Elliptic and parabolic problems in conformal geometry

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Abstract. I will review recent results regarding two questions that arise in connection with the Yamabe problem. The first problem is concerned with the conformal deformation of Riemannian metrics by their scalar curvature. This leads to a parabolic evolution equation, which can be interpreted as the flow of steepest descent for the Yamabe functional. I will provide conditions which guarantee that the flow converges to a metric of constant scalar curvature as $t \to \infty$. The second problem is concerned with the set of constant scalar curvature metrics in a given conformal class. I will discuss under what conditions this set is compact. A recurring theme in the study of both problems is that blow-up can be ruled out by means of the positive mass theorem provided that the dimension is less than 6, whereas the Weyl tensor plays a crucial role in higher dimensions.

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1. The uniformization theorem and the Ricci flow in dimension 2

To begin with, I will discuss some results concerning the Ricci flow in dimension 2 and their relation to the uniformization theorem.

Theorem 1.1. *Let M be a compact manifold of dimension 2 without boundary. Given any metric g_0 on M, there exists a metric g which is pointwise conformal to g_0 and has constant curvature.*

Recall that two metrics g_0 and g on a manifold M are said to be pointwise conformal if there exists a smooth function $w \colon M \to \mathbb{R}$ such that $g = e^{2w}\, g_0$. If M is two-dimensional, then the Gaussian curvature of g is related to the Gaussian curvature of g_0 by the formula

$$K_g\, e^{2w} = K_{g_0} - \Delta_{g_0} w, \tag{1}$$

where Δ_{g_0} denotes the Laplacian relative to the metric g_0. Hence, the uniformization theorem in dimension 2 is equivalent to the solvability of the nonlinear elliptic equation

$$\Delta_{g_0} w - K_{g_0} + k\, e^{2w} = 0, \tag{2}$$

where k is a constant. Solutions to (2) can be constructed using variational methods (see e.g. [18], [19]). This approach is based on the observation that every solution

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of (2) is a critical point of the functional

$$E_{g_0}(w) = \frac{1}{2} \int_M |dw|_{g_0}^2 \, d\,\mathrm{vol}_{g_0} + \int_M K_{g_0} \, w \, d\,\mathrm{vol}_{g_0} - \pi \, \chi(M) \, \log\bigg(\int_M e^{2w} \, d\,\mathrm{vol}_{g_0} \bigg), \tag{3}$$

where $\chi(M)$ denotes the Euler characteristic of M. The functional $E_{g_0}(w)$ was introduced by M. Berger in his work on the uniformization problem [5], and is known as the Liouville energy. It has a geometric interpretation in terms of the log determinant of the Laplacian associated with the conformal metric $e^{2w} \, g_0$ (see [27]).

The partial differential equation (2) is closely related to the Ricci flow in dimension 2. In dimension 2, the normalized Ricci flow takes the form

$$\frac{\partial}{\partial t} g(t) = -(K_{g(t)} - k_{g(t)}) \, g(t), \tag{4}$$

where $K_{g(t)}$ denotes the Gaussian curvature of $g(t)$ and

$$k_{g(t)} = \frac{2\pi \, \chi(M)}{\int_M d\,\mathrm{vol}_{g(t)}}$$

is the mean value of the Gaussian curvature on M. The evolution equation (4) can be reduced to a nonlinear partial differential equation of parabolic type. Hence, given any initial metric g_0 on M, the evolution equation (4) has a smooth solution on a small time interval. The evolution equation (4) was first studied by R. Hamilton [14]. The longtime behavior of the flow is characterized by the following result due to R. Hamilton [14] and B. Chow [9]:

Theorem 1.2. *Let M be a compact manifold of dimension* 2 *without boundary. Given any initial metric g_0 on M, the evolution equation* (4) *has a global solution. The solution converges exponentially to a metric of constant curvature as $t \to \infty$.*

The convergence of the Ricci flow follows from the maximum principle if $\chi(M) \leq 0$. The case $\chi(M) > 0$ is more subtle. To prove Theorem 1.2 in this case, Hamilton assumed that the initial metric has positive curvature. This condition is preserved by the flow, and guarantees that the entropy

$$\int_M K_g \log K_g \, d\,\mathrm{vol}_g$$

is well defined. It is shown in [14] that this functional is decreasing along the Ricci flow (see also [9]). In view of the Harnack inequality established in [14], this implies that the curvature is uniformly bounded from above. This is sufficient to prove Theorem 1.2 for initial metrics of positive curvature. The remaining cases were settled by B. Chow [9].

In a subsequent paper, Hamilton observed that the constant in the isoperimetric inequality improves along the Ricci flow [16]. This estimate can be used to give an alternative proof of Theorem 1.2. M. Struwe, building upon earlier work of X. Chen [8], provided another proof of Theorem 1.2 based on concentration-compactness arguments [39].

There are essentially two ways to generalize these results to higher dimensions. One is E. Calabi's problem concerning the existence of Kähler–Einstein metrics (or, more generally, extremal Kähler metrics). The other one is the Yamabe problem, which will be discussed below.

2. The Yamabe problem

In 1960, H. Yamabe [42] conjectured that the uniformization theorem can be generalized in the following way:

Theorem 2.1. *Let M be a compact manifold of dimension $n \geq 3$ without boundary, and let g_0 be a Riemannian metric on M. Then there exists a metric g which is pointwise conformal to g_0 and has constant scalar curvature.*

Theorem 2.1 was proved by T. Aubin [1], R. Schoen [29], and N. Trudinger [40]. A. Bahri gave an alternative proof of Theorem 2.1 in the locally conformally flat case [3].

As in the two-dimensional case, the Yamabe problem can be reduced to the solvability of a nonlinear elliptic equation. Indeed, if two metrics g_0 and g are related by $g = u^{\frac{4}{n-2}}\, g_0$ for a smooth positive function u, then the scalar curvature associated with g can be calculated from the scalar curvature associated with g_0 by means of the identity

$$R_g\, u^{\frac{n+2}{n-2}} = R_{g_0}\, u - \frac{4(n-1)}{n-2}\, \Delta_{g_0} u \tag{5}$$

(see [2]). Here, R_{g_0} denotes the scalar curvature associated with g_0, and Δ_{g_0} is the Laplace operator relative to g_0. Hence, if u is a positive solution of the partial differential equation

$$\frac{4(n-1)}{n-2}\, \Delta_{g_0} u - R_{g_0}\, u + r\, u^{\frac{n+2}{n-2}} = 0, \tag{6}$$

then the metric $g = u^{\frac{4}{n-2}}\, g_0$ has constant scalar curvature r. The solutions of (6) can be characterized as the critical points of the functional

$$E_{g_0}(u) = \frac{\int_M \big(\frac{4(n-1)}{n-2}\, |du|_{g_0}^2 + R_{g_0}\, u^2\big)\, d\,\mathrm{vol}_{g_0}}{\big(\int_M u^{\frac{2n}{n-2}}\, d\,\mathrm{vol}_{g_0}\big)^{\frac{n-2}{n}}}. \tag{7}$$

The functional $E_{g_0}(u)$ is called the Yamabe energy. It follows from (5) that $E_{g_0}(u) = \mathcal{E}(u^{\frac{4}{n-2}}\, g_0)$, where

$$\mathcal{E}(g) = \frac{\int_M R_g \, d\,\mathrm{vol}_g}{\big(\int_M d\,\mathrm{vol}_g\big)^{\frac{n-2}{n}}} \tag{8}$$

denotes the normalized Einstein–Hilbert action. The Yamabe constant of a metric g_0 is defined as the infimum of the Yamabe energy in the conformal class of g_0:

$$Y(M, g_0) = \inf_{0 < u \in C^\infty(M)} E_{g_0}(u). \tag{9}$$

The Yamabe constant is closely related to the optimal constant in the Sobolev embedding of $W^{1,2}(M, g_0)$ into $L^{\frac{2n}{n-2}}(M, g_0)$. It is not difficult to show that $Y(M, g_0) \leq Y(S^n)$, where $Y(S^n) = n(n-1)\,\omega_n^{\frac{2}{n}}$ denotes the Yamabe constant of the standard metric on S^n. The key step in the proof of Theorem 2.1 is the following result, which is due to Aubin and Schoen:

Theorem 2.2. *Let M be a compact manifold of dimension $n \geq 3$ without boundary, and let g_0 be a Riemannian metric on M. Suppose that (M, g_0) is not conformally equivalent to the standard sphere S^n. Then $Y(M, g_0) < Y(S^n)$.*

Aubin proved Theorem 2.2 under the additional assumption that $n \geq 6$ and (M, g_0) is not locally conformally flat. The remaining cases were solved by Schoen using the positive mass theorem.

3. The Yamabe flow

R. Hamilton proposed a heat flow approach to the Yamabe problem [15]. Hamilton considered the following evolution equation for the Riemannian metric g:

$$\frac{\partial}{\partial t} g(t) = -(R_{g(t)} - r_{g(t)})\, g(t). \tag{10}$$

Here, $R_{g(t)}$ denotes the scalar curvature associated with the metric $g(t)$. Moreover, the normalization constant $r_{g(t)}$ is defined as the mean value of the scalar curvature on M:

$$r_{g(t)} = \frac{\int_M R_{g(t)} \, d\,\mathrm{vol}_{g(t)}}{\int_M d\,\mathrm{vol}_{g(t)}}.$$

This choice of the normalization constant ensures that the volume of M does not change under the evolution. Note that the value of the normalization constant $r_{g(t)}$ may change during the evolution.

The evolution equation (10) can be viewed as a generalization of the Ricci flow in dimension 2 and is often referred to as the Yamabe flow. Like the Ricci flow in dimension 2, the Yamabe flow can be reduced to a parabolic equation for a scalar function. To see this, we fix a background metric g_0 in the conformal class of the initial metric. Since the Yamabe flow preserves the conformal structure, we may write the time-dependent metric in the form $g(t) = u(t)^{\frac{4}{n-2}}\, g_0$, where $u(t)$ is a positive function on M. Using the relation (5), one can show that the function u satisfies the partial differential equation

$$\frac{\partial}{\partial t} u^{\frac{n+2}{n-2}} = \frac{n+2}{4} \left(\frac{4(n-1)}{n-2}\, \Delta_{g_0} u - R_{g_0}\, u + r_g\, u^{\frac{n+2}{n-2}} \right), \tag{11}$$

which can be viewed as a parabolic analogue of the Yamabe equation (6). It follows from standard parabolic regularity theory that the Yamabe flow has a smooth solution which is defined on a small time interval. Hamilton proved that the Yamabe flow has a global solution for every initial metric [15]. Moreover, he showed that the scalar curvature satisfies the pointwise bound $|R_{g(t)}| \leq C$, where C is a constant that depends only on the initial metric (but not on t).

The discussion of the asymptotic behavior of the flow can be divided into two cases which are distinguished by the sign of the Yamabe constant $Y(M, g_0)$. If $Y(M, g_0) \leq 0$, one can apply the maximum principle to show that the Yamabe flow converges exponentially to a metric of constant scalar curvature.

One of the first results in the case of positive Yamabe constant is due to B. Chow [10]. Chow showed that the flow converges to a metric of constant sectional curvature provided that (M, g_0) is locally conformally flat and the initial metric has positive Ricci curvature. The proof is inspired by Hamilton's work on the Ricci flow in dimension 3 (see [13]), and is based on pinching estimates for the eigenvalues of the Ricci tensor.

R. Ye [43] proved the convergence of the flow for all initial metrics, assuming only that (M, g_0) is locally conformally flat. To this end, he established a gradient bound for the Yamabe flow on locally conformally flat manifolds. The proof of the gradient estimate is based on the method of moving planes and uses the injectivity of the developing map (see [34]).

A different approach was developed by H. Schwetlick and M. Struwe [36]. Among other things, Schwetlick and Struwe proved that the scalar curvature approaches a constant in the sense that

$$\lim_{t \to \infty} \int_M |R_{g(t)} - r_{g(t)}|^p \, d\,\mathrm{vol}_{g(t)} = 0 \tag{12}$$

for all $p \geq 1$. Using (12), they showed that the Yamabe flow cannot form a singularity unless volume concentration occurs. Moreover, if volume concentration occurs, then the metric can be rescaled in such a way that the rescaled metrics converge to the standard metric on S^n.

Lemma 3.1. *Let $\{t_\nu : \nu \in \mathbb{N}\}$ be a sequence of times such that $t_\nu \to \infty$ as $\nu \to \infty$. Moreover, let $u_\nu = u(t_\nu)$. After passing to a subsequence if necessary, we can find a non-negative integer m and sequences $\{x_{k,\nu} : \nu \in \mathbb{N}\}$, $\{\varepsilon_{k,\nu} : \nu \in \mathbb{N}\}$, $k = 1, \ldots, m$,* *such that*

$$u_\nu(x) - \sum_{k=1}^m \varphi_{x_{k,\nu}}(x) \left(\frac{4n(n-1)}{r_\infty}\right)^{\frac{n-2}{4}} \left(\frac{\varepsilon_{k,\nu}}{\varepsilon_{k,\nu}^2 + d(x_{k,\nu}, x)^2}\right)^{\frac{n-2}{2}} \to u_\infty(x)$$

in $W^{1,2}(M, g_0)$*. Here,* r_∞ *is defined as the limit of the normalization constant* $r_{g(t)}$ *as* $t \to \infty$*. The function* u_∞ *is a non-negative smooth solution of the partial differential equation*

$$\frac{4(n-1)}{n-2} \Delta_{g_0} u_\infty - R_{g_0} u_\infty + r_\infty u_\infty^{\frac{n+2}{n-2}} = 0.$$

Finally, $\varphi_{x_{k,\nu}}$ *is a cutoff function such that* $\varphi_{x_{k,\nu}}(x) = 1$ *for* $d(x_{k,\nu}, x) \leq \delta$ *and* $\varphi_{x_{k,\nu}}(x) = 0$ *for* $d(x_{k,\nu}, x) \geq 2\delta$*.*

This result is due to Schwetlick and Struwe (see [36], Lemma 3.4, and [38]). The proof uses a theorem due to M. Obata, which asserts that every conformal metric on S^n with constant scalar curvature has constant sectional curvature (see [12] or [26]).

Moreover, Schwetlick and Struwe were able to rule out volume concentration provided that $n \leq 5$ and the Yamabe energy of the initial metric is below a certain threshold. The latter condition precludes the formation of a singularity with more than one “bubble”. The formation of a singularity with exactly one “bubble” can be ruled out by means of the positive mass theorem.

It was shown in [7] that the condition $n \leq 5$ implies the convergence of the Yamabe flow for all initial metrics (regardless of their Yamabe energy):

Theorem 3.2. *Let* M *be a compact manifold of dimension* $n \geq 3$ *without boundary, and let* g_0 *be a Riemannian metric on* M*. Suppose that* $n \leq 5$ *or* (M, g_0) *is locally conformally flat. Given any initial metric in the conformal class of* g_0*, the Yamabe flow has a global solution which converges to a metric of constant scalar curvature as* $t \to \infty$*.*

In the locally conformally flat case, this provides an alternative proof of Ye's theorem. The proof of Theorem 3.2 rests on the following key lemma:

Lemma 3.3. *Suppose that* $n \leq 5$ *or* (M, g_0) *is locally conformally flat. Moreover, let* $\{g(t) : t \geq 0\}$ *be a solution of the Yamabe flow on* M*. Then there exist positive real numbers* γ*,* C*, and* t_0 *such that*

$$r_{g(t)} - r_\infty \leq C \left(\int_M |R_{g(t)} - r_{g(t)}|^{\frac{2n}{n+2}} \, d\text{vol}_{g(t)}\right)^{\frac{n+2}{2n}(1+\gamma)} \tag{13}$$

for $t \geq t_0$*.*

Suppose for simplicity that the volume is normalized to 1. Then the number $r_{g(t)}$ coincides with the Yamabe energy at time t. Hence, the number r_∞ can be interpreted as the limit of the Yamabe energy as $t \to \infty$. The difference $r_{g(t)} - r_\infty$ can be expressed as a space-time integral

$$r_{g(t)} - r_\infty = \frac{n-2}{2} \int_t^\infty \int_M (R_{g(\tau)} - r_{g(\tau)})^2 \, d\,\mathrm{vol}_{g(\tau)} \, d\tau. \tag{14}$$

Hence, we can use (13) and Hölder's inequality to derive a differential inequality for the function $r_{g(t)} - r_\infty$. This implies

$$\int_0^\infty \left(\int_M (R_{g(t)} - r_{g(t)})^2 \, d\,\mathrm{vol}_{g(t)} \right)^{\frac{1}{2}} dt < \infty. \tag{15}$$

This estimate can be used to rule out volume concentration. More precisely, given any positive real number η, we can find a real number $r > 0$ such that

$$\sup_{x \in M,\, t \geq 0} \int_{B_r(x)} u(t)^{\frac{2n}{n-2}} \, d\,\mathrm{vol}_{g_0} \leq \eta,$$

where $B_r(x)$ denotes a geodesic ball in the background metric g_0. In light of the results of Schwetlick and Struwe, it follows that

$$\sup_{x \in M,\, t \geq 0} u(x,t) < \infty.$$

Thus, the Yamabe flow converges to a metric of constant scalar curvature as $t \to \infty$. Moreover, if (13) holds with $\gamma = 1$, then the flow converges to the limiting metric at an exponential rate.

To prove Lemma 3.3, we need to find a positive real number ε_0 and a family of auxiliary functions $\bar{u}_{(p,\varepsilon)}$ ($p \in M$, $0 < \varepsilon < \varepsilon_0$) such that

$$E_{g_0}(\bar{u}_{(p,\varepsilon)}) \leq Y(S^n) \tag{16}$$

and $\bar{u}_{(p,\varepsilon)}$ has the "right" asymptotic behavior as $\varepsilon \to 0$. For $n \leq 5$ the existence of such a family of test functions follows from work of R. Schoen [29]. The same approach works if (M, g_0) is locally conformally flat.

Another ingredient in the proof of Lemma 3.3 is an inequality for real-analytic functions due to Lojasiewicz. This inequality was used in the work of L. Simon on the asymptotic behavior of gradient flows [37]. The Lojasiewicz inequality is typically applied to prove the uniqueness of the asymptotic limit of a gradient flow once a-priori estimates have been established.

We cannot, in general, expect (13) to be true for $\gamma = 1$. Indeed, if (13) holds for $\gamma = 1$, then the flow converges to the limiting metric at an exponential rate. This is unlikely to be true in the presence of degenerate solutions.

4. Convergence of the Yamabe flow in dimension greater or equal to 6

We next consider the case $n \geq 6$. In order to prove that the Yamabe flow approaches a metric of constant scalar curvature as $t \to \infty$, we need to construct a family of test functions $\bar{u}_{(p,\varepsilon)}$ that satisfy the inequality (16) among other technical conditions. To this end, it is natural to impose conditions on the Weyl conformal curvature tensor. For example, if we assume that $|W(p)| > 0$ for all $p \in M$, then we can use a result due to Aubin [1] to prove the existence of a family of auxiliary functions with the desired properties.

More generally, suppose that p is a point on M, and let $d = [\frac{n-2}{2}]$ be the largest integer less than or equal to $\frac{n-2}{2}$. If the Weyl tensor does not vanish to an order greater than $d-2$ at p, then we can take advantage of the local geometry to construct a test function with Yamabe energy less than $Y(S^n)$. On the other hand, if the Weyl tensor vanishes to an order greater than $d-2$ at p, we expect that the positive mass theorem can be used to push the energy of the test function below $Y(S^n)$. This motivates the following definition.

Definition 4.1. Let l be a positive integer. We denote by $Z_l(g_0)$ the set of all points $p \in M$ such that

$$\lim_{x \to p} d(x, p)^{2-l} \, |W(x)| = 0,$$

where $W(x)$ denotes the Weyl tensor associated with the metric g_0, and $d(\,\cdot\,,\cdot\,)$ denotes the geodesic distance relative to that metric.

Observe that the set $Z_l(g_0)$ depends only on the conformal class of g_0. It is easy to see that $Z_l(g_0)$ is a compact subset of M. Moreover, we have $M = Z_1(g_0) \supset Z_2(g_0) \supset \cdots$.

Theorem 4.2. *Suppose that $n \geq 6$ and $Z_d(g_0) = \emptyset$ for $d = [\frac{n-2}{2}]$. Then, for every initial metric in the conformal class of g_0, the Yamabe flow has a global solution which approaches a metric of constant scalar curvature as $t \to \infty$.*

To prove Theorem 4.2, we consider an arbitrary point $p \in M$. By a result of J. Lee and T. Parker, we can find a metric g in the conformal class of g_0 such that

$$\det g(x) = 1 + O(|x|^{2d+2}) \tag{17}$$

in geodesic normal coordinates around p (see [21] or [35]). This is called the conformal normal coordinate system. Working in conformal normal coordinates serves two purposes: first, the condition (17) allows us to simplify some of the calculations. Second, it can be shown that the metric agrees with the flat metric to the maximal order permitted by the Weyl tensor. More precisely, if $p \in Z_d(g_0)$, then $g_{ik}(x) = \delta_{ik} + O(|x|^{d+1})$ in conformal normal coordinates. (Conversely, if $g_{ik}(x) = \delta_{ik} + O(|x|^{d+1})$ for any metric conformally equivalent to g_0, then clearly p belongs to the set $Z_d(g_0)$.)

It is convenient to write the Riemannian metric in the form $g(x) = \exp(h(x))$, where $h(x)$ is a symmetric 2-tensor satisfying

$$\operatorname{tr} h(x) = O(|x|^{2d+2}). \tag{18}$$

We denote by $H_{ik}(x) = \sum_{2 \leq |\alpha| \leq d} h_{ik,\alpha}\, x^\alpha$ the Taylor polynomial of order d associated with the function $h_{ik}(x)$. Since $p \notin Z_d(g_0)$, at least one of the polynomials $H_{ik}(x)$ $(1 \leq i, k \leq n)$ is not identically zero.

Given a positive real number ε, we define a function $u_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ by

$$u_\varepsilon(x) = \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{\frac{n-2}{2}},$$

so that

$$\Delta u_\varepsilon + n(n-2)\, u_\varepsilon^{\frac{n+2}{n-2}} = 0. \tag{19}$$

Our aim is to construct a test function $\bar{u}_{(p,\varepsilon)}$ which is close to u_ε and has Yamabe energy less than $Y(S^n)$. To this end, we exploit the saddle point structure of the Einstein–Hilbert action near the standard metric on S^n (see [6], Section 4G). We first choose a vector field $V^\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\sum_{k=1}^n \partial_k \Big[u_\varepsilon^{\frac{2n}{n-2}} \big(H_{ik} - \partial_i V_k^\varepsilon - \partial_k V_i^\varepsilon + \frac{2}{n} \operatorname{div} V^\varepsilon\, \delta_{ik} \big) \Big] = 0 \tag{20}$$

for $i = 1, \ldots, n$. This is a linear elliptic system. In order to construct a solution to (20), it suffices to minimize the functional

$$\int_{\mathbb{R}^n} u_\varepsilon^{\frac{2n}{n-2}} \sum_{i,k=1}^n \Big(H_{ik} - \partial_i V_k - \partial_k V_i + \frac{2}{n} \operatorname{div} V\, \delta_{ik} \Big)^2$$

over all vector fields $V : \mathbb{R}^n \to \mathbb{R}^n$.

In the next step, we define a function $v_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ by

$$v_\varepsilon = \sum_{k=1}^n \partial_k u_\varepsilon\, V_k^\varepsilon + \frac{n-2}{2n}\, u_\varepsilon \operatorname{div} V^\varepsilon.$$

It follows from (20) that the function v_ε solves the linearized equation

$$\Delta v_\varepsilon + n(n+2)\, u_\varepsilon^{\frac{4}{n-2}}\, v_\varepsilon = \sum_{i,k=1}^n \frac{n-2}{4(n-1)}\, u_\varepsilon\, \partial_i \partial_k H_{ik}. \tag{21}$$

The fact that v_ε is a solution of the linearized equation (21) suggests that $u_\varepsilon + v_\varepsilon$ is a good candidate for a test function. Indeed, one can show that

$$\begin{aligned}
&\int_{B_\delta(0)} \Big(\frac{4(n-1)}{n-2}\,|d(u_\varepsilon + v_\varepsilon)|_g^2 + R_g\,(u_\varepsilon + v_\varepsilon)^2\Big) \\
&\quad \leq \Big(\int_{B_\delta(0)} (u_\varepsilon + v_\varepsilon)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\,Y(S^n) \\
&\qquad - \theta \sum_{2 \leq |\alpha| \leq d} \sum_{i,k=1}^n |h_{ik,\alpha}|^2\,\varepsilon^{2|\alpha|} + C(\delta)\,\varepsilon^{n-2}
\end{aligned} \tag{22}$$

if n is odd. A similar estimate holds if n is even: in this case, we have

$$\begin{aligned}
&\int_{B_\delta(0)} \Big(\frac{4(n-1)}{n-2}\,|d(u_\varepsilon + v_\varepsilon)|_g^2 + R_g\,(u_\varepsilon + v_\varepsilon)^2\Big) \\
&\quad \leq \Big(\int_{B_\delta(0)} (u_\varepsilon + v_\varepsilon)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\,Y(S^n) \\
&\qquad - \theta \sum_{2 \leq |\alpha| \leq d-1} \sum_{i,k=1}^n |h_{ik,\alpha}|^2\,\varepsilon^{2|\alpha|} \\
&\qquad - \theta \sum_{|\alpha| = d} \sum_{i,k=1}^n |h_{ik,\alpha}|^2\,\varepsilon^{n-2}\,\log\frac{1}{\varepsilon} + C(\delta)\,\varepsilon^{n-2}.
\end{aligned} \tag{23}$$

In both inequalities, θ is a positive constant that depends only on n. It remains to extend the function $u_\varepsilon + v_\varepsilon$ to all of M. In view of (22) and (23), this can be done in such a way that the Yamabe energy of the resulting function is less than $Y(S^n)$ if ε is sufficiently small. The details will appear in a forthcoming paper.

We expect that the conclusion of Theorem 4.2 is still valid without the condition $Z_d(g_0) = \emptyset$. A proof of this conjecture will likely involve the positive mass theorem. The positive mass theorem was first proved in dimension 3 by R. Schoen and S.T. Yau using minimal surface techniques [33]. This argument can be extended up to dimension 7, cf. [31]. E. Witten gave an alternative proof of the positive mass theorem based on spinor methods [41]. (This approach works only for spin manifolds.) R. Bartnik extended Witten's arguments to prove the positive mass theorem for spin manifolds of any dimension [4]. J. Lohkamp recently announced a proof of the positive mass theorem in arbitrary dimension, which does not require the manifold to be spin [24].

5. Compactness of the set of constant scalar curvature metrics in a given conformal class

In this section, I will review several recent results concerning the set of solutions to (6). In particular, I will discuss under what conditions the set of solutions is compact. These results are remarkably similar to those stated in the preceding sections. As above, we assume that M is a compact manifold of dimension $n \geq 3$ without boundary.

Theorem 5.1. *Suppose that $n \leq 7$ or (M, g_0) is locally conformally flat. Moreover, we assume that (M, g_0) is not conformally equivalent to the standard sphere S^n. Let r be a fixed real number, and let u be a positive solution of* (6). *Then there exists a constant C, depending only on g_0 and r, such that $\|u\|_{C^2(M,g_0)} \leq C$ and* $\inf_M u \geq 1/C$.

This compactness result was proved by R. Schoen in [30] (see also [31] and [32]). Y. Y. Li and M. Zhu gave an alternative proof in dimension 3 [23]. The extension to dimensions 4 and 5 is due to O. Druet [11]. The cases $n = 6$ and $n = 7$ were solved independently by Y.Y. Li and L. Zhang [22] and F. Marques [25].

M. Khuri and R. Schoen [20] recently established a compactness result in arbitrary dimension. Among other things, Khuri and Schoen proved that the Weyl tensor vanishes to an order greater than $[\frac{n-6}{2}]$ at each blow-up point. In particular, if $Z_d(g_0) = \emptyset$ for $d = [\frac{n-2}{2}]$, then blow-up cannot occur.

Theorem 5.2. *Suppose that $n \geq 8$ and $Z_d(g_0) = \emptyset$ for $d = [\frac{n-2}{2}]$. Moreover, let r be a fixed real number. Then there exists a constant C, depending only on g_0 and r, such that $\|u\|_{C^2(M,g_0)} \leq C$ and $\inf_M u \geq 1/C$ for every positive solution of* (6).

A special case of Theorem 5.2 was proved in a recent paper by Y. Y. Li and L. Zhang [22]. Li and Zhang assumed that $n \geq 8$ and $|W(p)| + |\nabla W(p)| > 0$ for all $p \in M$. (This condition is equivalent to $Z_3(g_0) = \emptyset$.) The proof of the main result in [22] uses special properties of conformal normal coordinates established by E. Hebey and M. Vaugon [17].

Moreover, Khuri and Schoen showed that the condition $Z_d(g_0) = \emptyset$ can be removed by means of the positive mass theorem. Since the positive mass theorem holds for spin manifolds of any dimension, this yields the following result:

Theorem 5.3. *Suppose that M is a spin manifold and (M, g_0) is not conformally equivalent to the standard sphere S^n. Moreover, suppose that u is a positive solution of* (6) *for some fixed real number r. Then $\|u\|_{C^2(M,g_0)} \leq C$ and $\inf_M u \geq 1/C$, where C depends only on g_0 and r.*

It is shown in [20] that the compactness result remains true if the equation (6) is replaced by a family of subcritical equations. This is useful in some applications. For example, this can be used to obtain results concerning the number of constant scalar curvature metrics in a given conformal class.

It is interesting to compare the results above to the following result of D. Pollack (see [28], Theorem 0.1):

Theorem 5.4. *Suppose that $Y(M, g_0) > 0$. Given any positive integer N, there exists a Riemannian metric g with the following properties:*

(i) $\|g - g_0\|_{C^0(M,g_0)} \leq 1/N$.

(ii) *The equation* $\frac{4(n-1)}{n-2}\,\Delta_g u - R_g\,u + u^{\frac{n+2}{n-2}} = 0$ *has at least N positive solutions.*

It follows from the results mentioned above that Theorem 5.4 cannot hold if the C^0-norm is replaced by the C^l-norm for a sufficiently large integer l (which may depend on the dimension n). The reason is that the a-priori estimates for solutions of (6) are stable under perturbations of the background metric that are small in the C^l-topology.

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