Lagrangian submanifolds: from the local model to the cluster complex

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Abstract. In these notes, I will present collaborative works on Lagrangian submanifolds that I realised mainly with Octav Cornea (the cluster complex, which naturally leads to a universal Lagrangian Floer theory), but also, at an earlier stage, with Jean-Claude Sikorav. To cover the subject in a more complete and adequate way, I will also mention very recent works by Barraud–Cornea and by Welschinger, closely related to the subject of these notes. The aim of the cluster machinery is to resolve the well known problem of real codimension 1 bubbling off of disks in the Gromov–Floer theory; see Fukaya–Oh–Ohta–Ono (especially the two lectures by Oh and Ono in these proceedings) for a different, earlier, approach.

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1. Introduction

By Weinstein's theorem, a sufficiently small neighbourhood of any Lagrangian submanifold $L \subset M$ of a symplectic manifold is symplectomorphic to a neighbourhood of the zero section of the cotangent T^*L of L through a diffeomorphism that sends L to the zero section. The study of Lagrangian submanifolds, which play a fundamental role in our understanding of symplectic geometry, therefore breaks down into two parts:

Local model: study of exact Lagrangian submanifolds in cotangent spaces.

General case: study of Lagrangian submanifolds in general symplectic manifolds.

From a heuristic point of view, these two cases correspond to the following dichotomy in studying Lagrangian submanifolds: either we are in a case where "real bubbling off" of disks is prohibited or not. The former case is much simpler, and the simplest case in which bubbling off cannot occur is the one of closed embedded exact Lagrangian submanifolds in cotangent spaces $L \subset T^*V$ where V is any (not necessarily closed) manifold and where T^*V is equipped with the exact symplectic form $\omega = d\lambda$, with λ the Liouville form on T^*V. Recall that λ is equal to the form $\sum_i p_i dq_i$ in local coordinates where the q_i's are coordinates on V and the p_i's the coordinates naturally induced on the fibers of the cotangent bundle by the q_i's. A

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submanifold $L \subset T^*V$ is *Lagrangian* if the restriction of ω to L vanishes identically, and it is *exact* if the restriction of λ (which is therefore closed on L) is exact on L. In a general symplectic manifold (M, ω), the notion of a Lagrangian submanifold is defined similarly, and the exactness is replaced by the notion of *weakly exact* meaning that the restriction of ω to $\pi_2(M, L)$ vanishes. It is obvious, by Stokes' Theorem, that an exact Lagrangian submanifold is weakly exact. However, as far as analytic techniques are concerned, there is not much difference between the two concepts (although the geometry might be quite different). See [10] for the definition of (generic) almost complex structures J's on a symplectic manifold tamed by ω. By definition, for such J's, J-invariant real 2-planes in the tangent space of any symplectic manifold are ω-positive – thus non-constant J-pseudoholomorphic curves, also called J-curves, (i.e. those for which the real differential commutes with the complex structure i on the Riemann surface at the source and the J-structure at the target space) have strictly positive symplectic area. *Real bubbling off* is the phenomenon that occurs when a J-curve with boundaries on a finite set of Lagrangian submanifolds of a symplectic manifold degenerates to the connected union of a J-curve connecting the same set of Lagrangian submanifolds and a J-pseudoholomorphic disk with boundary on one of the submanifolds. Thus real bubbling off can only happen if at least one of the classes in $\pi_2(M, L_i)$, for at least one i, has strictly positive symplectic area. Hence there is no bubbling off in (weakly) exact Lagrangian submanifolds, which makes the analytic study much simpler. See [2] for a survey on Lagrangian submanifolds.

2. Exact Lagrangian submanifolds

The most obvious examples of Lagrangian (exact) submanifolds are the graphs, in T^*V, of closed (exact) 1-forms defined on V. It turns out that exact Lagrangian submanifolds behave in a much more rigid way than other Lagrangian submanifolds, so much that the following statement seems a reasonable (although very strong) conjecture:

Conjecture A (see Lalonde–Sikorav [14]). Any closed exact Lagrangian submanifold of a cotangent space is isotopic, through a Hamiltonian isotopy, to the zero section.

A *Hamiltonian isotopy* is a very constraining condition: it requires that the isotopy be Lagrangian at each moment and that the symplectic area of the cylinder spanned by any non-contractible loop during any time interval in the isotopy be zero. Note that this conjecture implies that there is no closed exact Lagrangian submanifold in the cotangent of an open manifold, a fact proved in [14]. Actually, we proved in [14] that the projection $L \to V$ must be surjective in the exact case. This is a consequence of the following "Property 4" of [14]:

Let $K \subset V$ be a closed manifold. Then any closed exact Lagrangian submanifold L must intersect the conormal νK of K in the following two cases: (i) *K is the* *fiber of a submersion $V \to B$ where B is a closed submanifold;* (ii) *K is homotopic to a point in V.*

Since the conormal of a point $q \in V$ is the fiber T_q^*V at that point, surjectivity follows from (ii) (see below for a more direct proof). From (i), one may deduce that any two closed exact Lagrangian submanifolds L_0, L_1 must intersect in the cotangent of any homogeneous space $V = G/H$. Indeed, by fiber product, one easily assigns to L_0, L_1 two closed exact Lagrangian submanifolds L_0', L_1' of T^*G with a natural bijection between $L_0 \cap L_1$ and $L_0' \cap L_1'$. But the latter is in bijection with

$$(L_0' \times L_1') \cap \Delta_{T^*G} \subset T^*G \times T^*G$$

or, equivalently, with $(L_0' \times -L_1') \cap A$ where $A \subset T^*G \times T^*G$ is the anti-diagonal. But $T^*G \times T^*G = T^*(G \times G)$, A is the conormal of Δ_G which is a fiber of the submersion $(g, h) \mapsto gh^{-1}$, so the fact that $(L_0' \times -L_1') \cap A$ is not empty is a consequence of Property 4 (i) above.

Conjecture A has many other consequences, that could happen to be true while the conjecture itself might turn out to be false. Denoting by L a closed exact Lagrangian submanifold of T^*V, here are some of these consequences:

(I) L is diffeomorphic to V.

(II) The Maslov index of L vanishes.

(III) The map induced at the π_1-level by the projection $L \to V$ is onto.

(IV) The projection $L \to V$ has degree ± 1.

Except in cotangent spaces of surfaces ($n = 2$), the first of these consequences seems out of reach for the moment. Actually, in the case $n = 2$, many of these properties have been solved and all of them have been established for the torus. Indeed, it is not difficult to see that the formula for the number of double points of an immersed Lagrangian submanifold must be

$$\frac{d^2\chi(V) - \chi(L)}{2}$$

in the orientable case, which implies that the formula $\chi(L) = d^2\chi(V)$ must hold for an embedded Lagrangian submanifold (here d is the degree of the projection $L \to V$). Hence, in dimension 2, (I) is a consequence of (IV) (a similar formula mod 2 holds in the non-orientable case as well). It turns out that (IV) was established for T^2 in [14], and that, for surfaces, the degree $L \to V$ was proved to be always non-zero with the sole exception of an eventual exact degree 0 Lagrangian embedding of the 2-torus in the cotangent of the 2-sphere, a case ruled out by Viterbo a few years later (see [23]) using Floer techniques (or equivalently techniques related to the cohomology of the free loop space).

Let me now briefly explain, from [14], how one can prove all of these consequences, except the first one, in the cotangent of the n-torus, starting from the following basic results proved by Gromov using pseudoholomorphic curves:

Theorem 1 (Gromov [10]). (1) *A closed exact Lagrangian submanifold must intersect the zero section.*

(2) *A closed exact Lagrangian submanifold must intersect any submanifold obtained as the image of itself by a Hamiltonian isotopy.*

(The second statement in Gromov's theorem has been pushed much further by Floer, but this is not relevant at this moment).

We will also use Audin's conjecture, whose proof has recently been announced by several authors, stating that any Lagrangian embedding $L = T^n \to \mathbb{R}^{2n}$ has Maslov number 2 (i.e. there is a loop on L whose Maslov index is 2). Actually, we do not need such a strong conjecture for our present purposes: any bound, say the one found by Viterbo in [22], stating that there is a loop with Maslov index in $[2, n+1]$ for any such embedding, is enough.

Here is how to prove (III) for a closed exact Lagrangian submanifold L in $M = T^*T^n$. First note that the index of $\pi_\#(\pi_1(L))$ in $\pi_1(V)$ is always finite: otherwise, denoting by $V_1 \to V$ the covering corresponding to $\pi_\#(\pi_1(L))$, one could then lift L to a Lagrangian embedding $L_1 \subset T^*V_1$. If the index were infinite, one would then get a closed exact Lagrangian submanifold in the cotangent of an open manifold. The image of the projection $L_1 \to V_1$ would not be surjective. But this is impossible: indeed, if it were not onto, one could then define a Morse function f on V_1 with all of its critical points outside the subset $\pi(L_1)$ of V_1. Translating L_1 by a sufficiently large multiple of df in the fibers of T^*V_1 would produce an exact Lagrangian submanifold that would no longer intersect the zero section, a contradiction with Theorem 1 (1) above.

Now let us show that $f_\#\colon \pi_1(L) \to \pi_1(V)$ must actually be surjective when V is a Lie group. This will prove (III) (and therefore (IV) when V is a torus, if we know that L is also a n-torus). The basic geometric idea is that, if this map were not surjective, one would get more than one disjoint lifts of the same Lagrangian submanifold in the cotangent space of some covering V' of V corresponding to the subgroup $\pi_\#(\pi_1(L))$ in $\pi_1(V)$. But the deck transformations of that covering are induced by translations of the Lie group, and are therefore isotopic to the identity. Thus the group of automorphisms, at the cotangent level, of $T^*V' \to T^*V$, consists of Hamiltonian diffeomorphisms (this is because the differential of any diffeomorphism ϕ of a manifold V' induces a symplectic diffeomorphism of T^*V' and one sees easily that this symplectomorphism is Hamiltonian isotopic to the identity if ϕ is isotopic to the identity). This shows that one gets many disjoint exact Lagrangian embeddings of L, all Hamiltonian isotopic to each other, a contradiction with Theorem 1 (2).

Finally, the second consequence (the vanishing of the Maslov class) for an exact n-torus in the cotangent of the n-torus is proved using the geometric composition $j \circ i$ of two Lagrangian embeddings $i\colon L \to T^*V$ and $j\colon V \to (\mathbb{R}^{2n}, \omega_0)$ given by extending j to a small neighbourhood of the zero section by Weinstein's theorem and then composing after contraction of i in the fibers. The Maslov class formula is $\mu(j \circ i) = \mu(i) + f^*(\mu(j))$ where f is the projection of L on V. One can start with the standard embedding j and iterate the construction. By the surjectivity of $f_{\#}$ proved above, one deduces that $f^*\colon H^1(V,\mathbb{Z}) \to H^1(L,\mathbb{Z})$ is a bijection, which gives enough control to exceed any bound imposed by Audin's conjecture or by Viterbo's theorem if $\mu(i)$ does not vanish and if we iterate that construction a sufficient number of times (see [14] for more detail).

Using the Lagrangian surgery introduced in [14], and developed shortly afterwards by L. Polterovich, it was easy to say exactly in [14] which surfaces can be embedded in the cotangent of any surface (orientable or not), with the sole exception of an eventual local embedding of the Klein bottle, a difficult and classical problem whose solution has recently been explored by various methods, especially the ones of symplectic field theory.

2.1. Methods of symplectic field theory. There are, currently, two promising ways of establishing (some of) the above consequences (I)–(IV). The first one is symplectic field theory. The best results obtained so far are in real dimension 4 (cotangent of surfaces):

Let V be either the 2-sphere or the 2-torus. Then any closed orientable exact Lagrangian submanifold L in T^*V is Hamiltonian isotopic to the zero section.

To prove this, note that by Lalonde–Sikorav [14] and Viterbo [23] above, the degree cannot be zero which means, by the double point formula, that $\chi(L) = \chi(V)$; moreover, in the case $V = S^2$, the same formula implies that the degree is 1 while in the case $V = T^2$, this is Consequence (IV) proved above for tori. Thus L is diffeomorphic to V and is homologous to the zero section. On the other hand, building on a result by Eliashberg–Polterovich [7], Hind (preprint "Lagrangian unknottedness in Stein surfaces" based on [11]) and A. Ivrii [13] showed, using methods of SFT, that such a surface is Lagrangian isotopic to the zero section; since we have proved above that the projection in the torus case induces an isomorphism at the $H^1(\,\cdot\,;\mathbb{R})$-level, one may change the isotopy at each time by translating by the graph of an appropriate closed one-form in order to make the isotopy exact, which proves the statement. See the article by Eliashberg in these Proceedings for more on SFT.

2.2. A natural more algebraic approach. The second promising approach is implicit in the methods of [14]: one very simple idea that inspired the results of that paper is that if the projection $L \to V$ is a covering (without singularities), then the degree must be ± 1. The reason is that, if $|d|$ were larger than 1, the differences $x - y$ between all pairs of distinct points $x, y \in L \cap T_q^* V$ would form an exact Lagrangian submanifold $\mathcal{D}(L)$ of T^*V and the resulting submanifold would not meet the zero section, a contradiction. Unfortunately, this simple argument breaks down when singularities of the projection $L \to V$ occur, because some differences may vanish at a singularity. However, this is a frustrating problem since one should recognize the right pairs. A natural way to do this is to replace the difference of pairs by pseudo-holomorphic strips joining pairs of such points, with one boundary on L and the other on the fiber. What is needed is to update the paper [14] by replacing Gromov's theory by Floer's theory. Such an approach has been suggested by Fukaya for tori (private communication, Ringberg, 1997) and there is some hope that it will eventually lead to proofs of some of the consequences (II), (III) or (IV).

3. The cluster complex

The most useful algebraic tool for studying Lagrangian submanifolds in general symplectic manifolds is Floer homology. The goal is to assign to a pair L_0, L_1 of Lagrangian submanifolds a homology $FH_*(L_0, L_1)$ invariant under Hamiltonian deformations of any of the L_i (thus it would also assign an invariant to a single L by taking L and any of its images by a Hamiltonian isotopy). Assuming L_0, L_1 closed and meeting each other transversally, the complex is generated over $\mathbb{Q}$ by the finite set $I = L_0 \cap L_1$ and the differential is given by $da = \sum_{b \in I, \lambda} \mathrm{Card}(\mathcal{M}(a, b, \lambda; J)) b e^{\lambda}$ where λ keeps track of the homotopy class of strips joining a to b (parametrised say by the closed unit disk with two point p_-, p_+ removed, such that the lower boundary be sent to L_0, the upper one to L_1, and so that the map converge to a at p_- and to b at p_+) and where $\mathcal{M}(a, b, \lambda; J)$ is the moduli space of J-holomorphic strips joining a to b, with finite symplectic area, in the class λ, after quotient by the real 1-dimensional group of reparametrizations. The cardinality is taking over $\mathbb{Q}$ and is declared to be zero whenever the expected dimension of $\mathcal{M}(a, b, \lambda; J)$ is different from 0.

It is well known that this scheme does not work in general because d^2 is not always zero. The obstruction occurs when there is real bubbling off either on L_0 or on L_1. The reason is that the way to prove that $d^2 a$ vanishes follows the same argument as in ordinary Morse homology: an element in $d^2 a$ corresponds to two strips u_1 from a to b, and u_2 from b to c, each one belonging to a moduli space of dimension zero. The J-holomorphic surgery at b removes the singularity at b and exhibits the broken configuration (u_1, u_2) as the boundary of a real one-dimensional moduli space of strips u joining a to c. As this manifold $\mathcal{M}$ is compact, it must have another end: if that end is of the form (u_1', u_2') joining a to c through b', one sees that this cancels out the term ce^{λ} in $d^2 a$. However, unfortunately, $\mathcal{M}$ might have an end of a different nature if its one-parameter family of strips degenerates to a configuration made from one strip joining a to c in class $\lambda - \tau$ and one J-holomorphic disk with boundary on either of the L_i in class τ, and meeting the strip at one of its boundary points on L_i (which, generically, can be assumed to be distinct from a and c). In this case, the resulting broken configuration does not correspond anymore to an element of $d^2 a$ and the classical cancellation argument breaks down.

With Octav Cornea, we have recently proposed in [5] a solution to overcome this problem by introducing a broader algebra and larger moduli spaces, large enough so that the above undesirable broken configurations be realised as *interior* points in those larger moduli spaces. In order to carry out this programme, the first step consists in defining, for each of the L_i separately, a new complex, the *cluster complex* of L_i, leading to a well-defined homology assigned to each Lagrangian submanifold; the second step consists in using these complexes as coefficient rings to define a universal Floer homology, that we called the *Fine Floer homology* of the pair (L_0, L_1). From an algebraic point of view, the cluster complex has similarities with Chekanov's contact homology. From the perspective of the objectives, there are obvious similarities with the Fukaya–Oh–Ohta–Ono [8] approach – they introduced a universal object attached to each Lagrangian submanifold using an A^∞-approach and employed it as a coefficient ring; however our approach is based on a different geometric idea, leading to different moduli spaces and the relations between the cluster and the FOOO settings are still unclear.

Here is a description of the cluster homology, the Fine Floer homologies, and some of their applications.

We shall assume here that all Lagrangian submanifolds are compact, connected, orientable, and relative spin (recall that a Lagrangian submanifold $L \subset (M, \omega)$ is relative spin if the second Stiefel–Whitney class of L admits an extension to $H^2(M; \mathbb{Z}/2)$; a set of Lagrangian submanifolds is relative spin if their second Stiefel–Whitney classes admit a common extension to $H^2(M; \mathbb{Z}/2)$). In the notation L for a Lagrangian submanifold we always implicitly include the choices of an orientation and of a relative spin structure (the same applies for a set of such submanifolds) as described in [8]. The ambient symplectic manifold (M^{2n}, ω) is supposed to be compact or, if not, it should be geometrically bounded, so that no sequence of J-curves with boundary lying on a set of compact Lagrangian submanifolds $L_1, \dots, L_\ell \subset M$ can escape to infinity. In fact, the Lagrangian submanifolds need not all be compact, as long as the above control on sequences of J-curves is ensured.

The cluster complex is associated to a triple formed of (1) a Lagrangian embedding $L^n \hookrightarrow (M, \omega)$, equipped with a choice of an orientation and of a relative spin structure, (2) a generic almost complex structure J on M compatible with the symplectic form ω, and (3) a pair (f, g) with $f : L \to \mathbb{R}$ a Morse function and g a Riemannian metric on L so that (f, g) is Morse–Smale. The two conditions implicit in the notation L (i.e. the orientation and the relative spin structure) are needed to orient the clustered moduli spaces – to be roughly described below – in a coherent way.

This complex is denoted by $\mathcal{C}\ell_*(L; J, (f, g))$ and, with $\mathrm{Crit}(f)$ denoting the set of critical points of f, we set

$$\mathcal{C}\ell_*(L; J, (f, g)) = (S\mathbb{Q}\langle s^{-1}\,\mathrm{Crit}(f)\rangle \otimes \Lambda)^\wedge_*$$

where $s^{-1}\,\mathrm{Crit}(f)$ indicates that the natural index grading of $\mathrm{Crit}(f)$ is decreased by one unit, SV is the free, graded commutative algebra over the graded vector space V (as usual, the sign commutativity rule is $ab = (-1)^{|a||b|}ba$ for any two elements $a, b \in V$), Λ is the rational group ring of the quotient $\pi_2(M, L)/\sim$ where the equivalence relation $\sim$ is given by $\lambda \sim \tau$ iff $\omega(\lambda) = \omega(\tau)$ and $\mu(\lambda) = \mu(\tau)$, where ω and μ are the area and Maslov classes respectively. We write the elements of Λ in the form of finite sums $\sum_i c_i e^{\lambda_i}$, $c_i \in \mathbb{Q}$. The grading in Λ is given by $|e^\lambda| = -\mu(\lambda)$ for $\lambda \in \pi_2(M, L)/\sim$. With this convention, the grading of the cluster complex is given by the usual tensor product formula. Thus for $x_i \in \mathrm{Crit}(f)$, we have

$$|x_i| = \mathrm{ind}_f(x_i) - 1, \quad |x_1 \dots x_k e^{\lambda}| = \sum_{i=1}^{k} |x_i| - \mu(\lambda).$$

Finally, we describe the completion $^\wedge$. An element $m \in \mathcal{C}\ell\,(L; J, (f, g))$ can be written as a possibly infinite sum

$$m = m_0 + m_1 e^{\lambda_1} + \cdots + m_i e^{\lambda_i} + \cdots$$

where m_i are monomials in the elements of $\mathrm{Crit}(f)$ but, if this sum is infinite, then any infinite subsequence with $\omega(\lambda_i)$ bounded above, must have its corresponding word length sequence converging to infinity. Conversely, any formal sum verifying this condition belongs to the cluster complex.

We now give a rough idea of the construction of the cluster differential. The generic data $J, (f, g)$ on L being given, fix an order on the critical points of f. Choose any integer $k \geq 0$, any $x \in \mathrm{Crit}(f)$, any non-decreasing sequence of critical points $x_1, \dots, x_k$, and $\lambda \in \pi_2(M, L)/\sim$, with the constraint that the zero class $\lambda = 0$ is allowed only when k equals 1. The cluster differential

$$d\colon (S\mathbb{Q}\langle s^{-1}\,\mathrm{Crit}(f)\rangle \otimes \Lambda)^{\wedge}_{*} \to (S\mathbb{Q}\langle s^{-1}\,\mathrm{Crit}(f)\rangle \otimes \Lambda)^{\wedge}_{*-1}$$

is the unique commutative, graded differential algebra extension of

$$dx = \sum_{\substack{\lambda, k \geq 0 \\ x_1, \dots, x_k}} a^{x}_{x_1, \dots, x_k}(\lambda) x_1 \dots x_k e^{\lambda}, \tag{1}$$

where $x, x_1, \dots, x_k \in \mathrm{Crit}(f)$ have the property that $(x_1, \dots, x_k)$ respects the fixed order on $\mathrm{Crit}(f)$, $|x| - \sum_i |x_i| + \mu(\lambda) - 1 = 0$ and the coefficients $a^{x}_{x_1, \dots, x_k}(\lambda)$ count with signs the number of elements in the clustered moduli spaces

$${}^{\nu}\mathcal{M}^{x}_{x_1, \dots, x_k}(\lambda)$$

(due to the possible presence of multi-sections this number will belong in fact to $\mathbb{Q}$).

A rigorous description of the clustered moduli spaces appears in [5]. The main idea in their definition is simple: consider a one parametric family of J-disks of class λ and assume that the bubbling off of a J-disk of class λ' occurs in this family. The "bubbled configuration" formed by two touching J-disks in classes $\lambda'' = \lambda - \lambda'$ and λ' can of course be viewed as the limit of this bubbling off, but it can also be considered as the limit of a one parametric family of two J-disks in the same classes $\lambda'' = \lambda - \lambda'$ and λ' joined by a negative gradient flowline of f whose length tends to 0. By gluing these two one-parameter moduli spaces at their common limit, the above bubbled configuration becomes an interior point of the larger clustered moduli space.

By pursuing systematically this idea together with the usual description of stable maps (as, for example, in McDuff–Salamon [15]) we obtain moduli spaces of configurations modelled on oriented trees with edges of non-negative length so that each vertex corresponds to a J-disk (or sphere), and each edge corresponds to a negative flow line of f joining incidence points situated on the boundaries of the disks whose corresponding vertices are related by that edge. Moreover, to define ${}^{\nu}\mathcal{M}^{x}_{x_1,\dots,x_k}(\lambda)$, such a configuration is supposed to also carry $k+1$ additional marked points $z, z_1, \dots, z_k$ so that z belongs to the boundary of the root disk (which corresponds to the root vertex of the tree) and the z_i's belong to the boundaries of some of the disks involved so that z is in the unstable manifold of the critical point x and z_i is in the stable manifold of x_i; the sum of the classes of the disks and spheres involved has to be λ. There are, of course, appropriate stability conditions and, to insure a reasonable structure for these moduli spaces, we need to use a system ν of perturbations of the pseudoholomorphic equation. The role of ghost disks (for which the corresponding J-disk is constant) is particularly important as they allow to deal not only with the transversality issues due to multiple coverings but also with the crossing of some of the incidence or marked points.

The dimension of ${}^{\nu}\mathcal{M}^{x}_{x_1,\dots,x_k}(\lambda)$ equals

$$|x| - \sum_{i=1}^{k} |x_i| + \mu(\lambda) - 1. \tag{2}$$

These moduli spaces admit natural compactifications ${}^{\nu}\overline{\mathcal{M}}^{x}_{x_1,\dots,x_k}(\lambda)$. To describe its properties, introduce the notation S for a partially ordered set of critical points (in which repetitions are allowed) which, of course, can be identified with a unique non-decreasing sequence of points. If S', S'' are two such ordered subsets, we will denote by $\langle S' \cup S'' \rangle$ the partially ordered subset made of the elements in $S' \cup S''$. Letting S be the ordered set $\{x_1, \dots, x_k\}$, for those moduli spaces of dimension 1 (which are, in fact, branched 1-dimensional manifolds with rational weights) we have:

$$\partial({}^{\nu}\overline{\mathcal{M}}^{x}_{S}(\lambda)) = \bigcup_{\substack{S=\langle S' \cup S'' \rangle \\ y,\ \lambda'+\lambda''=\lambda}} ({}^{\nu}\overline{\mathcal{M}}^{x}_{\langle S', y \rangle}(\lambda')) \times ({}^{\nu}\overline{\mathcal{M}}^{y}_{S''}(\lambda'')). \tag{3}$$

Here ∂ represents the top dimensional stratum of the boundary, the summation is taken over all $y \in \mathrm{Crit}(f)$, all partitions of S into two subsets S', S'', all splittings of λ as sum of two classes λ', λ'' and all the ways to insert y in S' so as to get $\langle S', y \rangle$ (this is relevant if y is already present in S'; the number of these ways is $\ell + 1$ where ℓ is the number of appearances of y in S', and corresponds geometrically to the choices of Morse gluings at y). We also assume here that the set S does not contain a repetition of a critical point of odd degree. This ensures that ${}^{\nu}\overline{\mathcal{M}}^{x}_{S}(\lambda)$ admits an orientation and the above formula is verified by taking into account orientations with certain signs affecting the terms on the right hand side, while the orientation on the left side is, of course, the one induced on the boundary.

By Gromov's compactness theorem, only a finite number of the moduli spaces appearing on the right hand side of the last equation are non-empty.

This equation implies that the square of the cluster differential, as defined above, vanishes so that we deduce the first part of the following result:

Theorem 2. *The map d satisfies $d^2 = 0$. The resulting homology $\mathcal{C}\ell\, H_*(L)$ does not depend, up to isomorphism, on the choices of (f, g), J and ν.*

Notice that this statement implies that $\mathcal{C}\ell\, H_*(L) \simeq \mathcal{C}\ell\, H_*(\phi(L))$ for any symplectic diffeomorphism $\phi\colon M \to M$.

Due to the commutative DGA setting, proving the invariance part of this statement is more delicate than just the usual Floer type construction for comparison morphisms. Although the comparison morphism itself between two sets of auxiliary data is defined much as in the classical way, we need a spectral sequence argument to prove that it does induce an isomorphism on homology. Since this spectral sequence plays an important role in the theory, here is its description.

Denote by ε_D the infimum of $\int_{D^2} u^*\omega$ over the set of maps $u\colon (D^2, S^1) \to (M, L)$ which are J_t-holomorphic for some t in a compact set (for instance the set of J_t's in a one-parameter family joining the two auxiliary data J and J') and non-constant. This number is strictly positive and we use it to define the *weight* of a monomial $w(x_1 \dots x_k e^{\lambda}) = k + 2\frac{\omega(\lambda)}{\varepsilon_D}$. Then consider the following *word-area* filtration of the cluster complex $\mathcal{C}\ell\,(L; J, (f, g))$:

$$F^{\ell}\mathcal{C}\ell\,(f) = \mathbb{Q}\langle m = \; x_1 x_2 \dots x_k e^{\lambda} \mid w(m) \geq \ell \rangle.$$

Notice that the cluster differential preserves this filtration. If $m \in \mathcal{C}\ell\,(L; J, (f, g))$ is a monomial, then we may write $dm = d_0 m + \sum_i m_i$ with d_0 the Morse differential and the m_i are monomials with $w(m_i) \geq w(m) + 1$.

The spectral sequence $E^r(f)$ associated to the filtration $F^{\ell}\mathcal{C}\ell\,(f)$ is the *word-area spectral sequence*. The total vector space of the term $E^1(f)$ is isomorphic to $(S(s^{-1}H_*(L; \mathbb{Q})) \otimes \Lambda)^{\wedge}$ because the 0-order differential in the spectral sequence coincides with the Morse differential. It is this spectral sequence that enables us to establish the invariance of the cluster homology.

Remark 3.1. Note that, if the minimal Maslov number of L is at least 2, one may define a simpler version of the cluster complex defined on $\mathbb{Q}\langle \mathrm{Crit}(f) \rangle$ instead of its graded symmetric algebra, with the same Novikov ring, in which the differential is defined by counting only *linear* cluster trees joining two critical points, i.e. strings of gradient flowlines and J-holomorphic discs starting at x and ending at y. In this case, the word-area spectral sequence boils down to Oh's spectral sequence [18] that was especially useful in recent works of Biran.

Remark 3.2. a. A critical point y of index 0 can *never* appear as end in a 0-dimensional, non-empty moduli space of type

$${}^{\nu}\mathcal{M}^{x}_{\dots, y, \dots}(\lambda)$$

(except for usual Morse flow lines). This is because, if there were a negative gradient flowline from a non-constant loop in L to a local minimum, there would be a one-parameter family of these (in our construction all J-spheres that might appear are attached to some non-trivial J-disk which means that we may indeed assume the loop in question to be non-constant). Similarly, for such a moduli space, x cannot be a local maximum.

b. An element in the cluster complex, $\tau \in \mathcal{C}\ell\,(L; J, (f, g))$, is written as a sum $\tau = \sum_{\lambda}(a(\lambda) + m(\lambda))e^{\lambda}$ where $a(\lambda) \in \mathbb{Q}$ and $m(\lambda)$ is a sum of words (in the letters consisting of the critical points of f) of length at least one. We call each of the terms $a(\lambda)e^{\lambda}$ with $a(\lambda) \neq 0$ *a free term* of τ. If there is a critical point x of f whose differential contains at least one free term, we say that the *complex has free terms*. Further, if the Morse index of x above is larger or equal to 1, we say that *the cluster complex has high free terms*. Notice that, due to the fact that $\mu(\lambda)$ is even, if a critical point x verifies $dx = a_0 e^{\lambda} + \cdots$, $a_0 \neq 0$, then $\mathrm{ind}_f(x)$ is even. Moreover, in view of point a., $\mathrm{ind}_f(x) \neq \dim(L)$.

c. It is not difficult to verify that if the cluster complex is acyclic, i.e. $\mathcal{C}\ell\, H_*(L) = 0$, then the cluster has free terms. If the cluster complex $\mathcal{C}\ell\,(L; J, (f, g))$ has high free terms, then there are, moreover, some J-disks with Maslov index ≤ 0.

Example 3.3. If S^1 is a circle in $\mathbb{C}$ we have

$$\mathcal{C}\ell\, H_*(S^1) = 0.$$

Indeed, take on S^1 the perfect Morse function with one minimum m and one maximum M. There exists one pseudoholomorphic disk passing through m, of Maslov index 2, with a class in Λ that we will denote by λ_0. For the maximum, we have $dM = 0$. The differential of the minimum can be seen to be given by $dm = (1 + M + s)e^{\lambda_0}$ where s is a polynomial in M without constant or linear terms. This implies the claim because in our ring we may then find a series Q so that $Q(1 + M + s) = 1$ which means $d(Qme^{-\lambda_0}) = 1$ (and of course, the cluster homology vanishes identically iff 1 is a boundary).

Example 3.4. In the absence of bubbling (for example if $\omega|_{\pi_2(M,L)} = 0$), then

$$\mathcal{C}\ell\, H_*(L; J, (f, g)) \simeq S((s^{-1}H_*(L; \mathbb{Q})) \otimes \Lambda)^{\wedge}.$$

This happens because in this case the only component of the cluster differential is provided by the usual Morse differential.

4. Fine Floer homology

The purpose of this paragraph is to introduce the *fine Floer homology*, denoted $\mathbb{F}H_*(-)$, which was announced earlier in these notes.

4.1. Coefficient ring and moduli spaces. To define the fine Floer complex

$$\mathbb{F}C(L_0, L_1, \eta; J, (f_0, g_0), (f_1, g_1))$$

we first recall that the choices of orientations and of a relative spin structure for L_0, L_1 have been made and are included in the notation L_0, L_1. Besides this, we need auxiliary data as follows. First, as before, we need an almost complex structure J, Morse–Smale pairs (f_i, g_i) on L_i and coherent choices of perturbations. We also assume the f_i in generic position with respect to the intersection points $L_0 \cap L_1$ in the sense that these intersection points are included in the unstable manifolds of critical points of index 0 of f_i. We denote by $\Gamma = \{\alpha\colon [0,1] \to M : \alpha(i) \in L_i, i = 0, 1\}$ the space of continuous paths from L_0 to L_1. Here, η is an element in Γ – its choice means that we fix a basepoint for this space. We denote by Γ_η the connected component of Γ which contains η. We denote by $I(L_0, L_1)$ the intersection points between L_0 and L_1 and we let I_η be those intersection points which, viewed as constant paths, belong to Γ_η. The generators of the fine Floer complex will be precisely the elements of I_η. Up to a shift in degrees, the resulting fine Floer homology will only depend on L_0, L_1, the connected component of η and the choice of orientations and relative spin structures of L_0, L_1.

To continue the construction, note that there are two group morphisms

$$\omega\colon \pi_1\Gamma_\eta \to \mathbb{R}, \quad \mu\colon \pi_1\Gamma_\eta \to \mathbb{Z},$$

the first given by integration of ω and the second, a Maslov index type morphism, obtained in the usual way as in Robbin–Salamon [20].

We now define

$$\mathcal{R} = (S\mathbb{Q}\langle s^{-1}(\mathrm{Crit}(f_0) \cup \mathrm{Crit}(f_1))\rangle \otimes \bar{\Lambda})^\wedge \tag{4}$$

where $\bar{\Lambda}$ is the rational group ring of $\Pi = \mathrm{Im}(\omega \times \mu)$; the completion is as in the case of the cluster complex except that we take into consideration both critical points of f_0 and of f_1.

Notice that there are injective group morphisms $\phi_i\colon \pi_2(M, L_i)/\sim\, \to \Pi$ which are obtained by first assuming that η joins the base points in L_0 and L_1 and then viewing a disk with boundary in, say, L_0 as a cylinder whose end on L_1 is constant. Therefore, if we denote by Λ_i the group ring of $\pi_2(M, L_i)/\sim$, we have injective ring morphisms $\phi_i\colon \Lambda_i \to \bar{\Lambda}$. Thus, $\mathcal{R}$ is isomorphic to the obvious completion of

$$\mathcal{C}\ell\,(L_0; J, (f_0, g_0)) \otimes \mathcal{C}\ell\,(L_1; J, (f_1, g_1)) \otimes_{\Lambda_0 \otimes \Lambda_1} \bar{\Lambda}.$$

4.2. The fine Floer complex. This is the free differential graded module over $\mathcal{R}$ given by

$$\mathbb{F}C(L_0, L_1, \eta; J, (f_0, g_0), (f_1, g_1)) = (\mathcal{R} \otimes \mathbb{Q}\langle I_\eta\rangle, d_F).$$

The grading of the elements in I_η is obtained as follows: we consider lifts $\bar{a} \in \tilde{\Gamma}_\eta$ of the points $a \in I_\eta \subset \Gamma_\eta$ where $\tilde{\Gamma}_\eta$ is the regular covering of Γ_η associated to Π (here, as usual, the last inclusion means that we view intersection points as constant paths) and we define $|a| = \mu(\bar{a})$. This grading depends on the choices of the lifts, but different choices produce isomorphic complexes.

We now describe the differential d_F. We order the critical points in $\mathrm{Crit}(f_i)$. The differential d_F verifies the Leibniz formula and for an element $a \in I_\eta$ it is of the form:

$$d_F a = \sum_{\substack{\lambda,\ b,\ k\geq 0,\ l\geq 0\\ (x_1,\dots,x_k,y_1\dots,y_l)}} w^a_{x_1,\dots,x_k,y_1,\dots,y_l;b}(\lambda)\, x_1 \dots x_k y_1 \dots y_l b e^\lambda$$

where the x_i's belong to $\mathrm{Crit}(f_0)$, the y_j's to $\mathrm{Crit}(f_1)$, they respect the order, $\lambda \in \Pi$, and finally $b \in I_\eta$.

The coefficients $w^a_{x_1,\dots,x_k,y_1,\dots,y_l;b}(\lambda) \in \mathbb{Q}$ count the number of elements in certain 0-dimensional moduli spaces $\mathcal{W}^a_{x_1,\dots,x_k,y_1,\dots,y_l;b}(\lambda)$ (again after perturbation). These moduli spaces are defined in a way similar to the $\mathcal{M}_{\dots}(\lambda)$'s of §3. The starting point consists again of trees as in §3 but the root vertex of the tree no longer corresponds to a J-disk but rather to a J-strip (as in the usual Floer theory) which relates the two intersection points $a, b \in I_\eta$. Except for codimension two phenomena, all of the other vertices correspond to pseudoholomorphic disks with boundaries on one of the L_i's. Moreover, the gradient flows appearing in the construction correspond to one of the two functions f_i. In short, the elements of these moduli spaces are cluster trees on L_0 and L_1 that originate *at finite points* of the root strip. There may be finitely many such clusters attached to the boundary of the strip. These objects are called *cluster-strips*. It is easy to see how to associate a class $\lambda \in \Pi$ to such an object.

For generic choices of J, (f_i, g_i) and after perturbation, the dimension of the moduli space $\mathcal{W}^a_{x_1,\dots,x_k,y_1,\dots,y_l;b}(\lambda)$ is:

$$|a| - |b| - \sum |x_i| - \sum |y_j| + \mu(\lambda) - 1.$$

For one-dimensional moduli spaces, there is a formula analogous to (3). As a consequence, we have:

Theorem 3. *With the notation above, we have $d_F^2 = 0$. The resulting homology is called the fine Floer homology, $I\!\!F H_*(L_0, L_1, \eta)$. Up to isomorphism* (*and a possible shift in degrees*) *it does not depend on the choices made in its construction and if $\phi\colon M \to M$ is a Hamiltonian diffeomorphism, then we have isomorphisms*

$$I\!\!F H(L_0, L_1, \eta) \simeq I\!\!F H(\phi(L_0), L_1, \eta')$$

where η' corresponds to η via the Hamiltonian diffeomorphism.

Verifying $d_F^2 = 0$ is less immediate than for the cluster differential because, besides the usual breaking of clusters and of strips, there is a third potential way for boundary points to emerge: they correspond to some cluster tree attached to a strip at some moving point p which "slides" along the boundary of the strip to one of the ends of the strip. There are two reasons that make these boundary components disappear, one is purely algebraic and is a cancellation resulting from our graded commutative setting and the other one is analytic and consists in the fact that (as remarked by Oh [16]) the usual gluing argument applies (under generic conditions) to a J-disk passing (transversally) through a and to a itself viewed as a constant strip and produces a non-constant strip with the two ends at a. In contrast with [16], our cancellation argument applies without any hypothesis of symmetry between L_0 and L_1.

4.3. Symmetrization. A particular variant of the construction of the fine Floer homology is useful in applications. To describe it, assume that L_0 equals L_1 (we denote both Lagrangian submanifolds by L), and consider a generic time-dependent Hamiltonian $H_{t\in[0,1]}$, with Hamiltonian flow $\phi_{t\in[0,1]}$, and a generic family of compatible almost complex structures $J_{t\in[0,1]}$. Take as generators of the complex the trajectories I_η^H of ϕ_t starting at time 0 and ending at time 1 on L. We may choose the generic family $J_{t\in[0,1]}$ so that $J_0 = J_1$ (we will denote this almost complex structure by J).

The *symmetric fine Floer complex* appears in this setting by additionally choosing the pair (f_0, g_0) equal to the pair (f_1, g_1) (we denote both pairs by (f, g)). Since we have a differential graded algebra multiplication map:

$$\mathcal{C}\ell\,(L; J, (f, g)) \otimes \mathcal{C}\ell\,(L; J, (f, g)) \to \mathcal{C}\ell\,(L; J, (f, g))$$

and because $J_0 = J_1 = J$, we may replace the ring

$$\mathcal{R} = (\mathcal{C}\ell\,(L_0; J_0, (f_0, g_0)) \otimes \mathcal{C}\ell\,(L_1; J_1, (f_1, g_1)) \otimes \bar{\Lambda})^\wedge$$

which appears naturally in the definition of the fine Floer complex by the ring $\hat{\mathcal{R}} = (\mathcal{C}\ell\,(L; J, (f, g)) \otimes_\Lambda \bar{\Lambda})^\wedge$. To define a differential on

$$\hat{\mathcal{R}} \otimes \mathbb{Q}\langle I_\eta^H \rangle$$

we use moduli spaces consisting of configurations in which the root pseudoholomorphic strip is replaced by a semi-cylinder satisfying the usual (J, H)-Floer type equation, with its two ends coinciding with trajectories γ, γ' and the two side boundaries lying on L.

We denote by $(\hat{\mathbb{F}}C(L; H, J, (f, g)), d_{\hat{F}})$ this symmetric fine Floer homology. If no additional notation appears the path η used in this case is just the constant path.

This homology has the same type of invariance properties as the non-symmetric version and, moreover, it is independent of the choice of $(H, J, (f, g))$ as long as it remains generic.

The advantage of this construction is that it relates directly to the cluster complex as we shall see in the following.

4.3.1. A little algebra. Consider a commutative, differential graded algebra of the form $\mathcal{A} = (SV, d)$ with V a rational vector space. Write the elements of the SV-module $SV \otimes V$ in the form

$$x_1 \dots x_k \otimes v = x_1 \dots x_k \bar{v}$$

where $v, x_i \in V$. Define the linear map

$$\alpha \colon SV \to SV \otimes V$$

by letting $\alpha(v) = \bar{v}$, $\alpha(1) = 0$ where 1 is the unit in SV and extending this map by the formula

$$\alpha(ab) = a\alpha(b) + (-1)^{|a||b|} b\alpha(a).$$

Induction on the length of words easily shows that this map is well defined and that the formula above is verified for all homogeneous monomials a, b. Explicitly, we have

$$\alpha(x_1 \dots x_k) = \sum_i (-1)^{\sigma_i} x_1 \dots \hat{x}_i \dots x_k \bar{x}_i$$

where σ_i is the product of the degree of x_i with the sum of the degrees of the x_j, $j > i$. Define the map d on $SV \otimes V$ as the unique (SV, d)-module extension of

$$d\bar{v} = \alpha(dv)$$

which verifies the standard graded Leibniz rule.

One easily checks that d so defined is a differential and that α is a chain map. Denote by

$$\widetilde{\mathcal{A}} = (SV \otimes V, d)$$

the $\mathcal{A}$-differential module constructed in this way.

4.3.2. String-strip symmetrization. We return to our geometric setting. The algebraic construction above appears in the next proposition.

Proposition 4.1. *The symmetric fine Floer homology verifies:*

$$s^{-1}\hat{\mathbb{F}}H_*(L) \simeq H_*(\widetilde{\mathcal{C}\ell}\,(L; J, (f, g))).$$

This isomorphism is proved in two steps. First, a chain complex $\mathcal{C}$ is constructed by using, instead of root semi-cylinders, (J, h)-linear clusters: they consist of pairs of critical points of the additional Morse function h related via a linear sequence of negative flow lines of h and J-disks. Of course, there are still f-clusters attached to this linear cluster. This construction is possible even when $f = h$ and in that case the resulting complex is precisely (the suspension of) $\widetilde{\mathcal{C}\ell}\,(L)$. The second step is to define a Piunikin–Salamon–Schwarz [19] map that relates the complexes $\hat{\mathbb{F}}C_*(L, H)$ and $\mathcal{C}$ and prove that it induces an isomorphism in homology.

An immediate corollary of this proposition gives the description of the symmetric fine Floer homology if no bubbling is present.

Corollary 4.2. *If $\omega|_{\pi_2(M,L)} = 0$ then*

$$\hat{I\!\!F}H_*(L) \simeq (S(s^{-1}H_*(L;\mathbb{Q})) \otimes \Lambda)^\wedge \otimes H_*(L;\mathbb{Q}).$$

Clearly, if a Lagrangian is displaceable, then both the fine Floer homology and its symmetric version vanish.

5. Applications of cluster homology

We will describe three consequences of the cluster theory developed with Cornea.

5.1. The Gromov–Sikorav problem. As a first consequence, we analyze a plausible conjecture going back to Gromov's original paper [10] on pseudoholomorphic curves, stated orally by Sikorav in the late eighties in the following way: given any compact Lagrangian submanifold of $\mathbb{C}^n$, there is a holomorphic disk passing through each point of L.

Corollary 5.1. *Let $L \subset M$ be a compact, orientable, relative spin Lagrangian submanifold of any symplectic manifold M. Assume that $\hat{I\!\!F}H_*(L) = 0$ (for example if L is displaceable by a Hamiltonian isotopy). Then, for any generic almost complex structure J compatible with the symplectic form, one of the following holds:*

i. *There are J-holomorphic disks with boundary on L passing through a dense subset of points of L.*

ii. *The cluster complex $\mathcal{C}\ell\,(L, J, f)$ has high free terms for some Morse function f with a single local minimum and a single local maximum (in particular, there are J-disks of non-positive Maslov index).*

Proof. Assume that for any Morse function with a single local minimum and local maximum, the associated cluster complex does not have high free terms. Fix such a function f and denote its minimum by m. By Proposition 4.1, $s^{-1}\hat{I\!\!F}H_*(L)$ is isomorphic to $H_*(\widetilde{\mathcal{C}\ell}\,(L, J; (f, g)))$, which means that $H_*(\widetilde{\mathcal{C}\ell}\,(L, J; (f, g)))$ vanishes. Notice that if $d\bar{m} \neq 0$, then we also have $dm \neq 0$ in the cluster complex. Assume now that

$$\bar{m} \in \widetilde{\mathcal{C}\ell}\,(L, J; (f, g))$$

is a cycle. Using Remark 3.2, we see that $\bar{m}$ can be a boundary only if $\mathcal{C}\ell\,(-)$ has free terms: indeed, by this remark, the only possible primitive of $\bar{m}$ must have the form $\tau\bar{m}$ where τ is a primitive of the unit 1 in the cluster complex. This means that there is a free term in some dx for some $x \in \mathrm{Crit}(f)$. Once again by Remark 3.2, the index of this x cannot be n and it cannot be strictly between 0 and n by our assumption. Therefore, $x = m$ and $dm \neq 0$.

The fact that dm is different from 0 means that there exists a non-empty moduli space ${}^{\nu}\mathcal{M}^{m}_{x_1,\ldots,x_k}(\lambda)$ of dimension 0. But for a cluster tree to originate at the minimum

m, the root disk must go through m. As we may use a different function f to place m in any generic point in L, this implies the claim. □

The dichotomy in the statement of the previous corollary can be sometimes resolved by homological restrictions.

Corollary 5.2. *Suppose that L is orientable, relative spin and that $H_{2k}(L;\mathbb{Q}) = 0$ for $2k \notin \{0, \dim(L)\}$. If $\hat{\mathbb{F}}H_*(L) = 0$, then L verifies* (i) *of Corollary* 5.1 *above.*

Example 5.3. The homological restriction in the corollary above is serious but, still, there are many examples of such manifolds: $S^1 \times S^{n-1}$ and its connected sums with itself perhaps provide the simplest examples.

Using symmetrization, we can improve these results in the displaceable case by bounding from above the area of the disks detected in terms of the displacing energy. This upper bound and Gromov's compactness theorem then imply:

Corollary 5.4. *Suppose that the relative spin, orientable Lagrangian submanifold L is displaceable by a Hamiltonian isotopy and let $E(L)$ be its Hofer displacement energy. Any ω-tame almost complex structure J has the property that one of the following is true:*

i. *For any point $x \in L$ there exists a J-holomorphic disk of symplectic area at most $E(L)$ whose boundary rests on L and which passes through x.*

ii. *There exists a J-disk of Maslov index at most*

$$2 - \min\{2k \in \mathbb{N}\backslash\{0, \dim(L)\} : H_{2k}(L;\mathbb{Q}) \neq 0\}$$

and of symplectic area at most $E(L)$.

For a Lagrangian submanifold $L \subset (M,\omega)$, define, as in Barraud–Cornea [4], its real, or relative, Gromov radius as the supremum $r(L)$ of the positive constants r so that there is a symplectic embedding of the standard ball $(B(r),\omega_0) \overset{e}{\hookrightarrow} (M,\omega)$ with the property that $e^{-1}(L) = \mathbb{R}^n \cap B(r)$. Define its real Gromov capacity $c_G(L)$ as $\pi r^2(L)/2$.

Corollary 5.5. *If an orientable, relatively spin Lagrangian with $H_{2k}(L;\mathbb{Q}) = 0$ for $2k \notin \{0, \dim(L)\}$ is displaceable, then*

$$E(L) \geq c_G(L).$$

Actually, Barraud–Cornea [4] introduced an even more relative notion of Gromov capacity: if L_0, L_1 are two Lagrangian submanifolds, one may define

$$c_G(L_0, L_1) = \pi r(L_0, L_1)^2/2$$

where $r(L_0, L_1)$ is the supremum of the radii of symplectic balls, disjoint from L_1, whose real parts rest on L_0. They show in [4] that, if L_1 is the Hamiltonian image of L_0, and assuming that we are in a case where real bubbling off does not occur, an analogous energy–capacity inequality holds: $E \geq c_G(L_0, L_1)$ where E is the Hofer energy of the pair (L_0, L_1).

5.2. Constraints on Maslov indices. The cluster complex setting provides straightforward proofs of various constraints regarding Maslov indices of Lagrangian submanifolds. See the section on applications in [5]. However, it has not yet provided a proof of the Audin conjecture, except in dimension 2. Such an application is still premature before the algebraic operations on cluster homology have been well understood.

5.3. Detection of periodic orbits. In the presence of an orientable relative spin pair of Lagrangian submanifolds L, L', we show that, by replacing in the definition of the clustered moduli spaces one (and only one) of the J-disks by a pseudoholomorphic cylinder with one boundary on L and the other boundary on L', one can construct a chain map:

$$\mathrm{cyl} : \mathcal{C}\ell\,(L) \to \mathcal{C}\ell\,(L) \otimes \mathcal{C}\ell\,(L') \otimes \Lambda_{\Phi_0}$$

where Λ_{Φ_0} is an appropriate Novikov ring.

This map induces a morphism in homology whose non-triviality is used to detect the existence of non-constant periodic orbits of Hamiltonian diffeomorphisms that *separate* L from L'. These results, obtained in [5], generalize those in [9]. When L and L' are disjoint sections in a cotangent bundle, the non-triviality is easy to detect. But, in a general symplectic manifold, such a non-triviality could be detected by making use of a specific Hamiltonian H that does have periodic orbits in the prescribed classes, and using the Albers map from the symplectic Floer (or Floer–Novikov) homology of the ambient space (using H) to each of the cluster homologies of L and L'. The gluing of the two half cylinders at each of the orbits of H would then, plausibly, provide a non-trivial realisation of the above morphism, that could then be applied to prove the existence of orbits of any other separating Hamiltonian.

5.4. Concluding remarks on cluster homology. What remains to be done on the cluster theory is: (1) to establish the analysis of its moduli spaces on a solid ground, and (2) make it computable in non-trivial examples. Although (1) has not been entirely written down at the time of submitting these notes, it seems very likely that the transversalities issues fit very well in the Hofer–Wysocki–Zehnder polyfold setting so that, hopefully, (1) will be accomplished shortly after the HWZ approach fully appears in print. A second scheme to solve similar transversality problems in the absolute case has been very recently suggested by Cieliebak–Mohnke and could be likely adapted to the cluster context thanks to the existence of a Donaldson symplectic hypersurface in the complement of any Lagrangian L (established by Auroux–Gayet–Mohsen in [3]) whose role would be to stabilize the holomorphic discs (one would then use the abstract space of non-embedded cluster trees as the new parameter space in defining the dependence of the almost complex structure J). The first step in (2) will be to establish a formula for the cluster homology of the surgery on two Lagrangian submanifolds intersecting transversally. Such a formula requires understanding how to resolve the singularities of holomorphic surfaces with corners at the intersecting points – this was first explored in an appendix of FOOO [8].

Cluster homology may also play a role in the development of the new subject of relative (or Lagrangian) symplectic field theory.

6. The emerging field of real symplectic topology

Real symplectic topology is the study of triples (M, ω, c_M) where (M, ω) is a symplectic manifold and $c_M : M \to M$ is an anti-symplectic involution. It is easy to see that the fixed point set $\mathbb{R}M$ of c_M, called the real part of M, is a Lagrangian submanifold (when it is not empty). It is shown in [24] (see also [8]) that the space $\mathbb{R}\mathcal{J}(M)$ of ω-tame almost complex structures J on M for which c_M is anti-holomorphic is as generic as one could hope: it is both generic for the study of the full space of J-holomorphic curves in M, as well as for the subspace of *real* J-curves, i.e. the space of J-curves that are c_M-invariant.

Note that the results of the preceding sections always assumed that the Lagrangian submanifolds are orientable and relative spin. In real symplectic topology, the Lagrangian submanifold $\mathbb{R}M$ is often not orientable, for instance $\mathbb{R}P^2 \subset \mathbb{C}P^2$. Moreover, the "doubling construction" that assigns a c_M-invariant J-holomorphic 2-sphere to a disk with boundary on L only uses one of the anti-symplectic involutions of the 2-sphere, namely the reflection through the equator (leaving aside the antipodal map). For these reasons, there seems to be no point in studying real symplectic topology solely from the methods defined in the preceding sections.

A very interesting and recent development, due to Welschinger [24], provides a well-defined Gromov–Witten type invariant in real 4-dimensional symplectic manifolds for the space of real rational J-curves (for a generic J in $\mathbb{R}\mathcal{J}(M)$) in a class d passing through the right number ℓ of points in order to cut its dimension down to zero, assuming that these points are invariant under c_M. Denote by δ the number of double points of such a curve. Both ℓ and δ are topological invariants depending only on d. Let $r \leq \ell$ be the number of points belonging to $\mathbb{R}M$. For such a real rational J-curve, let $0 \leq m \leq \delta$, called the *mass*, be the number of its real isolated double points (i.e. those which belong to $\mathbb{R}M$ and which are isolated in $\mathbb{R}M$, being the intersection of complex conjugated branches of the curve). Finally, let $n_d(m)$ be the number of such real curves in class d passing through that generic set of points with m real isolated double points. Then the number $\chi_r^d = \sum_{m=0}^{\delta}(-1)^m n_d(m)$ is independent of both the choice of generic $J \in \mathbb{R}\mathcal{J}(M)$ and the c_M-invariant set of points, as long as the number r remains unchanged. Formally summing over r provides an invariant of the deformation class of (M, ω, c_M). Note that, obviously, χ_r^d is then a lower bound for the number of real rational curves, in particular χ_ℓ^d is a lower bound for the number of real rational curves in class d passing through ℓ real points. These numbers have been computed in various cases, notably by Mikhalkin for toric real surfaces (see his contribution in these proceedings).

These results have been extended to higher dimensions by Welschinger in [25]. Basically, these invariants are obtained by a procedure which has some similarities with the cluster complex, since they are extracted by gluing together various moduli spaces in an appropriate way. However, they do not point in the direction of a cluster homology, but rather suggest that there might exist some form of real quantum product related to mass. That is to say: the Lagrangian submanifolds occuring as real loci of real symplectic manifolds seem to be special enough so that one could overcome the real bubbling off phenomenon by gluing appropriate moduli spaces and extracting the desired invariants directly from these "clustered" moduli spaces much like in the complex (non-relative) case, without having to introduce a Floer or cluster complex.

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