Manifolds with positive curvature operators are space forms

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Abstract. We show that the normalized Ricci flow evolves metrics with positive curvature operator on compact manifolds to limit metrics of constant positive sectional curvature. In this note we only indicate the proof, the details will be published somewhere else.

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1. Introduction

The Ricci flow has been introduced by Hamilton in 1982 [H1] in order to prove that a compact 3-manifold admitting a Riemannian metric of positive Ricci curvature is a spherical space form. The unnormalized Ricci flow is the geometric evolution equation

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g) \tag{1}$$

for a curve g_t of Riemannian metrics on a compact manifold M^n.

Using moving orthonormal frames, this leads to the following evolution equation for the curvature operator $\mathrm{R}_t\colon \Lambda^2 T_pM \to \Lambda^2 T_pM \cong \mathfrak{so}(T_pM)$ of g_t (cf. [H2]):

$$\frac{\partial\,\mathrm{R}}{\partial t} = \Delta\,\mathrm{R} + 2(\mathrm{R}^2 + \mathrm{R}^{\#}), \tag{2}$$

where $\mathrm{R}^{\#} = \mathrm{ad} \circ (\mathrm{R} \wedge \mathrm{R}) \circ \mathrm{ad}^*$ and $\mathrm{ad}\colon \Lambda^2(\mathfrak{so}(T_pM)) \to \mathfrak{so}(T_pM)$ is the adjoint representation. For the details we refer the reader to Section 2. By Hamilton's maximum principle certain dynamical properties of the partial differential equation (2) can be derived from the dynamical properties of the corresponding ordinary differential equation

$$\frac{d\,\mathrm{R}}{dt} = \mathrm{R}^2 + \mathrm{R}^{\#}. \tag{3}$$

In dimension four Hamilton showed that compact 4-manifolds with positive curvature operators are spherical space forms as well [H2]. More generally, Chen showed that the same conclusion holds for compact 4-manifolds with 2-positive curvature

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operator [Che]. Recall that a curvature operator is called 2-positive, if the sum of its two smallest eigenvalues is positive. In arbitrary dimensions it was shown by Huisken [Hu], that there is an explicit open cone in the space of curvature operators such that the normalized Ricci flow evolves metrics whose curvature operator at each point is contained in that cone into metrics of constant positive sectional curvature.

All of these results are also based on the maximum principle. The main reason why they are more involved is that the ordinary differential equation (3) is not that well understood in dimensions $n > 3$ and in particular not for $n > 4$.

Hamilton conjectured that in all dimensions compact Riemannian manifolds with positive curvature operators must be space forms. We can confirm Hamilton's conjecture. More generally, we can show the following

Theorem 1. *Let (M, g) be a compact Riemannian manifold with 2-positive curvature operator. Then the normalized Ricci flow evolves g to a limit metric with constant positive sectional curvature.*

The theorem is known in dimensions below five and our proof only works in dimensions above two. Since the proof solely relies on Hamilton's maximum principle it carries over to orbifolds.

Let us mention that this is no longer true in dimension two. By the work of Hamilton [H4] and Chow [Cho] it is known that the normalized Ricci flow converges to a metric of constant curvature for any initial metric in the manifold case. However, there exist two dimensional orbifolds with positive sectional curvature which are not covered by a manifold. On such orbifolds the Ricci flow converges to a nontrivial Ricci soliton [CW].

There is a wealth of different techniques in geometry to prove sphere theorems. Here we only mention the theorem of Micallef and Moore [MM] that a simply connected manifold with positive isotropic curvature is a homotopy sphere. It is well known that a 2-positive curvature operator has positive isotropic curvature. However, the techniques of Micallef and Moore do not allow to get restrictions for the fundamental groups or the differentiable structure.

Let us turn to the proof of Theorem 1. The major obstacle is to understand the ordinary differential equation (3). Here we establish a new algebraic identity which should be useful in other context as well. We study how the differential equation changes if we pull it back by an equivariant linear map $\mathrm{l}\colon S^2_B(\mathfrak{so}(n)) \to S^2_B(\mathfrak{so}(n))$, where $S^2_B(\mathfrak{so}(n))$ denotes the space of curvature operators satisfying the Bianchi identity. For $n \geq 4$ this space decomposes into three pairwise inequivalent $\mathrm{O}(n)$-invariant irreducible subspaces

$$S^2_B(\mathfrak{so}(n)) = \langle I \rangle \oplus \langle \mathrm{Ric}_0 \rangle \oplus \langle \mathrm{W} \rangle.$$

Here $\langle I \rangle$ denotes multiples of the identity, $\langle \mathrm{W} \rangle$ curvature operators with vanishing Ricci curvature and $\langle \mathrm{Ric}_0 \rangle$ are the curvature operators of traceless Ricci type. Given such a curvature operator R we let R_I, $\mathrm{R}_{\mathrm{Ric}_0}$ and R_{W}, denote the projections onto $\langle I \rangle$, $\langle \mathrm{Ric}_0 \rangle$ and $\langle \mathrm{W} \rangle$, respectively.

Since we can always scale the linear map l by a factor it is not much of a restriction to consider maps of the form

$$\mathrm{l}_{a,b}(\mathrm{R}) = \mathrm{R} + 2(n-1)a\,\mathrm{R}_I + (n-2)b\,\mathrm{R}_{\mathrm{Ric}_0}\,. \tag{4}$$

Notice that $\mathrm{l}_{a,b}$ induces the identity on the space $\langle \mathrm{W}\rangle$ of Weyl curvature operators.

The following result is crucial.

Theorem 2. *Let* $\mathrm{R} \in S^2_B(\mathfrak{so}(n))$ *be a curvature operator,* $\mathrm{Ric} \in S^2\mathbb{R}^n$ *its Ricci curvature,* Ric_0 *the traceless part of* Ric*, and let*

$$D_{a,b} = \mathrm{l}_{a,b}^{-1}\big((\mathrm{l}_{a,b}\,\mathrm{R})^2 + (\mathrm{l}_{a,b}\,\mathrm{R})^{\#}\big) - \mathrm{R}^2 - \mathrm{R}^{\#}\,.$$

Then

$$\begin{aligned} D_{a,b} = {} & \big((n-2)b^2 - 2(a-b)\big)\,\mathrm{Ric}_0 \wedge \mathrm{Ric}_0 + 2a\,\mathrm{Ric}\wedge\mathrm{Ric} + 2b^2\,\mathrm{Ric}_0^2 \wedge \mathrm{id} \\ & + \frac{\mathrm{tr}(\mathrm{Ric}_0^2)}{n + 2n(n-1)a}\big(nb^2(1-2b) - 2(a-b)(1-2b+nb^2)\big)\, I. \end{aligned}$$

In particular, $D_{a,b}$ *is independent of the Weyl curvature of* R.

The theorem often allows to construct new invariant curvature conditions by considering the image of a known invariant curvature conditions under the linear map $\mathrm{l}_{a,b}$ for suitable constants $a, b \in \mathbb{R}$. Recall that an invariant curvature condition is a convex subset of $S_B(\mathfrak{so}(n))$ which is invariant under the ordinary differential equation (3) and hence, by Hamilton's maximum principle, invariant under the partial differential equation (2).

As already mentioned this note only contains indications of proofs. The details will be published somewhere else. We expect that the new algebraic identity on curvature operators (Theorem 2) and its Kähler analogue should give rise to further applications. This will be the subject of a forthcoming paper.

2. Preliminaries

For a Euclidean vector space V we let $\Lambda^2 V$ denote the exterior product of V. We endow $\Lambda^2 V$ with its natural scalar product; if $e_1, \dots, e_n$ is an orthonormal basis of V then $e_1 \wedge e_2, \dots, e_{n-1} \wedge e_n$ is an orthonormal basis of $\Lambda^2 V$. Notice that two linear endomorphisms A, B of V induce a linear map

$$A \wedge B \colon \Lambda^2 V \to \Lambda^2 V, \quad v \wedge w \mapsto \tfrac{1}{2}\big(A(v)\wedge B(w) + B(v) \wedge A(w)\big).$$

We will identify $\Lambda^2\mathbb{R}^n$ with the Lie algebra $\mathfrak{so}(n)$ by mapping the unit vector $e_i \wedge e_j$ onto the linear map $L(e_i \wedge e_j)$ of rank two which is a rotation with angle $\pi/2$ in the plane spanned by e_i and e_j. Notice that under this identification the scalar product on $\mathfrak{so}(n)$ corresponds to $\langle A, B\rangle = -1/2\,\mathrm{tr}(AB)$.

We let $S^2(\mathfrak{so}(n))$ denote the space of selfadjoint endomorphisms of $\mathfrak{so}(n)$ and $S^2_B(\mathfrak{so}(n))$ the subspace of operators satisfying the Bianchi identity. Recall that $S^2_B(\mathfrak{so}(n))$ is given by the orthogonal complement of $\Lambda^4\mathbb{R}^n$ in $S^2(\mathfrak{so}(n))$. We use the convention that the curvature operator of the round sphere is the identity of $\mathfrak{so}(n)$. This explains the extra factor 2 in the evolution equation (2), cf. [H2].

Hamilton observed that for two elements $\mathrm{S}, \mathrm{R} \in S^2(\mathfrak{so}(n))$ one can define a new element $\mathrm{S} \# \mathrm{R} \in S^2(\mathfrak{so}(n))$. This can be done invariantly by putting

$$\mathrm{S} \# \mathrm{R} := \mathrm{ad} \circ (\mathrm{S} \wedge \mathrm{R}) \circ \mathrm{ad}^*$$

where $\mathrm{ad}\colon \Lambda^2\mathfrak{so}(n) \to \mathfrak{so}(n),\ X \wedge Y \mapsto [X, Y]$ is the adjoint representation and ad^* is its dual. Following Hamilton we use the abbreviation $\mathrm{R}^\# = \mathrm{R} \# \mathrm{R}$. For a generic $\mathrm{R} \in S^2_B(\mathfrak{so}(n))$ neither R^2 nor $\mathrm{R}^\#$ are in $S^2_B(\mathfrak{so}(n))$. However, Hamilton showed that the sum is in $S^2_B(\mathfrak{so}(n))$ and hence the ordinary differential equation (3) leaves $S^2_B(\mathfrak{so}(n))$ invariant.

Recall that an $\mathrm{O}(n)$-invariant subset C in the space of curvature operators is invariant under the Ricci flow, if the Ricci flow evolves metrics on compact manifolds whose curvature operator lies in C at any point into metrics with the same property.

Theorem 2.1 (Hamilton, [H2]). *A closed convex* $\mathrm{O}(n)$*-invariant subset* $C \subset S_B(\mathfrak{so}(n))$ *of curvature operators is invariant under the Ricci flow if it is invariant under the ordinary differential equation*

$$\frac{d\mathrm{R}}{dt} = \mathrm{R}^2 + \mathrm{R}^\# .$$

We recall that if $e_1, \dots, e_n$ denotes an orthonormal basis, then

$$\mathrm{Ric}(\mathrm{R}^2 + \mathrm{R}^\#)_{ij} = \sum_{k,l} \mathrm{Ric}_{kl}\, \mathrm{R}_{kijl} \tag{5}$$

where $\mathrm{R}_{kijl} = \langle \mathrm{R}(e_i \wedge e_k), e_j \wedge e_l \rangle$, see [H1], [H2].

Finally let us mention that we learned from Huisken that the trilinear map

$$\mathrm{tri}(\mathrm{R}_1, \mathrm{R}_2, \mathrm{R}_3) := \mathrm{tr}\big((\mathrm{R}_1\, \mathrm{R}_2 + \mathrm{R}_2\, \mathrm{R}_1 + 2\, \mathrm{R}_1 \# \mathrm{R}_2)\, \mathrm{R}_3\big) \tag{6}$$

is symmetric in all three components $\mathrm{R}_1, \mathrm{R}_2, \mathrm{R}_3 \in S^2(\mathfrak{so}(n))$. Moreover (3) corresponds to the gradient flow of the function $\frac{1}{6}\,\mathrm{tri}(\mathrm{R}, \mathrm{R}, \mathrm{R})$.

3. On the proof of Theorem 2

In this section we show that the difference tensor $D = D_{a,b}$ does not depend on the Weyl curvature of R. The precise formula in Theorem 2 then follows from a calculation. We view D as quadratic form in R. By

$$B(\mathrm{R}, \mathrm{S}) := \tfrac{1}{4}\big(D(\mathrm{R} + \mathrm{S}) - D(\mathrm{R} - \mathrm{S})\big)$$

we get the corresponding bilinear form.

Let $\mathrm{S} = \mathrm{W} \in \langle \mathrm{W} \rangle$. We have to show $B(\mathrm{R}, \mathrm{W}) = 0$ for all $\mathrm{R} \in S^2_B(\mathfrak{so}(n))$. We start by considering $\mathrm{R} \in \langle \mathrm{W} \rangle$. Then $l_{a,b}(\mathrm{R} \pm \mathrm{W}) = \mathrm{R} \pm \mathrm{W}$. It follows from the formula (5) for the Ricci curvature of $\mathrm{R}^2 + \mathrm{R}^\#$ that $(\mathrm{R} \pm \mathrm{W})^2 + (\mathrm{R} \pm \mathrm{W})^\#$ has vanishing Ricci tensor. Hence $(\mathrm{R} \pm \mathrm{W})^2 + (\mathrm{R} \pm \mathrm{W})^\#$ is a Weyl curvature operator and accordingly fixed by $l^{-1}_{a,b}$. Therefore $B(\mathrm{R}, \mathrm{W}) = 0$.

Next we consider the case that $\mathrm{R} = I$ is the identity. Notice that

$$(I + \mathrm{W})^2 + (I + \mathrm{W})^\# - (I - \mathrm{W})^2 - (I - \mathrm{W})^\# = 4\,\mathrm{W} + 4\,\mathrm{W} \# I = 0.$$

The last equation follows by a straightforward computation for $n = 4$. Since there is a natural embedding of the Weyl tensors of $S^2_B(\mathbb{R}^4)$ to the Weyl tensors of $S^2_B(\mathbb{R}^n)$ the same holds for $n \geq 5$. Clearly the equation implies $B(W, I) = 0$.

It remains to consider the case of $\mathrm{R} \in \langle \mathrm{Ric}_0 \rangle$. Using the symmetry of the trilinear form (6) we see for each $\mathrm{W}_2 \in \langle \mathrm{W} \rangle$ that

$$\mathrm{tri}(\mathrm{W}, \mathrm{R}, \mathrm{W}_2) = \mathrm{tri}(\mathrm{W}, \mathrm{W}_2, \mathrm{R}) = 0$$

because $\mathrm{W}\,\mathrm{W}_2 + \mathrm{W}_2\,\mathrm{W} + 2\,\mathrm{W} \# \mathrm{W}_2$ lies in $\langle \mathrm{W} \rangle$ and $\mathrm{R} \in \langle \mathrm{Ric}_0 \rangle$. Combining this with $\mathrm{tri}(\mathrm{W}, \mathrm{R}, I) = 0$ gives that $\mathrm{W}\,\mathrm{R} + \mathrm{R}\,\mathrm{W} + 2\,\mathrm{W} \# \mathrm{R} \in \langle \mathrm{Ric}_0 \rangle$. Using once more that $l := l_{a,b}$ is the identity on $\langle \mathrm{W} \rangle$ we see that

$$l(\mathrm{W})\,l(\mathrm{R}) + l(\mathrm{R})\,l(\mathrm{W}) + 2\,l(\mathrm{W}) \# l(\mathrm{R}) = l(\mathrm{W}\,\mathrm{R} + \mathrm{R}\,\mathrm{W} + 2\,\mathrm{W} \# \mathrm{R}).$$

This clearly proves $B(\mathrm{W}, \mathrm{R}) = 0$.

4. On the proof of Theorem 1

We call a continuous family $C(t)_{t \in [0,1)} \subset S^2_B(\mathfrak{so}(n))$ of closed convex $\mathrm{O}(n)$-invariant cones of full dimension a pinching family, if

1. each $\mathrm{R} \in C(t) \setminus \{0\}$ has positive scalar curvature,
2. $\mathrm{R}^2 + \mathrm{R}^\#$ is contained in the interior of the tangent cone of $C(t)$ at R for all $\mathrm{R} \in C(t) \setminus \{0\}$ and all $t \in (0, 1)$,
3. $C(t)$ converges in the pointed Hausdorff topology to the one-dimensional cone $\mathbb{R}^+ I$ as $t \to 1$.

Using Theorem 2 we can show that

Theorem 4.1. *There is a pinching family $C(t)_{t \in [0,1)} \subset S^2_B(\mathfrak{so}(n))$ such that $C(0)$ is given by the cone of* 2 *nonnegative curvature operators.*

Outline of the proof of Theorem 4.1. Here we only show that we can stay away from the boundary of the cone C of 2 nonnegative curvature operators.

$$b \in \Big(0, \tfrac{\sqrt{2n(n-2)+4}-2}{n(n-2)}\Big] \quad \text{and} \quad 2a = 2b + (n-2)b^2$$

We claim that then the set $\mathrm{l}_{a,b}(C)$ is invariant under the ordinary differential equation. We have to show that for $\mathrm{R} \in C$ the curvature operator

$$X_{a,b} = \mathrm{l}_{a,b}^{-1}(\mathrm{l}_{a,b}(\mathrm{R})^2 + \mathrm{l}_{a,b}(\mathrm{R})^{\#})$$

is contained in the tangent cone $T_{\mathrm{R}}C$ of C at R. Notice that by assumption we have $\mathrm{R}^2 + \mathrm{R}^{\#} \in T_{\mathrm{R}}C$. Thus it suffices to show that $D_{a,b} = X_{a,b} - \mathrm{R}^2 - \mathrm{R}^{\#}$ lies in $T_{\mathrm{R}}C$. For that it is clearly sufficient to show that $D_{a,b}$ is positive definite. We know that $\mathrm{Ric}(\mathrm{R}) \geq 0$ for any $\mathrm{R} \in C$. Looking at the formula for $D_{a,b}$ in Theorem 2 it suffices to show that

$$0 \leq b^2\big(n(1-2b) - (n-2)(1-2b+nb^2)\big)$$

holds in the given range. This is a straightforward computation.

It is not hard to see that for $b = \frac{\sqrt{2n(n-2)+4}-2}{n(n-2)}$ and $n \geq 4$ the closed cone $\mathrm{l}_{a,b}(C)$ is contained in the open cone of positive curvature operators. Thus it suffices to prove the existence of a pinching family of cones with $C(0)$ being the cone of nonnegative curvature operators. Theorem 2 combined with a calculation, which will be carried out somewhere else, shows that such a family can be constructed in the form

$$C(t) := \mathrm{l}_{a(t),b(t)}\Big(\big\{\mathrm{R} \mid \mathrm{R} \geq 0, \mathrm{Ric}(\mathrm{R}) \geq p(t)\tfrac{\mathrm{scal}(\mathrm{R})}{n}\big\}\Big)$$

for suitable functions $a(t)$, $b(t)$ and $p(t)$. $\square$

Theorem 1 then follows from Theorem 4.1 combined with

Theorem 4.2. *Let $C(t)_{t\in[0,1)} \subset S^2_B(\mathfrak{so}(n))$ be a pinching family of closed convex cones, $n \geq 3$. Suppose that (M, g) is a compact Riemannian manifold such that the curvature operator of M at each point is contained in the interior of $C(0)$. Then the normalized Ricci flow evolves g to a constant curvature limit metric.*

On the proof of Theorem 4.2. Since M is compact and the family of cones is continuous we can assume that for a sufficiently large h_0 and a sufficiently small ε we have for the curvature operator R_p of (M, g) at each point $p \in M$ that

$$\mathrm{R}_p \in \{\mathrm{R} \mid \mathrm{scal} \leq h_0\} \cap C(\varepsilon).$$

We now consider to h_0 and ε the intersection F of all convex subsets which contain the above set and which are invariant under the ordinary differential equation (3). Clearly F is then invariant under the ordinary differential equation and one can show that for each t the set $F \setminus C(t)$ is relatively compact.

Thus F is a generalized pinching set similarly to Hamilton's concept.

From the maximum principle we know that the Ricci flow evolves g to metrics g_t whose curvature operators at each point are contained in F. We do also know that the solution of the Ricci flow exists as long as the curvature does not tend to infinity. Furthermore it follows from the maximum principle that the unnormalized Ricci flow exists only on a finite time interval $t \in [0, t_0)$. By Shi it follows from the maximum principle applied to the evolution equations for the i-th derivatives of the curvature tensor that

$$\max \|\nabla^i R_t\|^2 \leq C_i \max \|R_t\|^{i+2}$$

for all $t \in [t_0/2, t_0)$.

We now rescale each metric $g(t)$ to a metric $\tilde{g}(t)$ such that the maximal sectional curvature is equal to 1. From the above estimates it follows that we have a priori bounds for all derivatives of the curvature tensor of the metric $\tilde{g}(t)$ for $t \in [t_0/2, t_0)$.

We now look at a point $p_t \in (M, \tilde{g}_t)$ where the sectional curvature attains it maximum 1. We pull the metric via the exponential map back to the ball of radius π in $T_{p_t}M$. We identify this ball with the ball $B_\pi(0) \subset \mathbb{R}^n$ by choosing a linear isometry $\mathbb{R}^n \to T_{p_t}M$. We let $\bar{g}_t$ denote the induced metric on $B_\pi(0)$. From the above estimates on the derivatives of the curvature tensor it is clear that for any sequence t_n converging to t_0 there is a subsequence of $\bar{g}_{t_n}$ converging in the C^∞ topology.

The curvature operator at each point of the limit metric is contained in the set $\bigcap \frac{1}{\lambda_n^2} F = \mathbb{R}^+ I$, where λ_n denote the scaling factors which by assumption tend to infinity. Thus the limit metric on $B_\pi(0)$ has pointwise constant sectional curvature and by Schur's theorem it has constant curvature one, where we used $n \geq 3$.

It is now easy to deduce that the minimal sectional curvature of $(M, \tilde{g}_t)$ tends to 1 as well for $t \to t_0$. In particular for t close to t_0 we can apply Klingenberg's injectivity radius estimate for quarter pinched manifolds. Hence the universal cover of M has injectivity radius $\geq \pi$. This shows that the volume of the manifold does not converge to zero. It is then well known that we get a smooth limit space of constant curvature and that (M, g_t) is close in the C^∞-topology to this limit space. □

Finally we remark that the above proof carries over to orbifolds. In addition to Klingenberg's injectivity radius estimate one needs

Proposition 4.3. *Suppose (M, g) is compact orbifold with sectional curvature K. If $1/4 < K \leq 1$ and $\dim(M) \geq 3$ then M is the quotient of a Riemannian manifold by a finite isometric group action.*

The proof of the proposition is not related to the Ricci flow at all. It should rather be viewed as a generalization of Klingenberg's injectivity radius estimate.

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