# Generalized triangle inequalities and their applications

Michael Kapovich*

**Abstract.** We present a survey of our recent work on generalized triangle inequalities in infinitesimal symmetric spaces, nonpositively curved symmetric spaces and Euclidean buildings. We also explain how these results can be used to analyze some basic problems of algebraic group theory including the problem of decomposition of tensor products of irreducible representations of complex reductive Lie groups. Among the applications is a generalization of the Saturation Theorem of Knutson and Tao to Lie groups other than $\mathrm{SL}(n, \mathbb{C})$.

**Mathematics Subject Classification (2000).** Primary 22E46, 20G15, 53C20; Secondary 14L24, 20E42.

**Keywords.** Symmetric spaces, buildings, tensor product decomposition.

## 1. Introduction

As we learn in school, given 3 positive numbers $a$, $b$, $c$ satisfying the familiar triangle inequalities $a \le b + c$, etc., one can construct a triangle in the Euclidean plane whose side-lengths are $a$, $b$ and $c$. A brief contemplation shows that the same elementary geometry proof works in the hyperbolic plane and, more generally, in all simply-connected complete nonpositively curved Riemannian manifolds.

At the first glance, it appears that this is all one can say about the triangle inequalities. Note however that in all negatively curved simply-connected symmetric spaces, the *metric length* of a geodesic segment is a complete congruence invariant. On the other hand, in higher rank symmetric spaces, the congruence classes of oriented segments are parameterized by the Weyl chamber $\Delta$. We will refer to the parameter $\sigma(\gamma) \in \Delta$ corresponding to an oriented segment $\gamma$ as its $\Delta$-*length*. The same notion of $\Delta$-length can be defined in Euclidean buildings, where $\Delta$ is the Weyl chamber for the finite Weyl group in the associated Euclidean Coxeter complex. In this survey we discuss our recent work appearing in a series of papers with Bernhard Leeb and John Millson ([KLM1], [KLM2], [KLM3], [KM1], [KM2]). It originates with the following basic question:

**Question 1.1.** Suppose that $X$ is a nonpositively curved simply-connected symmetric space or a Euclidean building. What restrictions on the triples $(\lambda, \mu, \nu) \in \Delta^3$ are

*During the writing of this paper the author was partially supported by the NSF grant DMS-04-05180.

Proceedings of the International Congress
of Mathematicians, Madrid, Spain, 2006

necessary and sufficient for existence of an oriented geodesic triangle in $X$ whose $\Delta$-side lengths are $\lambda$, $\mu$, $\nu$?

We will see how this question (and related problems) connects to the theory of algebraic groups over real, complex and nonarchimedean valued fields as well as to representation theory of complex reductive Lie groups.

**Acknowledgments.** I am grateful to all my collaborators for the joint work. I am especially grateful to John Millson for our continuous collaboration in the last 13 years.

## 2. Metric spaces modelled on Coxeter complexes

Let $A$ be a (finite-dimensional) Euclidean space and $W_{\text{aff}}$ be a group of isometries of $A$ generated by reflections in a family of hyperplanes $H \subset A$ (called *walls*). A *half-apartment* is a closed half-space in $A$ bounded by a wall. By choosing the origin $o \in A$ and taking linear parts of the elements of $W_{\text{aff}}$ we obtain a group $W = W_{\text{sph}}$ fixing an origin $o$ in $A$. We require $W$ to be finite. Then $W$ is a finite Coxeter group and the group $W_{\text{aff}}$ is called an affine Coxeter (or Weyl) group. The pair $(A, W_{\text{aff}})$ is called a *Euclidean Coxeter complex*. We let $\Delta \subset A$ denote a fundamental domain of the reflection group $W$: it is a convex cone in $A$ with vertex at $o$. A point $x \in A$ is called *special* if its stabilizer in $W_{\text{aff}}$ is isomorphic to $W$. It turns out that each Euclidean Coxeter complex has a special point; in what follows we will always assume that $o \in A$ is special.

**Example 2.1.** Suppose that $W$ is a finite Coxeter group and $W_{\text{aff}} = W \ltimes V$, where $V$ is the full group of translations of $A$. Such affine Coxeter groups appear naturally in the context of symmetric spaces. Another useful example to keep in mind is given by $W \ltimes Q(R^\vee)$, where $W$ is the finite Weyl group associated with a root system $R \subset V^*$ and $Q(R^\vee) \subset V$ is the coroot lattice of $R$. In this case $W_{\text{aff}}$ is discrete. Such examples appear in the context of Bruhat–Tits buildings associated with groups $\underline{G}(\mathbb{K})$, where $\underline{G}$ is a reductive algebraic group and $\mathbb{K}$ is a field with discrete valuation.

Let $Z$ be a metric space. A *geometric structure* on $Z$ *modelled on* the Euclidean Coxeter complex $(A, W_{\text{aff}})$ consists of an atlas of isometric embeddings $\varphi\colon A \hookrightarrow Z$ satisfying the following compatibility condition:

> For any two charts $\varphi_1$ and $\varphi_2$, the transition map $\varphi_2^{-1} \circ \varphi_1$ is the restriction of an element of $W_{\text{aff}}$.

The charts and their images, $\varphi(A) \subset Z$, are called *apartments*. We will require that any two points in $Z$ lie in a common apartment. All $W_{\text{aff}}$-invariant notions introduced for the Coxeter complex $(A, W_{\text{aff}})$, such as walls, special points, etc., carry over to geometries modelled on $(A, W_{\text{aff}})$.

*Thickness* of a space $X$ modelled on $(A, W_{\mathrm{aff}})$ is the cardinality of the set of half-apartments adjacent to a wall in $X$. In all examples considered in this survey, thickness will be independent of the wall in $X$. The space $X$ is called *thick* if it has thickness $\geq 3$.

Examples of geometric structures modelled on $(A, W_{\mathrm{aff}})$ are provided by simply-connected symmetric spaces of nonpositive curvature (in which case $W_{\mathrm{aff}}$ acts transitively on $A$), Euclidean buildings and *infinitesimal symmetric spaces* $\mathfrak{p}$. The latter is equal to the tangent space to a symmetric space, $\mathfrak{p} = T_o X$. Apartments in $\mathfrak{p}$ correspond to Cartan subalgebras (i.e., the maximal abelian subalgebras) in $\mathfrak{p}$, where $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ is the Cartan decomposition of the Lie algebra $\mathfrak{g}$. Although, as a metric space, $\mathfrak{p}$ is nothing but a Euclidean space, its natural group of automorphisms is smaller than $\mathrm{Isom}(\mathfrak{p})$, it is the *Cartan motion group* $K \ltimes \mathfrak{p}$.

**Remark 2.2.** *Discrete* Euclidean buildings (i.e. the ones with the discrete structure group $W_{\mathrm{aff}}$) can be thought of as both geometric and combinatorial objects. From the combinatorial standpoint, one regards buildings as *polysimplical complexes*. The distance between cells is then a certain $W_{\mathrm{aff}}$-valued function, paths in buildings are replaced by *galleries*, etc. Geometric viewpoint appears more powerful as far as the problems raised in this paper are concerned. For one thing, one can do analysis (rather than combinatorics) on such spaces. As another example, one can *stretch* a piecewise-linear path in an apartment via a homothety while there is no obvious stretching construction for galleries. Importance of stretching will become apparent in Section 5.3.

An important example of a symmetric space to keep in mind is $\mathrm{Sym}_n$, the space of positive-definite symmetric $n \times n$ matrices with real coefficients. Then $\mathrm{Sym}_n = \mathrm{GL}(n, \mathbb{R})/\mathrm{O}(n)$ and the Weyl group of this space is the permutation group $S_n$.

Similarly, one defines metric spaces modelled on spherical Coxeter complexes. The most important examples of such spaces are spherical buildings.

For a metric space $Z$ modelled on $(A, W_{\mathrm{aff}})$, we define the $\Delta$-valued distance function

$$d_\Delta \colon Z \times Z \to \Delta$$

as follows:

> Given points $x, y \in Z$, find an apartment $\varphi\colon A \to A' \subset Z$ whose image contains $x$ and $y$. Then consider the vector $\varphi^{-1}(v)$ in $A$ with the tip $y$ and tail $x$ and project this vector to the Weyl chamber $\Delta$ via the quotient map $A \to A/W = \Delta$.

Clearly, this definition is independent of the choice of an apartment $\varphi(A)$ containing $x, y$.

**Example 2.3.** If $Z = \mathrm{Sym}_n$, then $d_\Delta(x, y)$ is the set of eigenvalues of the matrix $x^{-1}y$, counted with multiplicity and arranged in the decreasing order.

Given the notion of $\Delta$-valued distance between points in $Z$ we can also define the $\Delta$-*length* for piecewise-geodesic paths $p$ in $Z$ by taking the sum of the $\Delta$-lengths of the geodesic subsegments of $p$.

Observe that the $\Delta$-distance function $d_\Delta$ is not (in general) symmetric, however

$$d_\Delta(x, y) = (d_\Delta(y, x))^*,$$

where the vector $v^* = w_0(-v)$ is contragredient to the vector $v$. (Here $w_0$ is the longest element of $W$.)

It follows from the Cartan decomposition that in the case when $Z$ is a (nonpositively curved) symmetric space or an infinitesimal symmetric space, then $d_\Delta$ is a complete congruence invariant of an oriented geodesic segment $\overline{xy} \subset Z$:

> There exists an automorphism $g \in \mathrm{Aut}(Z)$ which carries $\overline{x_1 y_1}$ to $\overline{x_2 y_2}$ if and only if $d_\Delta(x_1, y_1) = d_\Delta(x_2, y_2)$.

The situation in the case of Euclidean buildings is more subtle, we will return to this in Section 4.

**Definition 2.4.** Given a space $X$ modelled on an affine Coxeter complex, we let $D_n(X)$ denote the collection of tuples $(\lambda_1, \dots, \lambda_n) \in \Delta^n$ such that there exists an oriented geodesic polygon in $X$ with the $\Delta$-side lengths $\lambda_1, \dots, \lambda_n$.

Thus Question 1.1 in the introduction is asking for a description of $D_3(X)$ for the given space $X$.

## 3. Generalized triangle inequalities

Suppose that $X$ is either a nonpositively curved simply-connected symmetric space, an infinitesimal symmetric space or a thick Euclidean building, modelled on $(A, W_{\mathrm{aff}})$. A priori, $D_n(X)$ is just a subset in $\Delta^n$. The following theorem establishes basic structural properties of this set.

**Theorem 3.1** ([KLM1], [KLM2]). 1. *$D_n(X)$ is a convex homogeneous polyhedral cone.*

2. *$D_n(X)$ depends only on the pair $(A, W)$ and nothing else, not even the type of the space $X$ (i.e., whether this is an infinitesimal symmetric space, symmetric space or a building).*

**Corollary 3.2.** 1. *If $\mathfrak{p} = T_o X$, where $X$ is a nonpositively curved symmetric space, then $D_n(\mathfrak{p}) = D_n(X)$.*

2. *Suppose that $\underline{G}$ is a split reductive algebraic group, $G_1 = \underline{G}(\mathbb{C})$, $G_2 = \underline{G}(\mathbb{R})$ and $K_i \subset G_i$ are maximal compact subgroups, $i = 1, 2$. Then*

$$D_n(G_1/K_1) = D_n(G_2/K_2).$$

Since $D_n(X)$ is a convex homogeneous cone, it is defined by a system of homogeneous linear inequalities which we will refer to as *generalized triangle inequalities*. Theorem 3.1 reduces the computation of these inequalities to the case of symmetric spaces. Since, clearly, $D_n(X \times \mathbb{R}^m) = D_n(X) \times \mathbb{R}^m$, it suffices to consider spaces $X = G/K$, so that the identity component $G$ of $\mathrm{Isom}(X)$ is semisimple. One of the main results of [KLM1] is a description of $D_n(X)$ in terms of the Schubert calculus in the Grassmannians associated to complex and real Lie groups $G$ (i.e., the quotients $G/P$ where $P$ is a maximal parabolic subgroup of $G$).

The *Tits boundary* $\partial_{\mathrm{Tits}} X$ of $X$ is a spherical building modelled on a spherical Coxeter complex $(S, W)$ with spherical Weyl chamber $\Delta_{\mathrm{sph}} \subset S$. It is formed by equivalence classes of geodesic rays in $X$; the metric on $\partial_{\mathrm{Tits}} X$ is given by the *Tits angle* $\angle_{\mathrm{Tits}}$, see for instance [Ba]. We identify $S$ with an apartment in $\partial_{\mathrm{Tits}} X$. Let $\Delta$ denote the Weyl chamber of $X$. We identify $\Delta_{\mathrm{sph}}$ with $\partial_{\mathrm{Tits}} \Delta$.

Let $B$ be the stabilizer of $\Delta_{\mathrm{sph}}$ in $G$. For each vertex $\zeta$ of $\partial_{\mathrm{Tits}} X$ one defines the generalized Grassmannian $\mathrm{Grass}_\zeta = G\zeta = G/P$. (Here $P$ is the maximal parabolic subgroup of $G$ stabilizing $\zeta$.) It is a compact homogeneous space stratified into $B$-orbits called *Schubert cells*. Every Schubert cell is of the form $C_\eta = B\eta$ for a unique vertex $\eta \in W\zeta \subset S^{(0)}$ of the spherical Coxeter complex. The closures $\overline{C}_\eta$ are called *Schubert cycles*. They are unions of Schubert cells and represent well defined elements in the homology $H_*(\mathrm{Grass}_\zeta, \mathbb{Z}_2)$.

For each vertex $\zeta$ of $\Delta_{\mathrm{sph}}$ and each $n$-tuple $\vec{\eta} = (\eta_1, \ldots, \eta_n)$ of vertices in $W\zeta$ consider the following homogeneous linear inequality for $\xi \in \Delta^n$:

$$\sum_i \xi_i \cdot \eta_i \leq 0 \qquad (*_{\zeta;\vec{\eta}})$$

Here we identify the $\eta_i$'s with unit vectors in $\Delta$.

Let $I_{\mathbb{Z}_2}(G)$ be the set consisting of all data $(\zeta, \vec{\eta})$ such that the intersection of the Schubert classes $[\overline{C}_{\eta_1}], \ldots, [\overline{C}_{\eta_n}]$ in $H_*(\mathrm{Grass}_\zeta, \mathbb{Z}_2)$ equals $[pt]$.

**Theorem 3.3** ([KLM1]). *$D_n(X) \subset \Delta^n$ consists of all solutions $\xi$ to the system of inequalities $(*_{\zeta;\vec{\eta}})$ where $(\zeta, \vec{\eta})$ runs through $I_{\mathbb{Z}_2}(G)$.*

**Remark 3.4.** This system of inequalities depends on the Schubert calculus for the generalized Grassmannians $G/P$ associated to the group $G$.

Typically, the system of inequalities in Theorem 3.3 is redundant. If $G$ is a *complex* Lie group one can use the complex structure to obtain a smaller system of inequalities. In this case, the homogeneous spaces $\mathrm{Grass}_\zeta$ are complex manifolds and the Schubert cycles are complex subvarieties and hence represent classes in *integral* homology. Let $I_{\mathbb{Z}}(G) \subset I_{\mathbb{Z}_2}(G)$ be the subset consisting of all data $(\zeta, \vec{\eta})$ such that the intersection of the Schubert classes $[\overline{C}_{\eta_1}], \ldots, [\overline{C}_{\eta_n}]$ in $H_*(\mathrm{Grass}_\zeta, \mathbb{Z})$ equals $[pt]$.

The following analogue of Theorem 3.3 was proven independently and by completely different methods in [BS] and in [KLM1]:

**Theorem 3.5** (Stability inequalities). *$D_n(X)$ consists of all solutions $\xi$ to the system of inequalities $(*_{\zeta;\vec{\eta}})$ where $(\zeta, \vec{\eta})$ runs through $I_{\mathbb{Z}}(G)$.*

As we will see in the next section, these inequalities generalize the system of inequalities used by Klyachko in [Kly1] to solve Weyl's problem on eigenvalues of sums of Hermitian matrices.

It was proven by Knutson, Tao and Woodward [KTW] that in the case $G = \mathrm{SL}(n, \mathbb{C})$ the system of inequalities appearing in Theorem 3.5 is irredundant; on the other hand, for the root systems $B_2, G_2, B_3, C_3$ this system is still redundant, see [KLM1] for the rank 2 computations and [KuLM] for the rank 3 computations. P. Belkale and S. Kumar in [BK] deformed the product structure on $H_*(\mathrm{Grass}_\zeta)$ to make a smaller system of inequalities defining $D_n(X)$, which is irredundant for all root systems of rank $\leq 3$. Conjecturally, the new system of inequalities is irredundant for all root systems.

## 4. Algebraic problems

Let $\mathbb{F}$ be either the field $\mathbb{R}$ or $\mathbb{C}$, and let $\mathbb{K}$ be a nonarchimedean valued field with discrete valuation ring $\mathcal{O}$ and the value group $\mathbb{Z}$. For simplicity, let us consider here only split reductive group $\underline{G}$ over $\mathbb{Q}$, we refer the reader to [KLM3] for the discussion in the general case. Below we consider the following four algebraic problems, labeled by *linear algebra* interpretation in the case when $\underline{G} = \mathrm{GL}(n)$. We refer the reader to Fulton's survey [Fu] for the detailed discussion of these linear algebra problems. We note here only that Problem Q1 in the case $G = \mathrm{GL}(n, \mathbb{C})$ is asking for the restrictions on eigenvalues of sums of Hermitian $n \times n$ matrices $A$ and $B$, provided that the eigenvalues of $A$ and $B$ are given.

- **Q1. Eigenvalues of a sum.** Set $G := \underline{G}(\mathbb{F})$, let $K$ be a maximal compact subgroup of $G$. Let $\mathfrak{g}$ be the Lie algebra of $G$, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be its Cartan decomposition. Give necessary and sufficient conditions on $\lambda, \mu, \nu \in \mathfrak{p}/Ad(K)$ in order that there exist elements $A, B, C \in \mathfrak{p}$ whose projections to $\mathfrak{p}/Ad(K)$ are $\lambda$, $\mu$ and $\nu$, respectively, so that
$$A + B + C = 0.$$
- **Q2. Singular values of a product.** Let $G$ and $K$ be the same as above. Give necessary and sufficient conditions on $\lambda, \mu, \nu \in K\backslash G/K$ in order that there exist elements $A, B, C \in G$ whose projections to $K\backslash G/K$ are $\lambda, \mu$ and $\nu$, respectively, so that
$$ABC = 1.$$
- **Q3. Invariant factors of a product.** Set $G := \underline{G}(\mathbb{K})$ and $K := \underline{G}(\mathcal{O})$. Give necessary and sufficient conditions on $\lambda, \mu, \nu \in K\backslash G/K$ in order that there

exist elements $A, B, C \in G$ whose projections to $K\backslash G/K$ are $\lambda$, $\mu$ and $\nu$, respectively, so that

$$ABC = 1.$$

- **Q4. Decomposing tensor products.** Let $\underline{G}^\vee$ be the Langlands dual group of $\underline{G}$. Give necessary and sufficient conditions on highest weights $\lambda, \mu, \nu$ of irreducible representations $V_\lambda, V_\mu, V_\nu$ of $G^\vee := \underline{G}^\vee(\mathbb{C})$ so that

$$(V_\lambda \otimes V_\mu \otimes V_\nu)^{G^\vee} \neq 0.$$

**Notation 4.1.** Throughout this paper we will denote by Sol(Qi, $G$) the set of triples $(\lambda, \mu, \nu)$ which are solutions of Problem Qi for the group $G$.

**Remark 4.2.** Assuming that the field $\mathbb{K}$ is local (and hence its residue field is a finite field of order $q$) we can reformulate the condition that $(\lambda, \mu, \nu) \in \mathrm{Sol}(\mathrm{Q3}, G)$ as

$$m_{\lambda,\mu}(\nu^*) \neq 0,$$

where the $m$'s are the structure constants of the spherical Hecke ring associated with the group $G = \underline{G}(\mathbb{K})$. We will discuss the geometric meaning of the constants $m$ in the end of this section.

It turns out that the first three algebraic problems are closely related to the geometric problems discussed in the previous section. Consider for instance Problem Q2. Let $X = G/K$ be the corresponding nonpositively curved symmetric space; the group $G$ acts on $X$ by left multiplication, preserving the function $d_\Delta$. Let $o \in X$ be the point stabilized by $K$. We identify $\Delta$ with the double coset $K\backslash G/K$.

Given elements $A, B, C \in G$ we define the polygonal chain in $X$ with the (four) vertices

$$o, x = A(o), \quad y = AB(o), \quad z = ABC(o).$$

Since $G$ preserves the $\Delta$-distance, we conclude that

$$d_\Delta(o, x) = \lambda, \quad d_\Delta(x, y) = \mu, \quad d_\Delta(y, z) = \nu,$$

where $A$, $B$, $C$ project to the vectors $\lambda$, $\mu$, $\nu$ in $\Delta$. If $ABC = 1$, the polygonal chain yields a geodesic triangle in $X$; conversely, if $z = o$ then, by multiplying $C$ by an element of $K$ if necessary, we get $ABC = 1$. Therefore

$$D_3(X) = \mathrm{Sol}(\mathrm{Q2}, G), \quad \text{for the symmetric space } X = G/K.$$

The same arguments work for the infinitesimal symmetric space $X' = T_oX$:

$$D_3(X') = \mathrm{Sol}(\mathrm{Q1}, G).$$

The situation in the case of Problem Q3 is more subtle. It is easy to see that

$$\mathrm{Sol}(\mathrm{Q3}, G) \subset D_3(X),$$

where $X$ is the (discrete) Bruhat–Tits building corresponding to the group $G$. There are two straightforward restrictions on the elements of $\mathrm{Sol}(\mathrm{Q3}, G) \subset D_3(X)$:

- **O1.** $\mathrm{Sol}(\mathrm{Q3}, G) \subset L^3$, where $L$ is the cocharacter lattice of a maximal torus in $\underline{G}$.
- **O2.** For each triple $\sigma = (\lambda, \mu, \nu) \in \mathrm{Sol}(\mathrm{Q3}, G)$ we have

$$T(\sigma) := \lambda + \mu + \nu \in Q(R^\vee).$$

We let $\Lambda \subset L^3$ denote the set of triples $\sigma$ satisfying $T(\sigma) \in Q(R^\vee)$. Then

**Theorem 4.3** ([KLM3]). 1. $(\lambda, \mu, \nu) \in \mathrm{Sol}(\mathrm{Q3}, G)$ *if and only if there exists a geodesic triangle* $\tau \subset X$ *whose vertices are* special points *of* $X$ *and whose* $\Delta$*-side lengths are* $\lambda, \mu, \nu$.

2. *"Conversely", if* $\sigma \in D_3(X) \cap \Lambda$ *then there exists a geodesic triangle* $\tau \subset X$ *whose vertices are* vertices *of* $X$ *and whose* $\Delta$*-side lengths are* $\lambda$*,* $\mu$*,* $\nu$*.*

Note that the vertices of the triangle $\tau$ in Part 2 of this theorem need not be special vertices of $X$, unless the root system $R$ is of type $A$ when *all vertices of* $X$ *are special*. The latter is ultimately responsible for the equivalence of all 4 algebraic problems in the case of $\underline{G} = \mathrm{GL}(n)$.

The basic reason why one should not expect the equality

$$\mathrm{Sol}(\mathrm{Q3}, G) = D_3(X) \cap \Lambda$$

in general, is lack of homogeneity of Euclidean buildings $X$: The $\Delta$-valued distance function is not a complete congruence invariant of pairs of points in $X$. One can remedy this by introducing the *refined distance function* $d_{\mathrm{ref}}(x, y)$ between points $x, y \in X$. The new distance function takes values in $A \times A/W_{\mathrm{aff}}$ (see [KLM2]).

**Theorem 4.4** (Transfer Theorem, [KLM2]). *Suppose that* $X$*,* $X'$ *are thick Euclidean buildings modelled on the same Euclidean Coxeter complex* $(A, W_{\mathrm{aff}})$*. Then for each geodesic polygon* $[x_1, \dots, x_n, x_{n+1} = x_1] \subset X$ *there exists a geodesic polygon* $[x'_1, \dots, x'_n, x'_{n+1} = x'_1] \subset X'$ *so that*

$$d_{\mathrm{ref}}(x_i, x_{i+1}) = d_{\mathrm{ref}}(x'_i, x'_{i+1}), \quad i = 1, \dots, n.$$

**Corollary 4.5** ([KLM3]). $\mathrm{Sol}(\mathrm{Q3}, G)$ *is independent of the field* $\mathbb{K}$*. If* $\mathbb{K}$*,* $\mathbb{K}'$ *are nonarchimedean valued fields with the valuation group* $\mathbb{Z}$*, then*

$$\mathrm{Sol}(\mathrm{Q3}, \underline{G}(\mathbb{K})) = \mathrm{Sol}(\mathrm{Q3}, \underline{G}(\mathbb{K}')).$$

**Relation to representation theory.** First notice that the Langlands dual group $G^\vee = \underline{G}(\mathbb{C})$ appears naturally in the context of problems Q3 and Q4: The character lattice of a maximal torus in $G^\vee$ is the cocharacter lattice of the corresponding maximal torus in $G$.

The following theorem was originally proven in [KLM3] via Satake correspondence; new proofs were given by Tom Haines in [Ha1] and in the work with John Millson [KM1] (along the same lines one can give yet another proof using the results of S. Gaussent and P. Littelmann [GL]).

**Theorem 4.6.** $\mathrm{Sol}(\mathrm{Q4}, G^\vee) \subset \mathrm{Sol}(\mathrm{Q3}, G)$.

Thus we can summarize the above results and Theorem 3.1 as:

**Theorem 4.7.**

$$\begin{aligned} D_3(X) &= \mathrm{Sol}(\mathrm{Q1}, \underline{G}(\mathbb{C})) = \mathrm{Sol}(\mathrm{Q1}, \underline{G}(\mathbb{R})) = \mathrm{Sol}(\mathrm{Q2}, \underline{G}(\mathbb{C})) \\ &= \mathrm{Sol}(\mathrm{Q2}, \underline{G}(\mathbb{R})) \supset \mathrm{Sol}(\mathrm{Q3}, \underline{G}(\mathbb{K})) \supset \mathrm{Sol}(\mathrm{Q4}, \underline{G}^\vee(\mathbb{C})), \end{aligned}$$

*where* $X = \underline{G}(\mathbb{C})/K$.

**Remark 4.8.** Different proofs of the equalities

$$\mathrm{Sol}(\mathrm{Q1}, \underline{G}(\mathbb{F})) = \mathrm{Sol}(\mathrm{Q2}, \underline{G}(\mathbb{F}))$$

were given (in case $\mathbb{F} = \mathbb{C}$) for classical groups by A. Klyachko [Kly2], for all complex Lie groups by A. Alexeev, E. Meinrenken, C. Woodward [AMW] and for all real groups by S. Evens, J.-H. Lu [EL].

**To which extent can the last two inclusions in Theorem 4.7 be reversed?** The restrictions O1, O2 provide necessary conditions for triples $(\lambda, \mu, \nu)$ to belong to $\mathrm{Sol}(\mathrm{Qi}, G)$, $i = 3, 4$. The natural question is if they are also sufficient. On the negative side:

**Theorem 4.9** ([KLM3]). 1. *In the case of the groups* $\mathrm{Sp}(4)$ *and* $G_2$ *the inclusion* $\mathrm{Sol}(\mathrm{Q2}, \underline{G}(\mathbb{R})) \cap \Lambda \supset \mathrm{Sol}(\mathrm{Q3}, \underline{G}(\mathbb{K}))$ *is proper.*

2. *For all non-simply laced groups the inclusion*

$$\mathrm{Sol}(\mathrm{Q3}, \underline{G}(\mathbb{K})) \supset \mathrm{Sol}(\mathrm{Q4}, \underline{G}^\vee(\mathbb{C}))$$

*is proper.*

It turns out however that one can reverse both inclusions at the expense of *multiplication by a saturation factor*. Let $\theta$ be the highest root of the root system $R$ (associated with the group $\underline{G}$)). Then we have the expansion

$$\theta = \sum_{i=1}^{\ell} m_i \alpha_i,$$

where the $\alpha_i$'s are the simple roots in $R$ (corresponding to the chamber $\Delta$).

**Definition 4.10.** Define the *saturation factor* $k_R$ to be the least common multiple of $m_1, \dots, m_\ell$.

Below are the values of $k_R$ for the irreducible root systems.

| Root system $R$ | Group $G$ | $k_R$ |
|---|---|---|
| $A_\ell$ | $\mathrm{SL}(\ell+1)$, $\mathrm{GL}(\ell+1)$ | 1 |
| $B_\ell$ | $\mathrm{SO}(2\ell+1)$ | 2 |
| $C_\ell$ | $\mathrm{Sp}(2\ell)$ | 2 |
| $D_\ell$ | $\mathrm{SO}(2\ell)$ | 2 |
| $G_2$ | $G$ | 6 |
| $F_4$ | $G$ | 12 |
| $E_6$ | $G$ | 6 |
| $E_7$ | $G$ | 12 |
| $E_8$ | $G$ | 60 |

**Theorem 4.11.** 1. $\mathrm{Sol}(\mathrm{Q2}, \underline{G}(\mathbb{R})) \cap k_R\Lambda \subset \mathrm{Sol}(\mathrm{Q3}, \underline{G}(\mathbb{K}))$ (*see* [KLM3].)

2. $k_R \cdot \mathrm{Sol}(\mathrm{Q3}, \underline{G}(\mathbb{K})) \subset \mathrm{Sol}(\mathrm{Q4}, \underline{G}^\vee(\mathbb{C}))$. *Moreover, if at least one of the weights* $\lambda, \mu, \nu$ *is a sum of minuscule dominant weights, then*

$$(\lambda, \mu, \nu) \in \mathrm{Sol}(\mathrm{Q3}, \underline{G}(\mathbb{K})) \iff (\lambda, \mu, \nu) \in \mathrm{Sol}(\mathrm{Q4}, \underline{G}^\vee(\mathbb{C})).$$

(*See* [KM1].)

3. *Therefore*

$$D_3(X) \cap k_R^2\Lambda \subset \mathrm{Sol}(\mathrm{Q4}, \underline{G}^\vee(\mathbb{C})),$$

*where* $X$ *is the symmetric space of* $\underline{G}(\mathbb{F})$.

In particular, in the case when $\underline{G}$ has type $A_\ell$ (e.g. $\underline{G} = \mathrm{SL}(\ell+1)$) we obtain a new proof of the *Saturation Theorem* of A. Knutson and T. Tao [KT] (another proof of this theorem was later given by H. Derksen and J. Whyman in [DW]):

**Theorem 4.12.** *The semigroup* $\Sigma := \mathrm{Sol}(\mathrm{Q4}, \mathrm{SL}(\ell+1, \mathbb{C}))$ *is* saturated*, i.e., a triple* $\sigma = (\lambda, \mu, \nu) \in \Lambda$ *belongs to* $\Sigma$ *if and only if there exists* $N \in \mathbb{N}$ *so that* $N\sigma$ *belongs to* $\Sigma$.

*Proof.* Since the cone $D_3(X)$ is homogeneous, the semigroup $D_3(X) \cap \Lambda$ is clearly saturated. However, according to Part 3 of Theorem 4.11, $\Sigma = D_3(X) \cap \Lambda$ in our case. □

**Remark 4.13.** The equality

$$\mathrm{Sol}(\mathrm{Q3}, \mathrm{GL}(\ell, \mathbb{C})) = \mathrm{Sol}(\mathrm{Q4}, \mathrm{GL}(\ell, \mathbb{C}))$$

was known since the 1960s, see [K1], [K2]. However these proofs do not generalize to other root systems.

Except for the case of the root system of type $A$, the constants which appear in Theorem 4.11 are (conjecturally) not optimal:

**Conjecture 4.14** ([KM1], [KM2]). 1. $D_3(X) \cap k\Lambda \subset \mathrm{Sol}(\mathrm{Q4}, \underline{G}^\vee(\mathbb{C}))$, where $k = 1$ in the case when the root system $R$ is simply laced and $k = 2$ otherwise.

2. Suppose that all three dominant weights $\lambda, \mu, \nu$ are *regular*, i.e., belong to the interior of the chamber $\Delta$. Then

$$(\lambda, \mu, \nu) \in D_3(X) \cap \Lambda \iff (\lambda, \mu, \nu) \in \mathrm{Sol}(\mathrm{Q4}, \underline{G}^\vee(\mathbb{C})).$$

This conjecture holds for the rank 2 simple Lie groups (see [KM2]); it is also supported by some computational experiments.

A less ambitious form of Conjecture 4.14 is

**Conjecture 4.15** (S. Kumar). If $\mathrm{Sol}(\mathrm{Q4}, \underline{G}^\vee(\mathbb{C})) \neq D_3(X) \cap L^3$, then there exists a triple

$$(\lambda, \mu, \nu) \in D_3(X) \cap L^3 \setminus \mathrm{Sol}(\mathrm{Q4}, \underline{G}^\vee(\mathbb{C})),$$

so that at least one of the vectors $\lambda, \mu, \nu$ is non-singular.

**Counting triangles.** Let $X$ be a Bruhat–Tits building of thickness $q + 1 < \infty$, i.e., $q + 1$ is the number of half-apartments adjacent to each wall in $X$. Equivalently, $q$ is the number of elements in the residue field of $\mathbb{K}$. Our goal is to relate the number of geodesic triangles in $X$ with the given $\Delta$-side lengths to the dimensions of the space of $G^\vee$–invariants

$$n_{\lambda,\mu,\nu}(0) = \dim(V_\lambda \otimes V_\mu \otimes V_\nu)^{G^\vee}.$$

Let $o \in X$ be a special point, for instance, the unique point stabilized by $\underline{G}(\mathcal{O})$. Let $f(q) := m_{\lambda,\mu,\nu}(0)$ denote the number of oriented geodesic triangles $[o, x, y]$ in $X$ with the $\Delta$-side lengths $\lambda, \mu, \nu \in P(R^\vee) \cap \Delta$.

**Remark 4.16.** The Hecke ring structure constant $m_{\lambda,\mu}(\nu^*)$ is the number of geodesic triangles as above for which the vertex $y$ is fixed.

Given the root system $R$ and the set $R^+$ of positive roots (determined by $\Delta$), let $\rho$ denote the half-sum of the positive roots.

**Theorem 4.17** ([KLM3]). *$f(q)$ is a polynomial function of $q$ of degree $\le q^{\langle\rho,\lambda+\mu+\nu\rangle}$ so that*

$$f(q) = n_{\lambda,\mu,\nu}(0)q^{\langle\rho,\lambda+\mu+\nu\rangle} + \textit{ lower order terms.}$$

## 5. Geometry behind the proofs

**5.1. Weighted Busemann functions and stability.** Let $X$ be a symmetric space of nonpositive curvature or a Euclidean building. Recall that the ideal boundary $B = \partial_{\text{Tits}} X$ has the structure of a spherical building, the metric on $B$ is denoted by $\angle_{\text{Tits}}$. Given a Weyl chamber $\Delta$ in $X$, we get a spherical Weyl chamber $\Delta_{\text{sph}} = \partial_\infty \Delta \subset \partial_{\text{Tits}} X$. We will identify $\Delta_{\text{sph}}$ with the unit vectors in $\Delta$. We have a canonical projection $\theta\colon \partial_{\text{Tits}} X \to \Delta_{\text{sph}}$.

Take a collection of weights $m_1, \dots, m_n \geq 0$ and define a finite measure space $(\mathbb{Z}/n\mathbb{Z}, \nu)$ where the measure $\nu$ on $\mathbb{Z}/n\mathbb{Z}$ is given by $\nu(i) = m_i$. An $n$-tuple of ideal points $(\xi_1, \dots, \xi_n) \in B^n$ together with $(\mathbb{Z}/n\mathbb{Z}, \nu)$ determine a *weighted configuration at infinity*, which is a map

$$\psi\colon (\mathbb{Z}/n\mathbb{Z}, \nu) \to \partial_{\text{Tits}} X.$$

The *type* $\tau(\psi) = (\tau_1, \dots, \tau_n) \in \Delta^n$ of the weighted configuration $\psi$ is given by $\tau_i = m_i \cdot \theta(\xi_i)$. Let $\mu = \psi_*(\nu)$ be the pushed forward measure on $B$. We define the *slope* of a measure $\mu$ on $B$ with finite total mass $|\mu|$ as

$$\text{slope}_\mu(\eta) = -\int_B \cos \angle_{\text{Tits}}(\xi, \eta)\, d\mu(\xi).$$

To see where the slope function comes from, consider the *$\mu$-weighted Busemann function* on $X$

$$b_\mu(x) := \int_B b_\xi(x) d\mu(\xi)$$

where $b_\xi\colon X \to \mathbb{R}$ is the Busemann function on $X$ corresponding to the point $\xi \in \partial_{\text{Tits}} X$. We normalize all Busemann functions to vanish at a certain point $o \in X$. The function $b_\mu$ is a convex $|\mu|$-Lipschitz function on $X$ which is *asymptotically linear* along each geodesic ray $\rho = \overline{o\eta}$ in $X$. Then

$$\text{slope}_\mu(\eta) = \lim_{t\to\infty} \frac{b_\mu(\rho(t))}{t}$$

is the *asymptotic slope* of $b_\mu$ in the direction of $\eta$.

**Remark 5.1.** Weighted Busemann functions are a powerful tool for studying asymptotic geometry of nonpositively curved spaces, see for instance [BCG].

In what follows we will consider only measures $\mu$ with finite support.

**Definition 5.2** (Stability). A measure $\mu$ on $B$ is called *semistable* if $\text{slope}_\mu(\eta) \geq 0$ and *stable* if $\text{slope}_\mu(\eta) > 0$ for all $\eta \in B$.

There is a refinement of the notion of semistability motivated by the corresponding concept in geometric invariant theory.

**Definition 5.3** (Nice semistability). A measure $\mu$ on $B$ (with finite support) is called *nice semistable* if $\mu$ is semistable and $\{\text{slope}_\mu = 0\}$ is a subbuilding or empty. In particular, stable measures are nice semistable.

A weighted configuration $\psi$ on $B$ is called *stable, semistable* or *nice semistable*, respectively, if the corresponding measure $\psi_* \nu$ has this property.

For our purposes, nice semistability is a useful concept in the case of symmetric spaces and infinitesimal symmetric spaces only. We note however that for these spaces, existence of a semistable configuration $\psi$ on $\partial_{\text{Tits}} X$ implies existence of a nice semistable configuration on $\partial_{\text{Tits}} X$, which has the same type as $\psi$, see [KLM1].

**Example 5.4.** Let $B$ be a 0-dimensional spherical building. Then a measure $\mu$ on $B$ is stable iff it contains no atoms of mass $\geq \frac{1}{2}|\mu|$, semistable iff it contains no atoms of mass $> \frac{1}{2}|\mu|$, and nice semistable iff it is either stable or consists of two atoms of equal mass.

Suppose now that $G$ is a reductive complex Lie group, $K \subset G$ is a maximal compact subgroup, $X = G/K$ is the associated symmetric space. Then the spaces of weighted configurations in $\partial_{\text{Tits}} X$ of the given type $\tau \in \Delta^n$ can be identified with products

$$F = F_1 \times \cdots \times F_n$$

where $F_i$'s are smooth complex algebraic varieties (generalized flag varieties) on which the group $G$ acts transitively. Hence $G$ acts on $F$ diagonally.

In case $X$ is the symmetric space associated to a complex Lie group, the notions of stability (semistability, etc.) introduced above coincide with corresponding notions from symplectic geometry, and, in the case when $\tau_i$'s are fundamental weights, they also coincide with the concepts of stability (semistability, etc.) used in Geometric Invariant Theory, see [KLM1].

Define the subset $\Delta^n_{ss}(B) \subset \Delta^n$ consisting of those $n$-tuples $\tau \in \Delta^n$ for which there exists a weighted semistable configuration on $B$ of type $\tau$. One of the central results of [KLM1] is

**Theorem 5.5.** *$\Delta^n_{ss}(B)$ is a convex homogeneous cone defined by the linear inequalities $(*_{\zeta;\vec{\eta}})$.*

This theorem generalizes the results of A. Klyachko [Kly1] (in the case of $\mathrm{GL}(n)$) and A. Berenstein and R. Sjamaar [BS] in the case of complex semisimple Lie groups.

## 5.2. Gauss maps and associated dynamical systems.

We now relate polygons in $X$ (where $X$ is an infinitesimal symmetric space, a nonpositively curved symmetric space or a Euclidean building) and weighted configurations on the ideal boundary $B$ of $X$, which plays a key role in [KLM1] and [KLM2].

Consider a (closed) polygon $P = [x_1, x_2, \dots x_n]$ in $X$, i.e., a map $\mathbb{Z}/n\mathbb{Z} \to X$. The distances $m_i = d(x_i, x_{i+1})$ determine a finite measure $\nu$ on $\mathbb{Z}/n\mathbb{Z}$ by $\nu(i) = m_i$. The polygon $P$ gives rise to a collection Gauss$(P)$ of *Gauss maps*

$$\psi : \mathbb{Z}/n\mathbb{Z} \to \partial_{\mathrm{Tits}} X \tag{1}$$

by assigning to $i$ an ideal point $\xi_i \in \partial_{\mathrm{Tits}} X$ so that the geodesic ray $\overline{x_i \xi_i}$ (originating at $x_i$ and asymptotic to $\xi_i$) passes through $x_{i+1}$.

**Remark 5.6.** This construction, in the case of the hyperbolic plane, appears in a letter of Gauss to Wolfgang Bolyai, [G]. I am grateful to Domingo Toledo for this observation.

Taking into account the measure $\nu$, we view the maps $\psi : (\mathbb{Z}/n\mathbb{Z}, \nu) \to \partial_{\mathrm{Tits}} X$ as *weighted configurations* of points on $\partial_{\mathrm{Tits}} X$. Note that if $X$ is a symmetric space and the $m_i$'s are all non-zero, there is a unique Gauss map. On the other hand, if $X$ is a Euclidean building then there are, in general, infinitely many Gauss maps. However, the corresponding weighted configurations are of the same type, i.e., they project to the same weighted configuration on $\Delta_{\mathrm{sph}}$.

The following crucial observation explains why the notion of semistability is important for studying closed polygons.

**Lemma 5.7** ([KLM1], [KLM2]). *For each Gauss map $\psi$ the push forward measure $\mu = \psi_* \nu$ is semistable. If X is a symmetric space or an infinitesimal symmetric space then the measure $\mu$ is nice semistable.*

**Polygons in infinitesimal symmetric spaces $X'$.** Let $X' = T_o X$ be the infinitesimal symmetric space. Then

**Theorem 5.8** ([KLM1]). 1. *$\psi$ is nice semistable iff the corresponding weighted Busemann function $b_\mu$ attains its minimum on X, iff $b_\mu$ has a critical point in X.*

2. *Suppose that $b_\mu$ attains its minimum at the origin $o \in X$. Identify the ideal points $\xi_i$ with the unit vectors $\bar{\xi}_i$ in the tangent space $X' = T_o X$ via the exponential map.*

*Then the gradient of $b_\mu$ at o satisfies*

$$0 = \nabla_o(b_\mu) = \sum_{i=1}^{n} m_i \bar{\xi}_i.$$

Thus, in Part 2 of the above theorem, we obtain a closed polygon in the infinitesimal symmetric space $X'$ whose $\Delta$-side lengths are $m_i \theta(\xi_i)$.

**Corollary 5.9.** $\Delta^n_{\mathrm{ss}}(\partial_{\mathrm{Tits}} X) = D_n(X')$.

**Polygons in nonpositively curved symmetric spaces and buildings.** Our goal is to "invert Gauss maps", i.e., given a semistable weighted configuration $\psi : (\mathbb{Z}/n, \nu) \to B$, we would like to find a closed geodesic $n$-gon $P$ so that $\psi \in \mathrm{Gauss}(P)$. The polygons $P$ correspond to the fixed points of a certain dynamical system. For $\xi \in \partial_{\mathrm{Tits}} X$ and $t \geq 0$, define the map $\phi := \phi_{\xi,t} : X \to X$ by sending $x$ to the point at distance $t$ from $x$ on the geodesic ray $\overline{x\xi}$. Since $X$ is nonpositively curved, the map $\phi$ is 1-Lipschitz. Then, given a weighted configuration $\psi : (\mathbb{Z}/n\mathbb{Z}, \nu) \to \partial_{\mathrm{Tits}} X$ with non-zero total mass, define the map

$$\Phi = \Phi_\psi : X \to X$$

as the composition

$$\phi_{\xi_n, m_n} \circ \cdots \circ \phi_{\xi_1, m_1}.$$

The fixed points of $\Phi$ are the first vertices of closed polygons $P = [x_1, \dots, x_n]$ so that $\psi \in \mathrm{Gauss}(P)$. Since the map $\Phi$ is 1-Lipschitz, and the space $X$ is complete and has nonpositive curvature, the map $\Phi$ has a fixed point if and only if the dynamical system $(\Phi^i)_{i \in \mathbb{N}}$ has a bounded orbit, see [KLM2]. Of course, in general, there is no reason to expect that $(\Phi^i)_{i \in \mathbb{N}}$ has a bounded orbit: For instance, if the support of the measure $\mu = \psi_*(\nu)$ is a single point, all orbits are unbounded.

**Problem 5.10.** Suppose that $X$ is a CAT(0) metric space and the weighted configuration $\psi$ is nice semistable. Is it true that $(\Phi^i)_{i \in \mathbb{N}}$ has a bounded orbit?

Although we do not know an answer to this problem in general, we have:

**Theorem 5.11.** 1. *Suppose that $X$ is a nonpositively curved simply-connected symmetric space. Then $\psi$ is nice semistable if and only if $(\Phi^i)_{i \in \mathbb{N}}$ has a bounded orbit, see* [KLM1].

2. *Suppose that $X$ is a Euclidean building. Then $\psi$ is semistable if and only if $(\Phi^i)_{i \in \mathbb{N}}$ has a bounded orbit. This was proven for locally compact buildings in the original version of* [KLM2], *for* 1*-vertex buildings in* [KLM2] *and by Andreas Balser* [B] *in the general case.*

**Corollary 5.12** ([KLM1], [KLM2]). *Suppose that $X$ is a symmetric space of nonpositive curvature or a Euclidean building. Then $D_n(X) = \Delta^n_{ss}(\partial_{\mathrm{Tits}} X)$.*

Now we can explain Part 2 of Theorem 3.1. For instance, let $X_i' = T_o(X_i)$, $i = 1, 2$, be infinitesimal symmetric spaces. Then

$$D_n(X_i') = D_n(X_i') = \Delta^n_{ss}(\partial_{\mathrm{Tits}} X_i), \quad i = 1, 2,$$

see Corollaries 5.9 and 5.12. Let $Y_i$ denote the 1-vertex Euclidean building which is the Euclidean cone over the spherical building $\partial_{\mathrm{Tits}} X_i$, $i = 1, 2$. Then, $\partial_{\mathrm{Tits}} Y_i = \partial_{\mathrm{Tits}} X_i$ and hence, according to Corollary 5.9,

$$D_n(Y_i) = \Delta^n_{ss}(\partial_{\mathrm{Tits}} X_i), \quad i = 1, 2.$$

Finally, according to the Transfer Theorem 4.4,

$$D_n(Y_1) = D_n(Y_2),$$

since these buildings are modelled on the same Euclidean Coxeter complex $(A, W)$, where $W$ is the finite Weyl group of $X_i$, $i = 1, 2$. By combining these equalities we obtain

$$D_n(X_1') = D_n(X_2').$$

**5.3. Relation to representation theory.** We now explain the connection of geodesic triangles in Euclidean buildings to the representation theory which appears in [KM1]. The key instrument here is *Littelmann's path model*. Given a thick Euclidean building $X$ modelled on $(A, W_{\mathrm{aff}})$ (and associated with a nonarchimedean Lie group $G = \underline{G}(\mathbb{K})$) and a special point $o \in X$, we define the projection

$$f : X \to \Delta,\ f(x) = d_\Delta(o, x).$$

This projection restricts to an isometry on each alcove $a \subset X$. Moreover, $f$ preserves the $\Delta$-length of piecewise-geodesic paths in $X$. Given a geodesic triangle $[o, x, y] \subset X$, we obtain a *broken triangle*

$$f([o, x, y]) \subset \Delta$$

which has two geodesic sides $\overline{of(x)}$, $\overline{f(y)o}$ and a broken side $p = f(\overline{xy})$. The $\Delta$-lengths of the above paths are $\lambda = d_\Delta(o, x)$, $\nu = d_\Delta(y, o)$ and $\mu = d_\Delta(x, y)$ respectively. One of the main results of [KM1] is an intrinsic description of the piecewise-geodesic paths $p$ which appear as the result of the above construction. They turn out to be closely related to *LS paths* introduced by Peter Littelmann in [Li].

The LS paths are defined by two axioms: The first one requires existence of a certain *chain* between the tangent vectors $p'_-(t)$, $p'_+(t)$ to the path $p$ at each breakpoint $p(t)$. The second axiom is a maximality condition for such a chain. This axiom is vacuously true if $p(t)$ is a special point of $(A, W_{\mathrm{aff}})$. In [KM1] we define *Hecke paths $p$* in $A$ as piecewise-geodesic paths satisfying the 1st of Littelmann's axioms. Below are the precise definitions.

Let $(A, W_{\mathrm{aff}})$ be a Euclidean Coxeter complex with $V$ the vector space underlying $A$; let $W' \subset W_{\mathrm{aff}}$ be the stabilizer of a point in $A$. By looking at the linear parts of the elements of $W'$, we identify $W'$ with a subgroup

$$l(W') \subset W \subset W_{\mathrm{aff}},$$

where $W$ is the stabilizer of the origin in $W_{\mathrm{aff}}$. Let $\Delta \subset V$ be a Weyl chamber of $W$. Then a *$W'$-chain in $V$ from $\eta_0 \in V$ to $\eta_m \in V$* is a sequence of pairwise distinct vectors

$$\eta_0, \dots, \eta_m \in V$$

so that for each $i$ there exists a reflection $\tau_i \in l(W')$ which sends $\eta_i$ to $\eta_{i+1}$ and whose fixed-point set separates $\eta_i$ from $\Delta$.

**Axiom 1.** A piecewise-linear path $p\colon I \to A$ is a *Hecke path* if for each $t \in I$ there is a $W_{\mathrm{aff},p(t)}$-chain from $p'_-(t)$ to $p'_+(t)$.

Here and below $W_{\mathrm{aff},x}$ is the stabilizer of $x$ in $W_{\mathrm{aff}}$.

**Axiom 2.** A path $p$ satisfies the *maximality property* if for each $p(t)$ the above $W_{\mathrm{aff},x}$-chain can be found which is a *maximal $W$-chain* from $p'_-(t)$ to $p'_+(t)$.

**Definition 5.13.** A path $p$ in $A$ is said to be an *LS path* if it satisfies Axioms 1 and 2.

Here maximality is understood in the set-theoretic sense.

**Theorem 5.14** ([KM1]). *A path $p$ in $\Delta$ is the projection* (*under $f$*) *of a geodesic path in $X$ if and only if $p$ is a Hecke path.*

**Remark 5.15.** An analogous result was independently proven in [GL] by Gaussent and Littelmann in the context of *folded galleries*.

On the other hand, Littelmann proved in [Li] the following fundamental

**Theorem 5.16.**

$$\dim((V_\lambda \otimes V_\mu \otimes V_\nu)^{G^\vee})$$

*is the number of polygons $P \subset \Delta$, each of which is the concatenation of the paths $\overline{o\lambda}$, $p$, $\overline{\nu^* o}$, where $p$ is an LS path of the $\Delta$-length $\mu$.*

Given this we immediately see that

$$\mathrm{Sol}(\mathrm{Q4}, G^\vee) \subset \mathrm{Sol}(\mathrm{Q3}, G),$$

since each LS path is also a Hecke path.

The converse relation is not as clear, since Hecke paths in general are not LS paths (maximality axiom may fail). Nevertheless, suppose for a moment that $\mu$ is an integer multiple of a fundamental coweight $\varpi_i$. Then the geodesic segment $\overline{xy}$ (having special end-points and $\Delta$-length $\mu$) crosses walls of $X$ only at vertices of $X$. Therefore, each break-point $p(t)$ of the path $p = f(\overline{xy})$ is a vertex of the Coxeter complex $(A, W_{\mathrm{aff}})$. One can easily see from the definition of the constant $k_R$ (Definition 4.10) that for each vertex $v$ of $(A, W_{\mathrm{aff}})$,

$$k_R v \text{ is a special vertex of } (A, W_{\mathrm{aff}}).$$

Therefore, the rescaled path $k_R \cdot p$ satisfies the 2nd Axiom of an LS path, while the 1st Axiom is preserved by integer scalings. Hence $k_R \cdot p$ is an LS path and thus, by Littelmann's Theorem

$$(V_{k_R\lambda} \otimes V_{k_R\mu} \otimes V_{k_R\nu})^{G^\vee} \neq 0.$$

This establishes Part 2 of Theorem 4.11 provided that the coweight $\mu$ is a multiple of some fundamental coweight. The general case is more subtle, we refer the reader to [KM1] for the details.

## 6. Other developments

**Restriction problems.** The *Restriction Problem* is, in a sense, even more fundamental for the representation theory than the tensor product decomposition problem:

$$(V_\lambda \otimes V_\mu \otimes V_\nu)^G \neq 0$$

if and only if the restriction of the product representation of $G \times G \times G$ to the diagonal $G \subset G \times G \times G$ has a nonzero invariant vector.

**Problem 6.1** (Restriction Problem). Let $H \subset G$ be a complex reductive subgroup in a complex reductive group $G$. Given an irreducible representation $V_\mu$ of $G$, decompose its restriction $V_\mu|H$ into irreducible factors.

Considerable progress on the general restriction problem and its infinitesimal geometric analogue (determining the projection of a cotangent orbit in the dual of the Lie algebra of a compact group to the dual of the Lie algebra of a subgroup) was made in [BS]. However it appears very difficult to prove saturation results in this generality. Below are two major obstacles in extending the results of [KLM1], [KLM2], [KLM3], [KM1] to the general restriction problem:

1. Given a subbuilding $Y \subset X$ in a Euclidean building $X$, there do not seem to be natural 1-Lipschitz retractions $X \to Y$. The nearest-point projection does not appear to be a good choice.

2. There is (at present) no analogue of Littelmann's path model for the general restriction problem: Littelmann's solution in [Li] of the restriction problem applies only to Levi subgroups.

It turns out, however, that the entire analysis of the *generalized triangle inequality* problems as well as the corresponding algebra problems Q1–Q4 outlined above, can be extended to the restriction problem in the case of Levi subgroups $H \subset G$. This generalization is carried out in the joint work with John Millson and Tom Haines [HKM]. Geodesic polygons are replaced with ideal geodesic polygons, for their infinite sides, the $\Delta$-valued distance function is replaced with $\Delta$-valued Busemann function, etc.

For instance, we obtain a generalization of the Saturation Theorem 4.11 described below.

Let $G$ be a complex reductive Lie group, $L$ be the character lattice of its maximal torus, $Q(R)$ be its root lattice. Let $M \subset G$ be a Levi subgroup; let $\Delta \subset \Delta_M$ be the Weyl chambers of $G$ and $M$. Set

$$\mathcal{L} := \{(\lambda, \mu) \in L \times L : \lambda + \mu^* \in Q(R^\vee)\}.$$

Let $k_R$ denote the saturation factor for the root system $R$ of the group $G$, see Definition 4.10. Then there exists a convex polyhedral cone $D(M, G) \subset \Delta_M \times \Delta$ so that:

**Theorem 6.2** ([HKM]). 1. *If* $\lambda, \mu$ *are dominant weights of* $M$ *and* $G$ *respectively so that*

$$V_\lambda \subset V_\mu | M,$$

*then* $(\lambda, \mu) \in \mathcal{L} \cap D(M, G)$.

2. *If* $(\lambda, \mu) \in \mathcal{L} \cap D(M, G)$, *then for* $k = k_R^2$,

$$V_{k\lambda} \subset V_{k\mu} | M.$$

**Remark 6.3.** In the case $R = A_\ell$, $k_R = 1$, and hence the above result is the analogue of the Knutson–Tao saturation theorem and in fact implies their saturation theorem.

**Problem 6.4.** Prove an analogue of Theorem 6.2 for arbitrary reductive subgroups $H$ of $G$ with constant $k_R^2$ replaced by a suitable number $k$ computable in terms of $G$ and $H$.

Note that even the case of $G = \mathrm{SL}(n, \mathbb{C})$ is extremely interesting in view of possible applications to P $\neq$ NP, see [MS1], [MS2].

**Structure of the sets Sol(Qi).** Despite the description of the convex cone Sol(Q1, $G$) via the linear inequalities $(*_{\zeta;\vec{\eta}})$, its structure remains somewhat mysterious. The case understood best is when $G = \mathrm{SL}(n, \mathbb{C})$.

1. For $\mathrm{SL}(n, \mathbb{C})$ there exists a procedure for computing the inequalities $(*_{\zeta;\vec{\eta}})$ by induction on the rank: This procedure was first conjectured by R. Horn in the 1960s; it was proven in a combination of works by A. Klyachko [Kly1] and A. Knutson and T. Tao [KT]. An alternative proof was later given by P. Belkale [Be2].

**Problem 6.5.** Generalize Horn's recursion formula to groups other than $\mathrm{SL}(n, \mathbb{C})$.

Such a generalization would give a more practical algorithm for computation of Sol(Q1, $G$) than the generalized Schubert calculus.

2. While the facets of the cone Sol(Q1, $G$) are given by the inequalities $(*_{\zeta;\vec{\eta}})$, the edges of this cone have not been described, except in the $\mathrm{SL}(n)$ case, see [KTW], [Be4].

3. Concerning Problems Q3 and Q4 one can reasonably ask "what a computation of the sets Sol(Q3), Sol(Q4) might mean?"

The following theorem was proven by C. Laskowski in [La] for Problem Q3 and in [KM2] for Problem Q4:

**Theorem 6.6.** *For each* $G$, *the set* Sol(Qi, $G$) $(i = 3, 4)$ *is a finite union of* elementary sets.

The notion of an elementary subset of $L^3$ comes from logic: It is a set given by a finite collection of linear inequalities (with integer coefficients) and congruences. Therefore one can interpret Problems Q3, Q4 as

**Problem 6.7.** Find the inequalities and congruences in the description of Sol(Q3) and Sol(Q4) as unions of elementary sets.

An example of such description is given in [KM2] for Sol(Q4, $G$), $G = \mathrm{Sp}(4, \mathbb{C})$, $G = G_2$. Note that for these groups, Sol(**Qi**, $G$), $i = 3, 4$, are not elementary sets themselves.

Note that if Conjecture 4.14 holds, then for each simply-laced group $\underline{G}$ the sets Sol(Q3), Sol(Q4) are elementary and are both equal to

$$D_3(X) \cap L^3,$$

where $X = \underline{G}(\mathbb{C})/K$.

**Quantum product problems.** Throughout this paper we restricted our attention only to noncompact Lie groups and nonpositively curved spaces. However Problem Q2 has a straightforward generalization in the case of compact Lie groups.

Let $K$ be a maximal compact subgroup in a complex semisimple Lie group $G$. Then an alcove $a$ of the associated affine Weyl group parameterizes conjugacy classes in $K$. The following is an analogue of Problem Q2.

**Problem 6.8.** Find necessary and sufficient conditions on elements $\lambda, \mu, \nu \in a$ in order that there exist elements $A, B, C \in K$ whose projections to $a$ are $\lambda$, $\mu$ and $\nu$, respectively, so that $ABC = 1$.

This problem was solved independently by S. Agnihotri and C. Woodward [AW] and P. Belkale ([AW], [Be1]) in the case $K = U(n)$. This solution was generalized to all simple groups $G$ by C. Teleman and C. Woodward in [TW]. The solution is given in a form of *nonhomogeneous* linear inequalities analogous to $(*_{\zeta;\vec{\eta}})$, where Schubert calculus is replaced with *quantum Schubert calculus*. An analogue of Horn recurrence formula for this problem was established by P. Belkale in [Be3].

**Problem 6.9.** 1. Solve the analogue of Problem 1.1 for compact symmetric spaces.
2. Is there an analogue of Problem Q3 in the setting of compact groups?

In the context of compact Lie groups, Problem Q4 generalizes to the problem about *product structure of the fusion ring $R_\ell(G)$ at level $\ell$*, which is a certain quotient of the representation ring of the group $G$. For the characters $\mathrm{ch}(V_\lambda)$, $\mathrm{ch}(V_\mu)$, $\mathrm{ch}(V_\nu)$ in the ring $R_\ell(G)$ consider the decomposition of the triple product:

$$\mathrm{ch}(V_\lambda) \cdot \mathrm{ch}(V_\mu) \cdot \mathrm{ch}(V_\nu) = \sum_\delta \tilde{n}_{\lambda,\mu,\nu,\ell}(\delta)\mathrm{ch}(V_\delta).$$

**Problem 6.10.** Give necessary and sufficient conditions on $\lambda$, $\mu$, $\nu$ in order that

$$\tilde{n}_{\lambda,\mu,\nu,\ell}(0) \neq 0.$$

P. Belkale proved in [Be3] an analogue of the Knutson–Tao saturation theorem for Problem 6.10, thereby establishing equivalence between Problem 6.10 and the multiplicative Problem 6.8:

**Theorem 6.11.** *For $\lambda$, $\mu$, $\nu$ so that $\lambda + \mu + \nu \in Q(R)$,*

$$\tilde{n}(N\lambda, N\mu, N\nu, N\ell) \neq 0, \quad \textit{for some } N \in \mathbb{N},$$

*if and only if*

$$\tilde{n}(\lambda, \mu, \nu, \ell) \neq 0.$$

**Conjecture 6.12** (C. Woodward). The above saturation theorem holds for all simply-laced groups.

## References


[AW] Agnihotri, S., and Woodward, C., Eigenvalues of products of unitary matrices and quantum Schubert calculus. *Math. Res. Lett.* **5** (1998), 817–836.

[AMW] Alekseev, A., Meinrenken, E., and Woodward, C., Linearization of Poisson actions and singular values of matrix products. *Ann. Inst. Fourier* (*Grenoble*) **51** (2001), no. 6, 1691–1717.

[Ba] Ballmann, W., *Lectures on spaces of nonpositive curvature*. With an appendix by Misha Brin, DMV Seminar 25, Birkhäuser, Basel 1995.

[B] Balser, A., Polygons with prescribed Gauss map in Hadamard spaces and Euclidean buildings. *Canad. Math. Bull.*, to appear.

[Be1] Belkale, P., Local systems on $\mathbb{P}^1$-$S$ for $S$ a finite set. *Compositio Math.* **129** (2001), 67–86.

[Be2] Belkale, P., Geometric proofs of Horn and Saturation conjectures. *J. Algebraic Geom.* **15** (1) (2006), 133–173.

[Be3] Belkale, P., Quantum generalization of Horn and Saturation Conjectures. Preprint math.AG/0303013, submitted.

[Be4] Belkale, P., Extremal unitary local systems on $\mathbb{P} - \{p_1, \dots, p_s\}$. In *Proceedings of the International Colloquium on Algebraic Groups*, Tata Institute 2004, to appear.

[BK] Belkale, P., Kumar, S., Eigenvalue problem and a new product in cohomology of flag varieties. Preprint, math.AG/0407034, 2004.

[BS] Berenstein, A., and Sjamaar, R., Coadjoint orbits, moment polytopes, and the Hilbert-Mumford criterion. *J. Amer. Math. Soc.* **13** (2000), no. 2, 433–466.

[BZ] Berenstein, A., and Zelevinsky, A., Tensor product multiplicities, canonical bases and totally positive varieties. *Invent. Math.* **143** (2001), 77–128.

[BCG] Besson, G., Courtois, G., and Gallot, S., Lemme de Schwarz réel et applications géométriques. *Acta Math.* **183** (1999), 145–169.

[DW] Derksen, H., Weyman, J., Semi-invariants of quivers and saturation for Littlewood-Richardson coefficients. *J. Amer. Math. Soc.* **13** (2000), 467–479.

[EL] Evens, S., Lu, J.-H., Thompson's conjecture for real semisimple Lie groups. In *The breadth of symplectic and Poisson geometry*, Progr. Math. 232, Birkhäuser, Boston 2005, 121–137.

[Fu] Fulton, W., Eigenvalues, invariant factors, highest weights, and Schubert calculus. *Bull. Amer. Math. Soc.* (*N.S.*) **37** (2000), no. 3, 209–249.

[G] Gauss, F., Letter to W. Bolyai. In *Collected Works*, Vol. 8, Georg Olms Verlag, Hildesheim 1973, 222–223.

[GL] Gaussent, S., and Littelmann, P., LS-Galleries, the path model and MV-cycles. *Duke Math. J.* **127** (2005), 35–88.

[Ha1] Haines, T., Structure constants for Hecke and representations rings. *Int. Math. Res. Not.* **39** (2003), 2103–2119.

[Ha2] Haines, T., Equidimensionality of convolution morphisms and applications to saturation problems. Preprint 2004, submitted.

[HKM] Haines, T., Kapovich, M., Millson, J. J., Ideal quadrilaterals in Euclidean buildings, constant term maps for spherical Hecke rings and branching to Levi subgroups. Preprint.

[KM1] Kapovich, M., Millson, J. J., A path model for geodesic in Euclidean buildings and its applications to the representation theory. Preprint, November 2004, submitted.

[KM2] Kapovich, M., Millson, J. J., Structure of the tensor product semigroup. *Asian Math. J.* (Chern memorial volume), to appear.

[KLM1] Kapovich, M., Leeb, B., Millson, J. J., Convex functions on symmetric spaces, side lengths of polygons and the stability inequalities for weighted configurations at infinity. Preprint, 2005, submitted.

[KLM2] Kapovich, M., Leeb, B., Millson, J. J., Polygons in buildings and their refined side-lengths. Preprint, 2005, submitted.

[KLM3] Kapovich, M., Leeb, B., Millson, J. J., The generalized triangle inequalities in symmetric spaces and buildings with applications to algebra. MPI Preprint, 2002; *Mem. Amer. Math. Soc.*, to appear.

[K1] Klein, T., The multiplication of Schur functions and extensions of $p$-modules. *J. London Math. Soc.* **43** (1968), 280–284.

[K2] Klein, T., The Hall polynomial. *J. Algebra* **12** (1969), 61–78.

[Kly1] Klyachko, A., Stable bundles, representation theory and Hermitian operators. *Selecta Math.* **4** (1998), 419–445.

[Kly2] Klyachko, A., Random walks on symmetric spaces and inequalities for matrix spectra. *Linear Algebra Appl.* **319** (Special Issue) (2000), 37–59.

[KT] Knutson, A., and Tao, T., The honeycomb model of $\mathrm{GL}_n(\mathbb{C})$ tensor products. I. Proof of the saturation conjecture. *J. Amer. Math. Soc.* **12** (1999), 1055–1090.

[KTW] Knutson, A., Tao, T., and Woodward C., The honeycomb model of $\mathrm{GL}_n(\mathbb{C})$ tensor products. II. Puzzles determine the facets of the Littlewood-Richardson cone. *J. Amer. Math. Soc.* **17** (2004), 19–48.

[KuLM] Kumar, S., Leeb, B., Millson, J. J., The generalized triangle inequalities for rank 3 symmetric space of noncompact type. In *Explorations in complex and Riemannian geometry* (Papers dedicated to Robert Greene), Contemp. Math. 332, Amer. Math. Soc., Providence, RI, 2003, 171–195.

[La] Laskowski, M. C. , An application of Kochen's Theorem. *J. Symbolic Logic* **68** (2003), no. 4, 1181–1188.

[Li] Littelmann, P., Paths and root operators in representation theory. *Ann. of Math.* (2) **142** (1995), no. 3, 499–525.

[MS1] Mulmuley, K., and Sohoni, M., Geometric complexity theory, P vs. NP and explicit obstructions. In *Advances in Algebra and Geometry* (ed. by C. Musili), Hindustan Book Agency, New Delhi 2003, 239–261.

[MS2] Mulmuley, K., and Sohoni, M., Geometric complexity theory III: on deciding positivity of Littlewood-Richardson coefficients. ArXiv preprint cs.CC/0501076.

[TW] Teleman, C., Woodward, C., Parabolic bundles, products of conjugacy classes and quantum cohomology. *Ann. Inst. Fourier* (*Grenoble*) **53** (2003), 713–748.

Department of Mathematics, 1 Shields Ave., University of California, Davis, CA 95616, U.S.A.

E-mail: kapovich@math.ucdavis.edu