Gromov–Witten invariants and moduli spaces of curves

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Abstract. The purpose of this note is to explain how much information of Gromov–Witten invariants of compact symplectic manifolds are determined by the geometry of moduli spaces of curves. In the case when a manifold has semisimple quantum cohomology, we believe that the information obtained this way might determine all higher genus Gromov–Witten invariants and could be used to study the Virasoro conjecture of Eguchi–Hori–Xiong and S. Katz.

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1. Introduction

Let V be a compact symplectic manifold with an almost complex structure which is compatible with its symplectic structure. Gromov–Witten invariants of V are certain intersection numbers on the moduli spaces of stable pseudo-holomorphic curves in V. Such invariants do not depend on the choice of the almost complex structure and are therefore symplectic invariants. The generating functions of Gromov–Witten invariants satisfy many interesting partial differential equations. A large class of such equations come from the study of the moduli spaces of stable curves $\overline{\mathcal{M}}_{g,k}$ which were introduced by Deligne and Mumford in [7]. There are natural geometric ways to construct cohomology classes on $\overline{\mathcal{M}}_{g,k}$. These classes generate a ring called the tautological ring. There is a canonical way to produce differential equations for generating functions of Gromov–Witten invariants from relations in the tautological ring of $\overline{\mathcal{M}}_{g,k}$. Such equations hold for the Gromov–Witten theory of all compact symplectic manifolds, and are therefore called universal equations. These equations can be used to compute Gromov–Witten invariants and study other properties of Gromov–Witten theory like the Virasoro conjecture of Eguchi–Hori–Xiong and S. Katz. The Virasoro conjecture predicts that the generating functions of Gromov–Witten invariants of smooth projective varieties are annihilated by an infinite sequence of differential operators which form a half branch of the Virasoro algebra. This is a generalization of Witten's conjecture, which was proved by Kontsevich, that the generating function of intersection numbers on $\overline{\mathcal{M}}_{g,k}$ is a τ-function of the KdV hierarchy. Moreover

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universal equations also hold for generating functions of intersection numbers on moduli spaces of spin curves. These equations can be used to study the generalized Witten conjecture which predicts that such generating functions are τ-functions of Gelfand–Dickey hierarchies.

On the other hand, it is a highly non-trivial problem to find explicit relations in the tautological ring of $\overline{\mathcal{M}}_{g,k}$, especially when the genus g is high. Knowledge of Gromov–Witten invariants of compact symplectic manifolds can be used to study this problem. We believe that relations in tautological rings are manifested in the properties of Gromov–Witten invariants of various compact symplectic manifolds. In a certain sense, one may think that Gromov–Witten theory of each compact symplectic manifold V gives a representation of the tautological ring of $\overline{\mathcal{M}}_{g,k}$. A good understanding of Gromov–Witten invariants of a class of manifolds might be sufficient to determine many relations in tautological rings.

In this expository article we will explain how much we know about universal equations and how much information can be obtained from such equations.

2. Tautological relations and universal equations

Let V^{2d} be a compact symplectic manifold with symplectic form ω and almost complex structure J such that the bilinear form $\omega(\cdot, J\cdot)$ defines an Riemannian metric on V. For simplicity, we assume that $H^{\text{odd}}(V;\mathbb{C}) = 0$. For each $A \in H_2(V;\mathbb{Z})$ and $g, k \in \mathbb{Z}_{\geq 0}$, let $\overline{\mathcal{M}}_{g,k}(V, A)$ be the moduli space of stable maps whose elements are of the form $(C; x_1, \dots, x_k; f)$ where C is a genus-g complex curve with at most nodal singularities, $x_1, \dots, x_k \in C$ are distinct smooth points which are called marked points, and f is a pseudo-holomorphic map from C to V such that $f_*[C] = A$ and f is stable in the sense that there is no infinitesimal deformation of f without moving marked points and their images. For each $i \in \{1, 2, \dots, k\}$, let $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,k}(V, A) \longrightarrow V$ be the evaluation map defined by

$$\mathrm{ev}_i(C; x_1, \dots, x_k; f) := f(x_i)$$

and let $E_i \longrightarrow \overline{\mathcal{M}}_{g,k}(V, A)$ be the tautological line bundle whose geometric fiber over $(C; x_1, \dots, x_k; f)$ is given by $T^*_{x_i}C$. For any $\gamma_1, \dots \gamma_k \in H^*(V;\mathbb{C})$ and $n_1, \dots, n_k \in \mathbb{Z}_{\geq 0}$, the associated Gromov–Witten invariant is defined to be

$$\langle \tau_{n_1}(\gamma_1) \dots \tau_{n_k}(\gamma_k) \rangle_{g,A} := \int_{[\overline{\mathcal{M}}_{g,k}(V,A)]^{\text{virt}}} \bigcup_{i=1}^{k} (c_1(E_i)^{n_i} \cup \mathrm{ev}_i^*(\gamma_i)),$$

where $[\overline{\mathcal{M}}_{g,k}(V, A)]^{\text{virt}}$ is the virtual fundamental class of degree

$$2\{(d-3)(1-g) + c_1(V)(A) + k\}$$

(cf. [30], [31], and [3]).

To define the generating functions, we need to fix a basis $\{\gamma_1, \ldots, \gamma_N\}$ of $H^*(V; \mathbb{C})$ with γ_1 equal to the identity of the cohomology ring of V. For each symbol $\tau_n(\gamma_\alpha)$ we associate a parameter t_n^α. The collection of all such parameters is denoted by

$$t = (t_n^\alpha \mid n \in \mathbb{Z}_{\geq 0},\ \alpha = 1, \ldots, N).$$

We can think of these parameters as coordinates on an infinite dimensional vector space called the *big phase space*. The subspace $\{t \mid t_n^\alpha = 0 \text{ if } n > 0\}$ is called the *small phase space*. Note that the small phase space can be canonically identified with $H^*(V; \mathbb{C})$. We also need the Novikov ring which is the completion of the multiplicative ring generated by monomials $q^A := d_1^{a_1} \ldots d_r^{a_r}$ over the ring of rational numbers, where $\{d_1, \ldots, d_r\}$ is a fixed basis of $H_2(V; \mathbb{Z})$ and $A = \sum_{i=1}^r a_i d_i$. The generating function of genus-g Gromov–Witten invariants is then defined by

$$F_g(t) := \sum_{k \geq 0} \frac{1}{k!} \sum_{\substack{\alpha_1, \ldots, \alpha_k \\ n_1, \ldots, n_k}} t_{n_1}^{\alpha_1} \ldots t_{n_k}^{\alpha_k} \sum_{A \in H_2(V; Z)} q^A \langle \tau_{n_1}(\gamma_{\alpha_1}) \ldots \tau_{n_k}(\gamma_{\alpha_k}) \rangle_{g,A}.$$

The function F_g is understood as a formal power series of t with values in the Novikov ring.

For $k,\ g \geq 0$, define a k-tensor $\langle\!\langle \underbrace{\cdots}_{k} \rangle\!\rangle_g$ to be the k-th covariant derivative of F_g with respect to the trivial connection on the big phase space. More precisely,

$$\langle\!\langle \mathcal{W}_1 \ldots \mathcal{W}_k \rangle\!\rangle_g := \sum_{m_1, \alpha_1, \ldots, m_k, \alpha_k} f_{m_1,\alpha_1}^1 \cdots f_{m_k,\alpha_k}^k \frac{\partial^k}{\partial t_{m_1}^{\alpha_1} \ldots \partial t_{m_k}^{\alpha_k}} F_g \tag{1}$$

for vector fields $\mathcal{W}_i = \sum_{m,\alpha} f_{m,\alpha}^i \frac{\partial}{\partial t_m^\alpha}$ where $f_{m,\alpha}^i$ are functions on the big phase space. This tensor is called the *k-point* (*correlation*) *function*. We will identify $\frac{\partial}{\partial t_n^\alpha}$ with $\tau_n(\gamma_\alpha)$ and set $\tau_0(\gamma_\alpha) = \gamma_\alpha$ and $\tau_n(\gamma_\alpha) = 0$ if $n < 0$. We call γ_α a *primary vector field*, and $\tau_n(\gamma_\alpha)$ a *descendant vector field* with descendant level n. We use τ_+ and τ_- to denote the operator which shift the level of descendants, i.e.

$$\tau_\pm \Big(\sum_{n,\alpha} f_{n,\alpha} \tau_n(\gamma_\alpha) \Big) = \sum_{n,\alpha} f_{n,\alpha} \tau_{n\pm 1}(\gamma_\alpha)$$

where $f_{n,\alpha}$ are functions on the big phase space. Let

$$\eta_{\alpha\beta} = \int_V \gamma_\alpha \cup \gamma_\beta$$

be the intersection form on $H^*(V, \mathbb{C})$. We will use $\eta = (\eta_{\alpha\beta})$ and $\eta^{-1} = (\eta^{\alpha\beta})$ to lower and raise indices. For example,

$$\gamma^\alpha := \eta^{\alpha\beta} \gamma_\beta.$$

Here we are using the summation convention that repeated indices (in this formula, β) should be summed over their entire ranges.

When V is a point, the moduli space $\overline{\mathcal{M}}_{g,k}(\{\mathrm{pt}\}, 0)$ is exactly the moduli space of genus-g stable curves with k-marked points constructed by Deligne and Mumford [7]. This space is usually denoted by $\overline{\mathcal{M}}_{g,k}$. Let ψ_i be the first Chern class of the tautological line bundle E_i over $\overline{\mathcal{M}}_{g,k}$. These classes are called ψ-classes. There are natural forgetful maps $\overline{\mathcal{M}}_{g,k+n} \longrightarrow \overline{\mathcal{M}}_{g,k}$ which forgets the last n marked points of a stable curve. There are also two types of natural gluing maps between moduli spaces of stable curves. The first type is $\overline{\mathcal{M}}_{g_1,k_1+1} \times \overline{\mathcal{M}}_{g_2,k_2+1} \longrightarrow \overline{\mathcal{M}}_{g_1+g_2,k_1+k_2}$ which glues the last marked points of two stable curves to form a new stable curve. The second type is $\overline{\mathcal{M}}_{g,k+2} \longrightarrow \overline{\mathcal{M}}_{g+1,k}$ which glues the last two marked points on a stable curve to form a new stable curve. The tautological ring of $\overline{\mathcal{M}}_{g,k}$, denoted by $R^*(\overline{\mathcal{M}}_{g,k})$, is the smallest $\mathbb{Q}$-subalgebra of the Chow ring of $\overline{\mathcal{M}}_{g,k}$ which contains all ψ-classes and is closed under the push-forward of forgetting maps and gluing maps. There is also a natural stratification of $\overline{\mathcal{M}}_{g,k}$ which is labeled by dual graphs of stable curves. The dual graph G of a stable curve C is defined in the following way: The vertices of G are irreducible components of C labeled by their genera. Two vertices are connected by an edge if the corresponding irreducible components intersects at a node of C. Therefore edges of G are one to one correspondent to nodes of C. Finally each marked point of C gives a tail of G emanating from a vertex whose corresponding irreducible component of C contains this marked point. The set of all genus-g stable curves with k marked points whose dual graph is isomorphic to G is denoted by $\mathcal{M}_{g,k}(G)$. The closure of this space, i.e. $\overline{\mathcal{M}}_{g,k}(G)$, gives a class in the tautological ring $R^*(\overline{\mathcal{M}}_{g,k})$. Such classes are called boundary classes. We will call any relation among powers of ψ-classes and boundary classes a *tautological relation*.

There is a natural map

$$\mathrm{St}\colon \overline{\mathcal{M}}_{g,k}(V, A) \longrightarrow \overline{\mathcal{M}}_{g,k}$$

which forgets the map f in $(C; x_1, \dots, x_k; f) \in \overline{\mathcal{M}}_{g,k}(V, A)$ and stabilizes

$$(C; x_1, \dots, x_k)$$

by squeezing unstable components to points. A tautological relation on $\overline{\mathcal{M}}_{g,k}$ can be pulled back to $\overline{\mathcal{M}}_{g,k}(V, A)$ which in turn gives relations between various Gromov–Witten invariants of V. Such relations can be described by partial differential equations for $F_0, \dots, F_g$. Equations obtained in this way hold for all compact symplectic manifolds and therefore are called *universal equations*. It is very convenient to write universal equations as equations for tensors $\langle\langle \cdots \rangle\rangle_g$.

There is a canonical way to translate tautological relations to universal equations. A boundary class $\overline{\mathcal{M}}_{g,k}(G)$ in a tautological relation corresponds to a product of correlation functions in the universal equation according to the following rules: Each tail of G is assigned an arbitrary vector field $\mathcal{W}_i$ on the big phase space. Each edge of G is assigned a pair of primary vector fields γ_α and γ^α, one for each half edge. Each genus-h vertex of G is assigned a correlation function of the form

$$\langle\!\langle \mathcal{W}_{i_1} \dots \mathcal{W}_{i_m}\ \gamma_{\alpha_1} \dots \gamma_{\alpha_n}\ \gamma^{\beta_1} \dots \gamma^{\beta_p} \rangle\!\rangle_h$$

if the corresponding tails and half edges are connected to this vertex. Then the boundary class $\overline{\mathcal{M}}_{g,k}(G)$ is assigned the product of all correlation functions associated with all vertices of G. The coefficient of this function in the universal equation should be the coefficient of $\overline{\mathcal{M}}_{g,k}(G)$ in the tautological relation divided by the number of elements in the automorphism group of G. To interpret ψ-classes in a tautological relation, an operator T was introduced in [33] which is defined by

$$T(\mathcal{W}) := \tau_+(\mathcal{W}) - \langle\!\langle \mathcal{W}\,\gamma^\alpha \rangle\!\rangle_0 \gamma_\alpha$$

for any vector field $\mathcal{W}$. If $\psi_i^{n_i}$ is involved in a tautological relation, then in the corresponding universal equation, $\mathcal{W}_i$ should be replaced by $T^{n_i}(\mathcal{W}_i)$. The reason for this is that the cohomology class $c_1(E_i) - St^*(\psi_i)$ on $\overline{\mathcal{M}}_{g,k}(V, A)$ is represented by a cycle consisting of elements $(C; x_1, \dots, x_k; f)$ where C has a genus-0 component with only one marked point, the i-th marked point, and only one node connecting this genus-0 component to another component of C (cf. [17] and [28]).

The simplest tautological relation is $\psi_1 = 0$ on $\overline{\mathcal{M}}_{0,3}$ since $\dim \overline{\mathcal{M}}_{0,3} = 0$. This relation can be translated into the universal equation

$$\langle\!\langle T(\mathcal{W}_1)\, \mathcal{W}_2\, \mathcal{W}_3 \rangle\!\rangle_0 = 0$$

for arbitrary vector fields $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$. This equation is called the genus-0 topological recursion relation. It was observed in [44] that derivatives of this equation gives the generalized WDVV equation:

$$\langle\!\langle \mathcal{W}_1\, \mathcal{W}_2\, \gamma^\alpha \rangle\!\rangle_0 \langle\!\langle \gamma_\alpha\, \mathcal{W}_3\, \mathcal{W}_4 \rangle\!\rangle_0 = \langle\!\langle \mathcal{W}_1\, \mathcal{W}_3\, \gamma^\alpha \rangle\!\rangle_0 \langle\!\langle \gamma_\alpha\, \mathcal{W}_2\, \mathcal{W}_4 \rangle\!\rangle_0$$

for arbitrary vector fields $\mathcal{W}_1, \dots, \mathcal{W}_4$. Because of this equation, we can define an associative product

$$\mathcal{W}_1 \circ \mathcal{W}_2 := \langle\!\langle \mathcal{W}_1\, \mathcal{W}_2\, \gamma^\alpha \rangle\!\rangle_0 \gamma_\alpha \tag{2}$$

for any vector fields $\mathcal{W}_1$ and $\mathcal{W}_2$ on the big phase space. This product is called the *quantum product on the big phase space* (cf. [33]). When restricted to the tangent bundle of the small phase space, this product is exactly the product for the quantum cohomology of V. Unlike the quantum product on the small phase space which has an identity element γ_1, the quantum product on the big phase space does not have an identity. Let

$$\mathcal{S} := -\sum_{m,\alpha} \tilde{t}_m^\alpha \tau_{m-1}(\gamma_\alpha)$$

be the *string vector field*, where $\tilde{t}_m^\alpha := t_m^\alpha - \delta_{m,1}\delta_{\alpha,1}$. The *string equation* has the form

$$\langle\langle \mathcal{S} \rangle\rangle_g = \frac{1}{2}\delta_{g,0}\eta_{\alpha\beta}t_0^\alpha t_0^\beta$$

for $g \geq 0$. By the second derivative of the genus-0 string equation, we can rewrite the operator T as

$$T(\mathcal{W}) = \tau_+(\mathcal{W}) - \mathcal{S} \circ \tau_+(\mathcal{W}).$$

Therefore T measures the difference between $\mathcal{S}$ and the identity of the quantum product on the big phase space, which actually does not exist by our definition of the quantum product. This gives an algebraic interpretation for the operator T.

Universal equations are very powerful for a class of manifolds for which the quantum product defined by equation (2) is semisimple in the following sense: Define the *Euler vector field* on the big phase space by

$$\mathcal{X} := -\sum_{m,\alpha} (m + b_\alpha - b_1 - 1)\,\tilde{t}_m^\alpha\,\tau_m(\gamma_\alpha) - \sum_{m,\alpha,\beta} \mathcal{C}_\alpha^\beta \tilde{t}_m^\alpha\,\tau_{m-1}(\gamma_\beta),$$

where $(\mathcal{C}_\alpha^\beta)$ is the matrix of multiplication by $c_1(V)$ in the ordinary cohomology ring of V with respect to the basis $\{\gamma_1, \dots \gamma_N\}$ and $b_\alpha = \frac{1}{2}(\dim(\gamma_\alpha) - d + 1)$ for a compact symplectic manifold V of dimension $2d$. In case that V is a smooth projective variety, we can choose $\gamma_\alpha \in H^{p_\alpha, q_\alpha}(V)$ and b_α should be modified as $b_\alpha = p_\alpha - \frac{1}{2}(d-1)$. This modification is necessary for formulating the Virasoro conjecture below. The Euler vector field satisfies the following *quasi-homogeneity equation*

$$\langle\langle \mathcal{X} \rangle\rangle_g = (3-d)(1-g)F_g + \frac{1}{2}\delta_{g,0}\sum_{\alpha,\beta}\mathcal{C}_{\alpha\beta}t_0^\alpha t_0^\beta - \frac{1}{24}\delta_{g,1}\int_V c_1(V) \cup c_{d-1}(V).$$

The quantum multiplication by $\mathcal{X}$ is an endomorphism on the vector space spanned by primary vector fields on the big phase space. We say that V has *semisimple* quantum cohomology if this endomorphism has distinct eigenvalues at generic points. Since our definition of quantum product on the big phase space coincides with usual quantum product when restricted to the small phase space, and the restriction of $\mathcal{X}$ to the small phase space also coincides with the Euler vector field for usual quantum cohomology, this definition of semisimplicity is a generalization of the semisimple condition used by Dubrovin in [10]. Under the semisimplicity assumption, there exists vector fields $\mathcal{E}_1, \dots, \mathcal{E}_N$ on the big phase space which are linear combinations of primary vector fields such that

$$\mathcal{X} \circ \mathcal{E}_i = u_i \mathcal{E}_i, \quad \mathcal{E}_i \circ \mathcal{E}_j = \delta_{ij}\mathcal{E}_i$$

for every i and j where u_i are functions on the big phase space (cf. [36]). These vector fields are called *idempotents* of the quantum product on the big phase space. When restricted to the small phase space, $(u_1, \dots, u_N)$ is precisely the *canonical coordinate system* for the semisimple Frobenius structure studied in [10]. On the big phase space these functions are not sufficient to give a nice coordinate system. The idempotents will play the role of the canonical coordinate system. We refer to [36] for more properties of the idempotents on the big phase space.

It is well known that the quantum cohomology on the small phase space defines a Frobenius manifold structure (cf. [10]). However the quantum product on the big phase space defined be equation (2) does not give an infinite dimensional Frobenius manifold structure. To obtain a Frobenius structure on the big phase space, we should modify the definition of the quantum product in the following way: For any vector fields $\mathcal{W}_1$ and $\mathcal{W}_2$ on the big phase space define

$$\mathcal{W} \diamond \mathcal{V} := \sum_{k=0}^{\infty} \langle\langle \tau_-^k(\mathcal{W}) \ \tau_-^k(\mathcal{V}) \ \gamma^\alpha \rangle\rangle_0 \ T^k(\gamma_\alpha).$$

This product is commutative and associative. The associativity follows from the fact that $\langle\langle \{\tau_-^k T^l(\gamma_\alpha)\} \ \mathcal{W} \ \mathcal{V} \rangle\rangle_0 = 0$ if $k \neq l$. Moreover for any primary vector fields $\mathcal{W}$ and $\mathcal{V}$,

$$T^k(\mathcal{W}) \diamond T^l(\mathcal{V}) = \delta_{kl} \ T^k(\mathcal{W} \circ \mathcal{V}).$$

This product has an identity

$$\hat{\mathcal{S}} := \sum_{k=0}^{\infty} \langle\langle \mathcal{S} \mathcal{S} \gamma^\alpha \rangle\rangle_0 T^k(\gamma_\alpha) = \sum_{k=0}^{\infty} T^k(\mathcal{S} \circ \mathcal{S}).$$

Define inner product on the big phase space by

$$(\mathcal{W}, \ \mathcal{V}) := \sum_{k=0}^{\infty} \langle\langle \mathcal{S} \ \tau_-^k(\mathcal{W}) \ \tau_-^k(\mathcal{V}) \rangle\rangle_0.$$

This is a symmetric non-degenerate bilinear form on the big phase space. Moreover,

$$(T^m(\gamma_\alpha), \ T^n(\gamma_\beta)) = \delta_{mn} \eta_{\alpha\beta}$$

for any $m, n \in \mathbb{Z}_{\geq 0}$ and $1 \leq \alpha, \beta \leq N$. The product "$\diamond$" is compatible with this inner product in the sense that

$$(\mathcal{W}_1 \diamond \mathcal{W}_2, \ \mathcal{W}_3) = (\mathcal{W}_1, \ \mathcal{W}_2 \diamond \mathcal{W}_3)$$

for all vector fields $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$ and therefore define a Frobenius algebra structure on tangent spaces of the big phase space. Note that if the quantum product "$\circ$" is semisimple in the above sense and $\mathcal{E}_1, \ldots, \mathcal{E}_N$ are the idempotents of "$\circ$", then vector fields

$$\{T^n(\mathcal{E}_i) \mid n \in \mathbb{Z}_{\geq 0}, \ i = 1, \ldots, N\}$$

are idempotents for "$\diamond$" and give a frame for the tangent bundle of the big phase space. This also justifies the notion of semisimplicity introduced above. We also note that this frame is not commutative with respect to the Lie bracket and therefore does not come from coordinate vector fields for any coordinate system on the big phase space. Since the product "$\circ$" is much easier to use than the product "$\diamond$", we will not use the product "$\diamond$" in this paper.

In [27] and [42], the WDVV equation has been used to compute genus-0 Gromov–Witten invariants. Higher genus analogues of genus-0 universal equations can also be used to compute higher genus Gromov–Witten invariants. Lets first see what happens for genus-1 and genus-2 cases.

The genus-1 analogue of the genus-0 topological recursion relation is the following

$$\langle\langle T(\mathcal{W})\rangle\rangle_1 = A_0(\mathcal{W}) := \frac{1}{24}\langle\langle \mathcal{W}\,\gamma^\alpha\,\gamma_\alpha\rangle\rangle_0.$$

This equation is translated from the following tautological relation on $\overline{\mathcal{M}}_{1,1}$ (cf. [8]):

$$\psi_1 = \frac{1}{12}\{\text{boundary stratum of } \overline{\mathcal{M}}_{1,1}\}.$$

The genus-1 analogue of the WDVV equation is the following equation discovered by Getzler [16]: For any vector fields $\mathcal{W}_1, \ldots, \mathcal{W}_4$,

$$\begin{aligned}\sum_{\sigma\in S_4} \big\{&4\langle\langle\{\mathcal{W}_{\sigma(1)}\circ\mathcal{W}_{\sigma(2)}\circ\mathcal{W}_{\sigma(3)}\}\,\mathcal{W}_{\sigma(4)}\rangle\rangle_1 \\ &- 3\langle\langle\{\mathcal{W}_{\sigma(1)}\circ\mathcal{W}_{\sigma(2)}\}\{\mathcal{W}_{\sigma(3)}\circ\mathcal{W}_{\sigma(4)}\}\rangle\rangle_1 \\ &+ \langle\langle\{\mathcal{W}_{\sigma(1)}\circ\mathcal{W}_{\sigma(2)}\}\,\mathcal{W}_{\sigma(3)}\,\mathcal{W}_{\sigma(4)}\,\gamma^\alpha\rangle\rangle_0\langle\langle\gamma_\alpha\rangle\rangle_1 \\ &-2\langle\langle\mathcal{W}_{\sigma(1)}\,\mathcal{W}_{\sigma(2)}\,\mathcal{W}_{\sigma(3)}\,\gamma^\alpha\rangle\rangle_0\langle\langle\{\gamma_\alpha\circ\mathcal{W}_{\sigma(4)}\}\rangle\rangle_1\big\} = B_0(\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4)\end{aligned}$$

where B_0 is a symmetric 4-tensor which consists of 3 terms involving only genus-0 data (see, for example, [36] for the precise form of B_0 where we used notation G_0 for this tensor). This equation corresponds to a tautological relation on $\overline{\mathcal{M}}_{1,4}$.

Let F_1^s be the restriction of F_1 to the small phase space. Using the genus-1 topological recursion relation, Dijkgraaf and Witten [9] obtained a formula, called the *genus-*1 *constitutive relation*, for computing F_1 from F_1^s. In [11], Dubrovin and Zhang observed that if the quantum cohomology of V is semisimple, then in the canonical coordinate system $(u_1, \ldots, u_N)$ on the small phase space, Getzler's equation gives all second order partial derivatives $\frac{\partial^2 F_1^s}{\partial u_i \partial u_j}$ in terms of genus-0 data. They then use the quasi-homogeneity equation to obtain a formula for all first order derivatives of F_1^s. This formula determines F_1^s in terms of genus-0 functions up to an additive constant. It was observed in [36] that one can solve Getzler's equation on the big phase using idempotents $\mathcal{E}_1, \ldots, \mathcal{E}_N$ and obtain

$$\langle\langle\mathcal{E}_i\rangle\rangle_1 = \frac{1}{24}\Big\{\langle\langle\tau_-(\mathcal{L}_0)\,\gamma_\alpha\,\gamma^\alpha\,\mathcal{E}_i\rangle\rangle_0 - \sum_j u_j B_0(\mathcal{E}_j, \mathcal{E}_j, \mathcal{E}_j, \mathcal{E}_i)\Big\}$$

where $\mathcal{L}_0 := -\mathcal{X} - (b_1 + 1)T(\mathcal{S})$. Let ∇ be the trivial connection on the big phase space which is uniquely characterized by the requirement that all vector fields $\tau_n(\gamma_\alpha)$ are parallel. Then $\nabla_{\mathcal{E}_i} \tau_-(\mathcal{L}_0) = 0$. Moreover, second order derivatives of the string equation imply that $A_0(\mathcal{S})$ is a constant. So the above equation can be written in the following form:

Theorem 2.1 ([36]). *For manifolds with semisimple quantum cohomology,*

$$\mathcal{E}_i F_1 = \frac{1}{24}\Big\{\mathcal{E}_i \; A_0\left(\tau_-(\mathcal{L}_0) + \mathcal{S}\right) - \sum_j u_j \, B_0(\mathcal{E}_j, \mathcal{E}_j, \mathcal{E}_j, \mathcal{E}_i)\Big\}$$

for $i = 1, \ldots, N$.

Together with the genus-1 topological recursion relation, this equation gives all the first order derivatives of F_1 and therefore completely determines F_1 up to a constant. We also note that when restricted to the small phase space, this equation is equivalent to the equation obtained in [11].

The genus-2 analogue of the topological recursion relation is

$$\langle\!\langle T^2(\mathcal{W}) \rangle\!\rangle_2 = A_1(\mathcal{W})$$

where A_1 is a 1-tensor which consists of 5 terms involving only genus-0 and genus-1 data (see, for example, [33] for the precise form of A_1). We call this equation Mumford equation since it corresponds to a tautological relation on $\overline{\mathcal{M}}_{2,1}$ of the form

$$\psi_1^2 = \text{linear combinations of boundary strata of } \overline{\mathcal{M}}_{2,1},$$

which was proved in [40]. Mumford's tautological relation was first translated to a universal equation by Getzler [17] following some observations by Faber.

The genus-2 analogue of the WDVV equation is the following equation due to Belorousski and Pandharipande [4]: For arbitrary vector fields $\mathcal{W}_1$, $\mathcal{W}_2$, and $\mathcal{W}_3$ on the big phase space we have

$$\begin{aligned} &2\langle\!\langle \{\mathcal{W}_1 \circ \mathcal{W}_2 \circ \mathcal{W}_3\} \rangle\!\rangle_2 - 2\langle\!\langle \mathcal{W}_1 \, \mathcal{W}_2 \, \mathcal{W}_3 \, \gamma^\alpha \rangle\!\rangle_0 \langle\!\langle T(\gamma_\alpha) \rangle\!\rangle_2 \\ &\quad + \frac{1}{2} \sum_{\sigma \in S_3} \langle\!\langle \mathcal{W}_{\sigma(1)} \, T(\mathcal{W}_{\sigma(2)} \circ \mathcal{W}_{\sigma(3)}) \rangle\!\rangle_2 - \langle\!\langle T(\mathcal{W}_{\sigma(1)}) \, \{\mathcal{W}_{\sigma(2)} \circ \mathcal{W}_{\sigma(3)}\} \rangle\!\rangle_2 \\ &= B_1(\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3), \end{aligned}$$

where B_1 is a symmetric 3-tensor which consists of 16 terms involving only genus-0 and genus-1 data (see, for example, [33] for the precise form of B_1 where we used the notation B for this tensor). This equation corresponds to a tautological relation on $\overline{\mathcal{M}}_{2,3}$. There is also another genus-2 equation due to Getzler [17] which corresponds to a tautological relation on $\overline{\mathcal{M}}_{2,2}$. It was proved in [33] and [35] that Getzler's genus-2 equation can be derived from Mumford equation and Belorousski-Pandharipande equation. So we will not give the precise form of this equation. Note that all genus-2 universal equations involve gravitational descendants and can not be reduced to equations on the small phase space. This is the main reason for introducing the quantum product on the big phase space. In [37], we proved that in the semisimple case F_2 can be solved from the above genus-2 universal equations and obtained

Theorem 2.2 ([37]). *For manifolds with semisimple quantum cohomology,*

$$F_2 = \frac{1}{6}\Big\{A_1\left(2\,\tau_-^2(\mathcal{L}_0) + 3\,\tau_-(\mathcal{S})\right) - \sum_{i=1}^{N} u_i B_1(\mathcal{E}_i, \mathcal{E}_i, \mathcal{E}_i)\Big\}.$$

This formula is very similar to the solution of genus-1 equations if we ignore the vector field $\mathcal{E}_i$ in the formula in Theorem 2.1.

For genus bigger than 2, we do not have enough information about universal equations due to the lack of understanding of the tautological rings of moduli spaces of stable curves. So far the only non-trivial universal equation with genus bigger than 2 is the following genus-3 analogue of the topological recursion relation proved by Kimura and the author in [25]: For any vector field $\mathcal{W}$ on the big phase space

$$\langle\langle T^3(\mathcal{W})\rangle\rangle_3 = A_2(\mathcal{W})$$

where A_2 is a tensor which consists of 29 terms involving only data of genus less than or equal to 2. (See [25] for the precise form of this tensor. An equivalent formula was derived in [2] using the invariance conjecture which has not yet been proved.) This equation corresponds to a tautological relation on $\overline{\mathcal{M}}_{3,1}$ of the form

$$\psi_1^3 = \text{linear combinations of boundary strata of } \overline{\mathcal{M}}_{3,1}.$$

There is a conjecture by Getzler [17] that any degree g monomial of ψ-classes on $\overline{\mathcal{M}}_{g,k}$ should be supported on the boundary of $\overline{\mathcal{M}}_{g,k}$. This conjecture was proved by Ionel [22]. Faber and Pandharipande [15] further proved that degree g monomials of ψ-classes on $\overline{\mathcal{M}}_{g,k}$ are equal to some tautological classes on the boundary of $\overline{\mathcal{M}}_{g,k}$. Note that tautological rings also contain push-forward classes of powers of ψ-classes, i.e. the so called κ-classes, which do not appear in the definition of Gromov–Witten invariants. These results do not yet guarantee the existence of corresponding universal equations. Nevertheless, as mentioned in [33], we do expect that the following conjecture should be true.

Conjecture 2.3. For all $g \geq 1$, there is a universal equation of the form

$$\langle\langle T^g(\mathcal{W})\rangle\rangle_g = A_{g-1}(\mathcal{W})$$

where A_{g-1} is a tensor which only involve data of genus less than or equal to $g-1$.

Such equations should be the genus-g analogue of the topological recursion relation. Based on our experience from genus-1 and genus-2, we also expect that there should be a genus-g analogue of the WDVV equation whose lower genus part should be a symmetric tensor, written as B_{g-1}, which only involves data of genus less than or equal to $g-1$. However, at this stage, we even do not have a good prediction for the top genus part of this equation for $g \geq 3$. Despite of these difficulties we believe that the following conjecture should be true.

Conjecture 2.4. For manifolds with semisimple quantum cohomology, F_g can be solved explicitly from universal equations for $g \geq 2$.

Based on Theorems 2.1 and 2.2, we may even speculate that the form of F_g obtained by solving universal equations should be

$$F_g = A_{g-1}\big(a_g\, \tau_-^g(\mathcal{L}_0) + b_g\, \tau_-^{g-1}(\mathcal{S})\big) - c_g \sum_{i=1}^{N} u_i\, B_{g-1}(\mathcal{E}_i, \ldots, \mathcal{E}_i)$$

where a_g, b_g, c_g are some constants depending only on g.

One might also expect that there exist universal equations of the form

$$\langle\langle T^{n_1}(\mathcal{W}_1) \ldots T^{n_k}(\mathcal{W}_k) \rangle\rangle_g = \text{an expression involving at most genus-}(g-1)\text{ data} \quad (3)$$

for $n_1 + \cdots + n_k = g$. This statement is somewhat stronger than Getzler's conjecture on the tautological ring of $\overline{\mathcal{M}}_{g,k}$. Since there are only finitely many strata on $\overline{\mathcal{M}}_{g,k}$ with a fixed degree, one can explicitly write out the right-hand side of equation (3) with certain undetermined coefficients. It might be possible that in many cases these coefficients can be fixed by known Gromov–Witten theory. For example, to obtain the genus-3 topological recursion relation in [25], we only need the Gromov–Witten theory of a point and $\mathbb{CP}^1$. However, coefficients obtained in this way are quite mysterious. It would be very interesting to give a better explanation to such coefficients.

3. The Virasoro conjecture

In [14] Eguchi, Hori and Xiong constructed a sequence of differential operators on the big phase space. With a slight modification proposed by S. Katz (cf. [6]), these operators satisfy the Virasoro bracket relation when the underlying manifold is a smooth projective variety and therefore form a half branch of the Virasoro algebra. They conjectured that these operators annihilate the generating function of the Gromov–Witten invariants for all smooth projective varieties. This conjecture is now known as the *Virasoro conjecture*. It is a far-reaching generalization of a conjecture of Witten, which was proved by Kontsevich, that the generating function of intersection numbers of ψ-classes on $\overline{\mathcal{M}}_{g,k}$ is a τ-function of the KdV hierarchy (cf. [45], [26], and [45]). In [13], Dubrovin and Zhang proved that if the quantum cohomology of a projective variety V is semisimple, then higher genus Gromov–Witten invariants of V are determined by genus-0 invariants and the Virasoro conjecture. For manifolds with semisimple quantum cohomology, Givental [19] has conjectured a form of higher genus generating functions in terms of genus-0 data and the τ-function in Witten's conjecture and showed that his conjectural formula satisfies the Virasoro constraints (cf. [20]). Consequently, Givental's conjecture is equivalent to the Virasoro conjecture for projective varieties with semisimple quantum cohomology. In the case that the underlying manifold has a torus action with isolated fixed points and also has semisimple quantum cohomology, Givental has a scheme to reduce his conjecture to the so-called *R-conjecture* on the fundamental solutions to the flat section equations of a one-parameter family of connections on the small phase space defined using quantum product. Note that localization techniques played a crucial role in this scheme. An outline for the proof of the R-conjecture for projective spaces were given in [20]. The R-conjecture has been verified for flag manifolds in [24] and for Grassmannians in [5]. Also using localization techniques, Okounkov and Pandharipande [41] proved the Virasoro conjecture for algebraic curves. In this paper we will focus on the approach to the Virasoro conjecture using universal equations instead of localization techniques. Since universal equations hold for all compact symplectic manifolds, this approach should apply to a larger class of manifolds. In particular, there is no need to assume the existence of torus actions on the manifolds. In [39], Tian and the author proved the following theorem using genus-0 topological recursion relation (see also [12], [18], [34], [21] for alternative proofs.):

Theorem 3.1 ([39]). *The genus-*0 *Virasoro conjecture holds for all compact symplectic manifolds.*

The genus-1 Virasoro conjecture for manifolds with semisimple quantum cohomology was proved by Dubrovin and Zhang [12] (see also [32] and [36]). The genus-2 analogue of this result was proved in [37] using Theorem 2.2.

Theorem 3.2 ([37]). *The genus-*2 *Virasoro conjecture holds for manifolds with semisimple quantum cohomology.*

This theorem implies in particular that Givental's conjectural formula is correct in the genus-2 case. An alternative approach to the genus-2 Virasoro conjecture for manifolds with semisimple quantum cohomology is to show that Givental's formula satisfies known genus-2 universal equations and then use the result in [33] that known genus-2 universal equations uniquely determine F_2 in the semisimple case. Givental's formula was constructed using actions of twisted loop groups on a product of copies of the τ-function in Witten's conjecture. The invariance of the genus-2 Mumford's relation under the action of Lie algebras of twisted loop groups was discussed in [29]. It was also claimed that the genus-2 equations of Getzler and Belorousski-Pandharipande are invariant in the same sense. In [32] and [33], the author proved that the genus-1 and genus-2 Virasoro conjecture for all smooth projective varieties can be reduced to an $\mathrm{SL}(2)$ symmetry for Gromov–Witten invariants. We will give the precise statement of this result later. Below we explain in more detail the relations between universal equations and the Virasoro conjecture.

The following operators were introduced in [33] as a convenient tool in the study of the Virasoro conjecture: For any vector field $\mathcal{W} = \sum_{m,\alpha} f_{m,\alpha}\tau_m(\gamma_\alpha)$ on the big phase space, define

$$G(\mathcal{W}) := \sum_{m,\alpha,\beta} (m + b_\alpha) f_{m,\alpha}\tau_m(\gamma_\alpha), \quad C(\mathcal{W}) := \sum_{m,\alpha,\beta} f_{m,\alpha}\mathcal{C}_\alpha^\beta \tau_m(\gamma_\beta),$$

and

$$R(\mathcal{W}) := (GT + C)(\mathcal{W}).$$

Starting from the string vector field $\mathcal{S}$, we can apply the operator R recursively to obtain a sequence of vector fields on the big phase space:

$$\mathcal{L}_n := -R^{n+1}(S)$$

for $n \geq -1$. It was proved in [33] that this sequence of vector fields satisfy the Virasoro bracket relation:

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}.$$

These vector fields are not exactly the Virasoro operators given in [14] which are second order differential operators. However the Virasoro conjecture can be rephrased using these vector fields in the following way. First, second order derivatives of the genus-0 Virasoro conjecture can be reinterpreted as

$$\mathcal{L}_n \circ \mathcal{W} = -\mathcal{X}^{n+1} \circ \mathcal{W} \tag{4}$$

for any vector field $\mathcal{W}$ and $n \geq 0$, where $\mathcal{W}^k$ is defined to be the quantum product of k copies of $\mathcal{W}$. This equation follows from the associativity of the quantum product and the following property of the operator R (cf. [33]):

$$R(\mathcal{V}) \circ \mathcal{W} = \mathcal{X} \circ \mathcal{V} \circ \mathcal{W}$$

for any vector fields $\mathcal{W}$ and $\mathcal{V}$. Using this property a proof for genus-0 Virasoro conjecture was obtained in [34] which is much simpler than the original proof in [39].

To interpret higher genus Virasoro conjecture, we also need the following operators:

$$Q_0 := \tau_-, \quad Q_1 := Q := G + C\tau_-, \quad Q_k := Q(Q-1)\dots(Q-k+1)\tau_+^{k-1}$$

for $k \geq 1$. These operators were used in [34] to simplify the proof of the genus-0 Virasoro conjecture. Define second order differential operators

$$W_n := \sum_{i=1}^{n} \{Q_i(\gamma^\alpha)\} \, \{Q_{n-i}\tau_+ Q(\gamma_\alpha)\}$$

for $n \geq 1$. Note that $Q_i(\gamma^\alpha)$ and $Q_{n-i}\tau_+ Q(\gamma_\alpha)$ are parallel vector fields with respect to the trivial connection ∇ on the big phase space. Therefore they commute with each other as first order differential operators. Then *Virasoro conjecture* for $g \geq 1$ can be formulated as

$$\Big\{\mathcal{L}_n + \frac{1}{2}\lambda^2 W_n + \frac{1}{2}(W_n F_0)\Big\} e^{\sum_{g=1}^{\infty}\lambda^{2g-2}F_g} = 0$$

for $n \geq -1$. Here $(W_n F_0)$ means the multiplication by the genus-0 function $W_n F_0$. The Virasoro conjecture for $g \geq 2$ can also be formulated as

$$\Big\{\mathcal{L}_n + \frac{1}{2}\lambda^2 e^{-F_1}\, W_n\, e^{F_1}\Big\} e^{\sum_{g=2}^{\infty}\lambda^{2g-2}F_g} = 0$$

for $n \geq -1$. Here we understand $W_n = 0$ for $n \leq 0$. In this formulation, $e^{-F_1}\, W_n\, e^{F_1}$ is understood as the composition of three operators acting on the space of functions.

The genus-g L_{-1}-constraint is simply the string equation which holds for all compact symplectic manifolds

$$\langle\langle \mathcal{L}_{-1} \rangle\rangle_g = -\frac{1}{2}\delta_{g,0}\eta_{\alpha\beta}t_0^{\alpha}t_0^{\beta}.$$

The genus-g L_0-constraint has the following form

$$\langle\langle \mathcal{L}_0 \rangle\rangle_g = -\frac{1}{2}\delta_{g,0}\mathcal{C}_{\alpha\beta}t_0^{\alpha}t_0^{\beta} + \frac{1}{24}\delta_{g,1}\left\{\int_V c_1(V)\cup c_{d-1}(V) - \frac{3-d}{2}\chi(V)\right\}.$$

This constraint follows from the quasi-homogeneity equation and the *dilaton equation*

$$\langle\langle T(\mathcal{S}) \rangle\rangle_g = (2g-2)F_g + \frac{1}{24}\chi(V)\delta_{g,1}$$

where $\chi(V)$ is the Euler characteristic number of V. These equations also hold for all compact symplectic manifolds.

For $g \geq 1$ and $n \geq 1$, the genus-g L_n-constraint in the Virasoro conjecture can be formulated as:

$$\begin{aligned}\langle\langle \mathcal{L}_n \rangle\rangle_g = -\frac{1}{2}\sum_{i=1}^{n}\big\{&\langle\langle Q_i(\gamma^{\alpha})\ \ \{Q_{n-i}\tau_+ Q(\gamma_{\alpha})\}\rangle\rangle_{g-1} \\ &+ \sum_{h=1}^{g-1}\langle\langle Q_i(\gamma^{\alpha})\rangle\rangle_h \langle\langle \{Q_{n-i}\tau_+ Q(\gamma_{\alpha})\}\rangle\rangle_{g-h}\big\}.\end{aligned}$$

Note that the right-hand side of this formula only involves data of genus less than g.

For any compact symplectic manifold V, we can use universal equations to obtain recursion relations among $\langle\langle \mathcal{L}_n \rangle\rangle_g$ for $g = 1, 2$. In genus-1 case, we have (cf. [32])

$$\langle\langle \mathcal{L}_n \rangle\rangle_1 + \frac{n+1}{2}(\mathcal{S} \circ \mathcal{L}_{n-1})\langle\langle \mathcal{L}_1 \rangle\rangle_1 \equiv 0 \ \text{ mod } \ \{\text{genus 0 data}\}.$$

In genus-2 case, we have (cf. [33])

$$\langle\!\langle \mathcal{L}_n \rangle\!\rangle_2 + \frac{n+1}{2(n-1)} T(\mathcal{S} \circ \mathcal{L}_0) \langle\!\langle \mathcal{L}_{n-1} \rangle\!\rangle_2 \equiv 0 \mod \{\text{genus} \le 1 \text{ data}\}.$$

The right-hand sides of these two equations are explicit functions only involving lower genus data (See [32] and [33] for the precise forms). As a consequence of these two formulas, we have the next result.

Theorem 3.3 ([32], [33]). *For any compact symplectic manifold, $\langle\!\langle \mathcal{L}_n \rangle\!\rangle_g$ can be computed from $\langle\!\langle \mathcal{L}_1 \rangle\!\rangle_g$ and lower genus data for all $n \ge 2$ and $g = 1, 2$.*

In case that the manifold is a projective variety we can further verify the following:

Theorem 3.4 ([32], [33]). *For any smooth projective variety, the genus-1 and genus-2 Virasoro conjecture follows from the L_1-constraint.*

Note that the first 3 Virasoro operators (as well as vector fields $\mathcal{L}_{-1}$, $\mathcal{L}_0$ and $\mathcal{L}_1$) form a 3-dimensional subalgebra which is isomorphic to $\mathrm{SL}(2)$. Since the L_{-1} and L_0 constraints are satisfied for all compact symplectic manifolds, the proof of the above theorem can be interpreted as saying that for genus-1 and genus-2 cases universal equations upgrade an $\mathrm{SL}(2)$ symmetry for Gromov–Witten invariants to the Virasoro conjecture. We believe that this should also be true for higher genus cases:

Conjecture 3.5. For all compact symplectic manifolds V, $\langle\!\langle \mathcal{L}_n \rangle\!\rangle_g$ can be computed from $\langle\!\langle \mathcal{L}_1 \rangle\!\rangle_g$ and lower genus data for $n \ge 2$ and $g \ge 1$. In case that V is a smooth projective variety the Virasoro conjecture follows from universal equations and the L_1- constraint.

Note that the above two theorems do not need semisimplicity. In case that the quantum cohomology is semisimple, we can actually recover the action of $\mathcal{L}_1$ on F_g from the action of $\mathcal{L}_n$ on F_g for $n \ge 2$ due to algebraic relations among these vector fields (cf. [32] and [33] for details). This is the reason that the genus-1 and genus-2 Virasoro conjecture hold for manifolds with semisimple quantum cohomology. In fact, one can actually use the formulas in Theorems 2.1 and 2.2 to prove the Virasoro conjecture in these cases (cf. [36] and [37] for details). Based on these results, it is reasonable to believe that the following conjecture should be true.

Conjecture 3.6. For all smooth projective varieties with semisimple quantum cohomology, the Virasoro conjecture follows from universal equations.

We also note that besides vector fields $\mathcal{L}_n$, $n \ge -1$, there are also other natural vector fields on the big phase space which satisfy the Virasoro bracket relation. For example, if we define $\overline{\mathcal{X}}^k := \mathcal{S} \circ \mathcal{X}^k$, then these vector fields satisfy the bracket relation

$$[\overline{\mathcal{X}}^m, \overline{\mathcal{X}}^k] = (k-m)\overline{\mathcal{X}}^{m+k-1}$$

for all $m, k \geq 0$ (cf. [33]). Here $\mathcal{X}^0$ is understood to be the string vector field $\mathcal{S}$. The relation between these vector fields and the Virasoro conjecture is explained in equation (4) and in more detail in [33].

Another sequence of vector fields is $T^n(\mathcal{X})$, $n \geq 0$, which satisfy the relation

$$[T^k(\mathcal{X}),\ T^m(\mathcal{X})] = (m-k)T^{m+k}(\mathcal{X})$$

for all $m, k \geq 0$. This relation follows from the following properties of the covariant differentiation

$$\nabla_{\mathcal{W}}\mathcal{X} = -Q(\mathcal{W}) + (b_1+1)\mathcal{W}, \quad \nabla_{\mathcal{V}}\, T^k(\mathcal{W}) = T^k(\nabla_{\mathcal{V}}\mathcal{W}) - T^{k-1}(V \circ W)$$

for all vector fields $\mathcal{V}$ and $\mathcal{W}$, as well as the following properties for operators T and Q

$$T^k Q T^m - T^m Q T^k = (m-k)T^{m+k}$$

for $m, k > 0$, and

$$(QT^k - T^k Q)(\mathcal{W}) = k\, T^k(\mathcal{W}) + T^{k-1}(\mathcal{X} \circ \mathcal{W})$$

for $k > 0$ and any vector field $\mathcal{W}$. The relation between vector fields $T^n(\mathcal{X})$ and the Virasoro conjecture is not clear at this moment.

4. Universal equations and spin curves

Universal equations also arise in the study of intersection theory on moduli spaces of spin curves. As mentioned above, the generating function for intersection numbers on the moduli spaces of stable curves is a τ-function of KdV hierarchy by Kontsevich–Witten theorem. In [46], Witten also proposed an algebraic geometric way to produce a τ-function for more general Gelfand–Dickey hierarchies by considering intersection numbers on the moduli spaces of spin curves. Assume that $r \geq 2$ is an integer and $\boldsymbol{m} = (m_1, \dots, m_k)$ is a collection of integers with $0 \leq m_i \leq r-2$. When r is prime, an r-spin structure of type $\boldsymbol{m}$ over a smooth stable curve $(C; x_1, \dots, x_k)$ is a line bundle $L \longrightarrow C$ together with an isomorphism $L^{\otimes r} \longrightarrow \omega\big(-\sum_{i=1}^k m_i x_i\big)$ where ω is the canonical line bundle over C. For degree reasons such a line bundle exists only if $\big(2g - 2 - \sum_{i=1}^k m_i\big)/r \in \mathbb{Z}$ where g is the genus of C. If r is not prime, then all d-th roots of $\omega\big(-\sum_{i=1}^k m_i x_i\big)$ should be considered for all d which divides r. If C is not smooth the definition of an r-spin structure is more involved (cf. [23] for details). A stable curve with an r-spin structure is called an r-spin curve. Let $\overline{\mathcal{M}}_{g,k}^{1/r}(\boldsymbol{m})$ be the moduli space of all genus-g r-spin curves of type $\boldsymbol{m}$ and with k-marked points. Let Ψ_i be the first Chern class of the line bundle over $\overline{\mathcal{M}}_{g,k}^{1/r}(\boldsymbol{m})$ whose geometric fiber over each r-spin curve is given by $T_{x_i}^* C$.

Let $e_0, \ldots, e_{r-2}$ be some abstract symbols. Define

$$\langle \tau_{n_1}(e_{m_1}) \ldots \tau_{n_k}(e_{m_k}) \rangle_{g,r} := r^{1-g} \int_{\overline{\mathcal{M}}^{1/r}_{g,k}(\boldsymbol{m})} c^{1/r}_{g,n}(\boldsymbol{m}) \cup \Psi_1^{n_1} \cup \cdots \cup \Psi_k^{n_k}$$

where $c^{1/r}_{g,n}(\boldsymbol{m})$ is a rational cohomology class on $\overline{\mathcal{M}}^{1/r}_{g,n}(\boldsymbol{m})$ of degree

$$\frac{2}{r}\Big\{(r-2)(g-1) + \sum_{i=1}^{n} m_i\Big\}.$$

The Poincaré dual of $c^{1/r}_{g,n}(\boldsymbol{m})$ corresponds to the virtual fundamental class in the Gromov–Witten theory. The genus-g generating function $\Phi_{g,r}$ for such numbers is then a function of parameters t_n^m with $0 \le m \le r-2$ and $n \in \mathbb{Z}_{\ge 0}$. Let $\Phi_r = \sum_{g=0}^{\infty} \Phi_{g,r}$. The *generalized Witten conjecture* predicts that $\exp(\Phi_r)$ is a τ-function of r-th KdV hierarchy (also called the Gelfand–Dickey hierarchy). More precisely, consider a differential operator

$$L = D^r - \sum_{i=0}^{r-2} u_i(x)\ D^i, \quad \text{where } D := \frac{\sqrt{-1}}{\sqrt{r}} \frac{\partial}{\partial x}.$$

An r-th root of L is a pseudo-differential operator of the form

$$L^{1/r} = D + \sum_{i>0} w_i(x) D^{-i}$$

whose r-th power is L and w_i are differential polynomials in $u_0, \ldots u_{r-2}$. Assume that L also depends on infinitely many parameters t_n^m with $0 \le m \le r-2$ and $n \ge 0$. The variable x is usually identified with t_0^0. We say that L satisfies the *r-th KdV hierarchy* if

$$\sqrt{-1}\ \frac{\partial L}{\partial t_n^m} = \frac{k_{n,m}}{\sqrt{r}} \big[\big(L^{n+\frac{m+1}{r}}\big)_+,\ L\big] \tag{5}$$

for all m and n, where

$$k_{n,m} = \frac{(-1)^n r^{n+1}}{(m+1)(r+m+1)\ldots(nr+m+1)}$$

and $\big(L^{n+\frac{m+1}{r}}\big)_+$ is a differential operator obtained by discarding all negatives powers of D in $L^{n+\frac{m+1}{r}}$. The *generalized Witten conjecture* says that there exists an operator L which satisfies the r-th KdV hierarchy such that

$$\frac{\partial^2 \Phi_r}{\partial t_0^0\, \partial t_n^m} = -k_{n,m}\, \mathrm{Res}\,\big(L^{n+\frac{m+1}{r}}\big) \tag{6}$$

for all m and n. Here “Res” means taking the coefficient of D^{-1}.

In some sense the generating functions $\Phi_{g,r}$ behaves like the generating function F_g for Gromov–Witten invariants of a fictitious manifold of rational dimension $2(r-2)/r$ and with the first Chern class equal to 0. Each abstract symbol e_m plays the role of a cohomology class of this fictitious manifold with rational degree $2m/r$. One can use the bilinear form defined by $(e_i,\, e_j) := \delta_{i+j,r-2}$ as a substitute for the Poincaré pairing. In this way, all structures of Gromov–Witten invariants mentioned in previous sections can make sense for intersection numbers on the moduli spaces of spin curves. In particular, $\Phi_{g,r}$ also satisfies the string equation. Together with the initial condition $\Phi(0) = 0$, the generalized Witten conjecture and the string equation completely determine the intersection numbers $\langle \tau_{n_1}(e_{m_1}) \ldots \tau_{n_k}(e_{m_k}) \rangle_{g,r}$. It is also well known that the generalized Witten conjecture and the string equation are equivalent to Virasoro constraints formulated in a similar way (See for example [1]). There is also a canonical map $\overline{\mathcal{M}}_{g,k}^{1/r}(\boldsymbol{m}) \longrightarrow \overline{\mathcal{M}}_{g,k}$ which forgets spin structures on underlying stable curves. Therefore the functions $\Phi_{g,r}$ also satisfy universal equations for Gromov–Witten invariants which are obtained from tautological relations on $\overline{\mathcal{M}}_{g,k}$ (cf. [23]). In particular, a proof for the Virasoro conjecture only using universal equations also gives a proof to the generalized Witten conjecture.

On the other hand, if we assume that the generalized Witten's conjecture is true, we should also get a lot of information for universal equations for Gromov–Witten invariants. For this purpose it is desirable to write equations (5) and (6) in a form closer to universal equations. Analogous to Gromov–Witten theory, we define $\langle\!\langle \mathcal{W}_1 \ldots \mathcal{W}_k \rangle\!\rangle_g$ and $\langle\!\langle \mathcal{W}_1 \ldots \mathcal{W}_k \rangle\!\rangle$ as in equation (1) with F_g replaced by $\Phi_{g,r}$ and Φ_r respectively. We also define the grading operator G as a linear operator on the space of vector fields on the big phase space by

$$G(\tau_n(e_m)) := (n + b_m)\, \tau_n(e_m), \quad \text{with } b_m := \frac{m+1}{r}.$$

For any vector field $\mathcal{W}$ on the big phase space define

$$\tilde{T}(\mathcal{W}) := \tau_+(\mathcal{W}) - \sum_{m=0}^{r-2} \langle\!\langle \mathcal{W}\, e_m \rangle\!\rangle e_{r-2-m} \quad \text{and} \quad \tilde{R}(\mathcal{W}) := G\, \tilde{T}(\mathcal{W}).$$

The difference of operator $\tilde{T}$ and the operator T is that the coefficient of e_{r-2-m}, i.e. $\langle\!\langle \mathcal{W}\, e_m \rangle\!\rangle$, contains information for all genera, not only genus-0. Note that we think of $c_1 = 0$ for the theory of spin curves, so $\tilde{R}$ is a direct analogue of R with T replaced by $\tilde{T}$. In particular, we have

$$\tilde{R}(\tau_n(e_m)) = \left(n + 1 + \frac{m+1}{r}\right) \tau_{n+1}(e_m) - \sum_{j=0}^{r-2} \frac{r-1-j}{r} \langle\!\langle \tau_n(e_m)\, e_j \rangle\!\rangle e_{r-2-j}.$$

For any vector field $\mathcal{W}$ on the big phase space and $0 \le m \le r-2$, define

$$\begin{aligned}\Omega_m(\mathcal{W}) := \langle\!\langle \tilde{R}(\mathcal{W})\, e_0\, e_m \rangle\!\rangle - \sum_{i=0}^{r-2} b_i^2 b_{r-2-i} \langle\!\langle \mathcal{W}\, e_0\, e_m\, e_i\, e_{r-2-i} \rangle\!\rangle \\ - \sum_{i=0}^{r-2} \frac{b_{r-2-i}+b_m}{2} \langle\!\langle \mathcal{W}\, e_0\, e_i \rangle\!\rangle \langle\!\langle e_{r-2-i}\, e_m \rangle\!\rangle \\ - \sum_{i=0}^{r-2} \frac{b_{r-2-i}+b_0}{2} \langle\!\langle \mathcal{W}\, e_m\, e_i \rangle\!\rangle \langle\!\langle e_{r-2-i}\, e_0 \rangle\!\rangle. \end{aligned} \tag{7}$$

For each r, the generalized Witten conjecture can be reformulated in the form

$$\Omega_m(\mathcal{W}) = \cdots \tag{8}$$

for all vector field $\mathcal{W}$, where the right-hand side of this equation is an expression each term of which involves only $\mathcal{W}$ and at least 6 primary vector fields. For example, when $r = 2$, Witten's formulation of his KdV conjecture in [44] is equivalent to

$$\Omega_0(\mathcal{W}) = 0 \tag{9}$$

for all vector field $\mathcal{W}$. Shadrin [43] shows that the generalized Witten conjecture for $r = 3$ also implies a formula which is equivalent to equation (9). In [38], it is proved that the generalized Witten conjecture for $r = 3$ also implies that

$$\Omega_1(\mathcal{W}) = \frac{1}{108}\left\{-\langle\!\langle \mathcal{W}\, e_0^2 \rangle\!\rangle \langle\!\langle e_0^4 \rangle\!\rangle - \langle\!\langle \mathcal{W}\, e_0^3 \rangle\!\rangle \langle\!\langle e_0^3 \rangle\!\rangle + \langle\!\langle \mathcal{W}\, e_0^4 \rangle\!\rangle \langle\!\langle e_0^2 \rangle\!\rangle\right\}$$

for all vector field $\mathcal{W}$. In fact, this equation and equation (9) together are equivalent to the generalized Witten conjecture for $r = 3$. In this formula and the formula below e_i^k does not mean the quantum power of e_i. It simply means e_i repeating k times.

It seems quite hard to give a general formula for the right-hand side of equation (8) for all r, even for the special case $m = 0$. Equation (9) only holds for $r = 2$ and $r = 3$. It does not hold when $r > 3$. For example, when $r = 4$, we have

$$\Omega_0(\mathcal{W}) = \frac{1}{192}\left\{\langle\!\langle \mathcal{W}\, e_0^2 \rangle\!\rangle \langle\!\langle e_0^4 \rangle\!\rangle + \langle\!\langle \mathcal{W}\, e_0^3 \rangle\!\rangle \langle\!\langle e_0^3 \rangle\!\rangle - \langle\!\langle \mathcal{W}\, e_0^4 \rangle\!\rangle \langle\!\langle e_0^2 \rangle\!\rangle\right\}$$

as a consequence of the generalized Witten conjecture (cf. [38]). However, from all examples computed in [38], we expect that at the origin of the big phase space $\Omega_m(\mathcal{W}) = 0$ for all $r \ge 2$ and all vector fields $\mathcal{W}$ on the big phase space. Combining with the string equation, the equation $\Omega_0(\mathcal{W}) = 0$ at the origin gives the following recursion formula for intersection numbers:

$$\left(n + 1 + \frac{m+2}{r}\right)\langle \tau_n(e_m) \rangle = \sum_{j=0}^{r-2} \frac{(j+1)^2(r-1-j)}{r^3} \langle \tau_{n-1}(e_m)\, e_j\, e_{r-2-j} \rangle$$

for all r, m and n. This formula has been checked for $r \leq 7$ in [38] using the generalized Witten conjecture. In particular, we used this formula computed $\langle \tau_n(e_m) \rangle_{3,r}$. The results match with the computations using the genus-3 topological recursion relation in [25].

Starting from any vector field $\mathcal{W}$ on the big phase space, the above equations obtained from the generalized Witten conjecture are recursion relations involving $\tilde{R}(\mathcal{W})$. One can also write them as recursion relations involving $R(\mathcal{W})$ in more complicated forms. In comparison, universal equations obtained from tautological relations are recursion relations involving $T(\mathcal{W})$. In this sense the recursion relations from the generalized Witten conjecture are closer to the Virasoro constraints rather than universal equations. It would be interesting to find out relations between these two types of recursion relations. In [44], Witten showed how to check the compatibility of the equation (9) for $r = 2$ and the topological recursion relations of genus-0 and genus-1. He employed the constitutive relations which were obtained in [9] using genus-0 and genus-1 topological recursion relations. So far there is no analogue for the constitutive relations when the genus is bigger than one. This makes it harder to check the compatibility even for the case when $r = 2$ if the genus is bigger than 1. We believe that it is very important to understand the relations between the generalized Witten conjecture (or τ-functions of Gelfand–Dickey hierarchies) and universal equations of Gromov–Witten invariants. Such relations should be crucial in understanding the structures of the complicated systems of universal equations.

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