, $\mathcal{C}_s$ is very nice: it is smooth and linearly normal, among other things. This motivates us to develop the theory of prolongations of linear automorphisms of such nice projective varieties, using projective geometry instead of Lie algebra. In this theory, one makes use of the rich results on projective geometry such as [32] to replace the computational tool of semi-simple Lie algebras. This theory is developed in [19] to the extent needed for the proof of Conjecture 3.1 for S of $b_2 = 1$. As a byproduct, one can give new geometric proofs of many results on prolongations of reductive linear Lie algebras, too.

Readers must have noticed that our proof of Conjecture 3.1 for $b_2(S) = 1$ is not very uniform. A proof of Conjecture 2.2, or a further development of the prolongation theory of [19] would give a more uniform proof of Conjecture 3.1.

Conjecture 3.1 for the case of $b_2(S) > 1$ is wide open, except for some special cases. To attack this general situation, we would need to develop the theory of the variety of minimal rational tangents for Fano manifolds with $b_2 > 1$. Fano manifolds with $b_2 > 1$ admit non-trivial Mori contractions, which are crucial in understanding their geometry. Thus the geometry of the variety of minimal rational tangents has to be combined with Mori theory. This will be an interesting direction for future research.

4. The Campana–Peternell conjecture

The generalized Frankel conjecture which was proved by Mok in [23] states that a Fano manifold with a Kähler metric with nonnegative holomorphic bisectional curvature is a Hermitian symmetric space of compact type. In [1], Campana and Peternell proposed an algebraic generalization of this. Here we will discuss the following slightly stronger form of their conjecture:

Conjecture 4.1. A Fano manifold X is homogeneous if all rational curves on X are free.

A rational curve $\nu\colon \mathbb{P}_1 \to X$ is free if $\nu^* T(X)$ is a semi-positive vector bundle. There are many reasons to believe that Conjecture 4.1 is one of the central problems of the uniformization theory in several complex variables. In [5], it is proved that Conjecture 4.1 would give a complete description of Kähler manifolds with semi-positive curvature in a most reasonable sense.

One can check that Conjecture 4.1 is true for $\dim X \leq 3$ from the classification of Fano threefolds (cf. [1], [33]). However, in higher dimensions, very few results on Conjecture 4.1 are known. In the rest of the paper we will give a proof of the following which illustrates how the variety of minimal rational tangents can be used in rigidity problems.

Theorem 4.2. *Conjecture* 4.1 *is true in dimension* 4.

The proof uses the works of [2], [4], [22] and [24]. First of all, using Mori theory, [2] showed that Theorem 4.2 is true for Fano 4-folds with $b_2 > 1$. Thus we may assume that $b_2(X) = 1$. Pick a family of minimal rational curves on X and consider the variety of minimal rational tangents $\mathcal{C}_x$ at a general point $x \in X$. Suppose $\dim \mathcal{C}_x = 3$. Then $X = \mathbb{P}_4$ by Theorem 2.3. Suppose $\dim \mathcal{C}_x = 2$. Then $X = \boldsymbol{Q}_3$ by Theorem 2.4. If $\dim \mathcal{C}_x = 0$, the freeness of all minimal rational curves implies $\mathcal{C}$ is an etale cover of X. Since a Fano manifold is simply connected, this shows that $\mathcal{C}$ is biholomorphic to X by the natural projection. This will give a regular foliation of X by minimal rational curves, a contradiction to $b_2(X) = 1$. Thus we are left with the case of $\dim \mathcal{C}_x = 1$. To handle this case, we use the following result of Mok from [24].

Theorem 4.3. *Let X be a Fano manifold with $b_2(X) = 1$. Assume that all rational curves on X are free and* $\dim \mathcal{C}_x = 1$ *for a general point $x \in X$. Assume in addition that $b_4(X) = 1$. Then X is homogeneous.*

Let us give a brief sketch of Mok's proof. By the freeness of rational curves, the space $\mathcal{K}$ of minimal rational curves is a projective manifold and the associated universal family morphisms $\rho\colon \mathcal{U} \to \mathcal{K}$, $\mu\colon \mathcal{U} \to X$ are smooth morphisms whose fibers are curves. By definition, ρ is a $\mathbb{P}_1$-bundle. If the fibers of μ have genus ≥ 1, μ must be a trivial fiber bundle over minimal rational curves on X. This contradicts some basic geometric feature of the morphism μ. Thus μ is a $\mathbb{P}_1$-bundle and the variety of minimal rational tangents at each $x \in X$ is a rational curve. The key point of [24] is to show that this rational curve $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ has degree $d \leq 3$. Using this bound, one can see that the geometric structure defined by $\mathcal{C}$ is isomorphic to the one modelled on $\mathbb{P}_2$, $\boldsymbol{Q}_3$, or the 5-dimensional homogeneous contact manifold. Thus by the discussion in Section 2, X must be the model.

To get the bound on the degree d of $\mathcal{C}_x \subset \mathbb{P}T_x(X)$, Mok observed that there exists a stable vector bundle of rank 2 on $\mathcal{K}$ whose projectivization is $\rho\colon \mathcal{U} \to \mathcal{K}$. The crucial part of Mok's work is to show that the Chern number inequality for this stable bundle on $\mathcal{K}$ gives the bound $d \leq 3$. The additional assumption $b_4(X) = 1$ was used in this step. $b_4(X) = 1$ implies $b_4(\mathcal{K}) = 1$, which makes it possible to translate the Chern number inequality to the inequality $d \leq 3$.

In order to use Theorem 4.3 to complete the proof of Theorem 4.2, it suffices to show that the additional assumption $b_4(X) = 1$ in Theorem 4.3 can be removed. Since this part has not appeared in print, we will give full details.

First recall from [6] that the i-th Chow group $A_i(Z)$ of a variety Z is the abelian group of algebraic cycles of dimension i on Z modulo the rational equivalence. The i-th rational Chow group $A_i(Z)_{\mathbb{Q}}$ is the tensor product of $A_i(Z)$ with $\mathbb{Q}$. As mentioned above, the condition $b_4(X) = 1$ in the proof of Theorem 4.3 was used to get $b_4(\mathcal{K}) = 1$, which was necessary for the Chern class computation to get $d \leq 3$. Since Chern classes can be defined as algebraic cycles, it suffices to have $A_2(\mathcal{K})_{\mathbb{Q}} \cong \mathbb{Q}$ to conclude $d \leq 3$.

To prove $A_2(\mathcal{K})_{\mathbb{Q}} \cong \mathbb{Q}$ in the setting of Theorem 4.3 (without the assumption $b_4(X) = 1$), we use the double $\mathbb{P}_1$-bundle structure on $\mathcal{U}$ given by ρ and μ. We define, inductively, a sequence of smooth irreducible projective varieties Z_i, $i = 1, 2, 3 \ldots,$

of dimension i together with a morphism $\nu_i \colon Z_{i+1} \to Z_i$ and a morphism $\eta_i \colon Z_i \to \mathcal{U}$ as follows. Fix a general point $x \in X$. Let $Z_1 := \mu^{-1}(x)$ and $\eta_1 \colon Z_1 \to \mathcal{U}$ be the natural injection. Let Z_2 be the $\mathbb{P}_1$-bundle on Z_1 obtained as the pull-back of ρ by the morphism $\rho \circ \eta_1 \colon Z_1 \to \mathcal{K}$. Denote the bundle map $Z_2 \to Z_1$ by ν_1 and the natural map $Z_2 \to \mathcal{U}$ by η_2. Note that ν_1 has a natural section defined by η_1. Now let Z_3 be the $\mathbb{P}_1$-bundle over Z_2 obtained as the pull-back of μ by the morphism $\mu \circ \eta_2$. Denote the bundle map by $\nu_2 \colon Z_3 \to Z_2$ and the natural map $Z_3 \to \mathcal{U}$ by η_3. Then ν_2 has a natural section defined by η_2. Continuing in this manner, we define Z_{i+1} to be the $\mathbb{P}_1$-bundle over Z_i obtained as the pull-back of ρ or μ by $\rho \circ \eta_i$ or $\mu \circ \eta_i$ depending on whether i is odd or even. Denote the bundle map by $\nu_i \colon Z_{i+1} \to Z_i$ and the natural map to $\mathcal{U}$ by η_{i+1}. Then ν_i has a section defined by η_i.

Now we will apply the following lemma which is a simple consequence of Theorem 3.3 in [6].

Lemma 4.4. *Let $p \colon Z' \to Z$ be a $\mathbb{P}_1$-bundle with a section $\sigma \colon Z \to Z'$. Then any $\beta \in A_k(Z')$ is of the form $\beta = \sigma_*\alpha + p^*\gamma$ for some $\alpha \in A_k(Z)$ and $\gamma \in A_{k-1}(Z)$.*

Note that since X, $\mathcal{K}$ and Z_i are all rationally connected, $A_0(X)_{\mathbb{Q}} \cong A_0(\mathcal{K})_{\mathbb{Q}} \cong A_0(Z_i)_{\mathbb{Q}} \cong \mathbb{Q}$. Repeatedly applying Lemma 4.4, we see that $A_1(Z_i)$ is generated by curves which are sent by η_i to either a ρ-fiber or a μ-fiber in $\mathcal{U}$. Similarly, $A_2(Z_i)$ is generated by surfaces whose images in $\mathcal{U}$ under η_i are either a curve or surfaces of the form $\mu^{-1}(\mu(C))$ for some ρ-fiber C or $\rho^{-1}(\rho(C'))$ for some μ-fiber C'. It follows that the rank of the image of the push-forward

$$(\rho \circ \eta_i)_* \colon A_2(Z_i)_{\mathbb{Q}} \to A_2(\mathcal{K})_{\mathbb{Q}}$$

is ≤ 1. From $b_2(X) = 1$, there exists some ℓ such that $\eta_\ell \colon Z_\ell \to \mathcal{U}$ is surjective. Recall that when $\psi \colon Z \to Y$ is a proper surjective morphism of algebraic varieties, the induced push-forward map $\psi_* \colon A_i(Z)_{\mathbb{Q}} \to A_i(Y)_{\mathbb{Q}}$ is surjective. Thus

$$(\rho \circ \eta_\ell)_* \colon A_2(Z_\ell)_{\mathbb{Q}} \to A_2(\mathcal{K})_{\mathbb{Q}}$$

is surjective. It follows that $A_2(\mathcal{K})_{\mathbb{Q}} \cong \mathbb{Q}$. This finishes the proof of Theorem 4.2.

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