

Extremal metrics and stabilities on polarized manifolds

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Abstract. The Hitchin–Kobayashi correspondence for vector bundles, established by Donaldson, Kobayashi, Lübke, Uhlenbeck and Yau, states that an indecomposable holomorphic vector bundle over a compact Kähler manifold is stable in the sense of Takemoto–Mumford if and only if the vector bundle admits a Hermitian–Einstein metric. Its manifold analogue known as Yau’s conjecture, which originated from Calabi’s conjecture, asks whether “stability” and “existence of extremal metrics” for polarized manifolds are equivalent. In this note the recent progress of this subject, by Donaldson, Tian and our group, together with its relationship to algebraic geometry will be discussed.

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1. Introduction

Let M be a compact complex connected manifold. As an introduction to our subject we recall the following well-known conjecture of Calabi [5]:

Conjecture. (i) If $c_1(M)_{\mathbb{R}} < 0$, then M admits a unique Kähler–Einstein metric ω such that $\text{Ric}(\omega) = -\omega$.

(ii) If $c_1(M)_{\mathbb{R}} = 0$, then each Kähler class on M admits a unique Kähler–Einstein metric ω such that $\text{Ric}(\omega) = 0$.

(iii) For $c_1(M)_{\mathbb{R}} > 0$, find a suitable condition for M to admit a Kähler–Einstein metric ω such that $\text{Ric}(\omega) = \omega$.

Based on the pioneering works of Calabi [6], [7] and Aubin [1], a complete affirmative answer to (i) and (ii) was given by Yau [49] by solving systematically certain complex Monge–Ampère equations. However for (iii), sufficient conditions are known only partially by Siu [41], Nadel [37], Tian [42], [44], Tian and Yau [46], Wang and Zhu [52], where some necessary conditions were formulated as obstructions by

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Futaki [18] and Matsushima [34] (see also Lichnerowicz [23]). It is (iii) that mainly motivates our studies of stabilities and extremal metrics, while an analysis of destabilizing phenomena caused by the non-existence of Kähler–Einstein metrics allows us to obtain very nice by-products, such as Nadel’s vanishing theorem [37] via the use of multiplier ideal sheaves.

2. Stability for manifolds in algebraic geometry

In Mumford’s GIT [36], moduli spaces of algebraic varieties are constructed via the theory of invariants, where varieties are described by numerical data modulo actions of reductive algebraic groups. Then, roughly speaking, stable points are those away from very bad points in moduli spaces. For a precise definition consider a representation of a reductive algebraic group G on a complex vector space W . Let $w \in W$. We denote by G_w the isotropy subgroup of G at w .

Definition 2.1. A point $w \in W$ is said to be *stable* (resp. *properly stable*) if the orbit $G \cdot w$ is closed in W (resp. $G \cdot w$ is closed in W with $|G_w| < \infty$).

For moduli spaces of polarized varieties the Chow–Mumford stability and the Hilbert–Mumford stability are known. In what follows, by a polarized manifold (M, L) , we mean a very ample holomorphic line bundle L over a nonsingular projective algebraic variety M defined over \mathbb{C} , where the arguments in this section have nothing to do with the nonsingularity of M . Now for a polarized manifold (M, L) put $n := \dim M$ and let m be a positive integer. Then associated to the complete linear system $|L^m|$ we have the Kodaira embedding

$$\iota_m : M \hookrightarrow \mathbb{P}^*(V_m),$$

where $\mathbb{P}^*(V_m)$ denotes the set of all hyperplanes in $V_m := H^0(M, \mathcal{O}(L^m))$ through the origin. Let d_m be the degree of $\iota_m(M)$ in the projective space $\mathbb{P}^*(V_m)$. Put $G_m := \mathrm{SL}_{\mathbb{C}}(V_m)$ and $W_m := \{S^{d_m}(V_m)\}^{\otimes n+1}$, where $S^{d_m}(V_m)$ is the d_m -th symmetric tensor product of the space V_m . Take an element $M_m \neq 0$ in W_m^* such that the associated element $[M_m]$ in $\mathbb{P}^*(W_m)$ is the Chow point of the irreducible reduced algebraic cycle $\iota_m(M)$ on $\mathbb{P}^*(V_m)$. For the natural action of G_m on W_m^* we now apply Definition 2.1 to $G = G_m$ and $W = W_m$:

Definition 2.2. (a) (M, L^m) is said to be *Chow–Mumford stable* (resp. *Chow–Mumford properly stable*) if M_m in W_m^* is stable (resp. properly stable).

(b) (M, L) is said to be *asymptotically Chow–Mumford stable* (resp. *asymptotically Chow–Mumford properly stable*) if, for $m \gg 1$, (M, L^m) is Chow–Mumford stable (resp. Chow–Mumford properly stable).

Let m and k be positive integers. Then the kernel $I_{m,k}$ of the natural homomorphism of $S^k(V_m)$ to $V_{mk} := H^0(M, \mathcal{O}(L^{mk}))$ is the homogeneous ideal of degree k

defining M in $\mathbb{P}^*(V_m)$. Put $N_{mk} := \dim V_{mk}$ and $\gamma_{m,k} := \dim I_{m,k}$. Then $\wedge^{\gamma_{m,k}} I_{m,k}$ is a complex line in the vector space $W_{m,k} := \wedge^{\gamma_{m,k}}(S^k(V_m))$. Take an element $w_{m,k} \neq 0$ in $\wedge^{\gamma_{m,k}} I_{m,k}$. For the natural action of $G_m := \mathrm{SL}_{\mathbb{C}}(V_m)$ on $W_{m,k}$ we apply Definition 2.1 to $G = G_m$ and $W = W_{m,k}$:

Definition 2.3. (a) (M, L^m) is said to be *Hilbert–Mumford stable* (resp. *Hilbert–Mumford properly stable*) if $w_{\ell,k} \in W_{\ell,k}$ is stable (resp. properly stable).

(b) (M, L) is said to be *asymptotically Hilbert–Mumford stable* (resp. *asymptotically Hilbert–Mumford properly stable*) if, for all $m \gg 1$, (M, L^m) is Hilbert–Mumford stable (resp. Hilbert–Mumford properly stable).

A result of Fogarty [17] shows that if (M, L^m) is Chow–Mumford stable, then (M, L^m) is also Hilbert–Mumford stable. Though the converse has been unknown, the relationship between these two stabilities is now becoming clear (cf. [33]).

Stability for manifolds is an important subject in moduli theories of algebraic geometry. Recall, for instance, the following famous result of Mumford [35]:

Fact 1. *If L is an ample line bundle of degree $d \geq 2g + 1$ over a compact Riemann surface C of genus $g \geq 1$, then (C, L) is Chow–Mumford properly stable.*

For the pluri-canonical bundles $K_M^{\otimes m}$ on M , $m \gg 1$, using the asymptotic Hilbert–Mumford stability, Gieseker [20] generalized this result to the case where M is a surface of general type. For higher dimensions a stability result by Viehweg [48] is known in the case where the canonical bundle K_M is semipositive. However, for both the results of Gieseker and of Viehweg the proof of stability is fairly complicated, while the underlying manifold (or orbifold) admits a Kähler–Einstein metric.

3. The Hitchin–Kobayashi correspondence and its manifold analogue

For a holomorphic vector bundle E over an n -dimensional compact Kähler manifold (M, ω) we say that E is *Takemoto–Mumford stable* if

$$\frac{\int_M c_1(\mathcal{F})\omega^{n-1}}{\mathrm{rk}(\mathcal{F})} < \frac{\int_M c_1(E)\omega^{n-1}}{\mathrm{rk}(E)}$$

for every coherent subsheaf \mathcal{F} of $\mathcal{O}(E)$ satisfying $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(E)$. Recall the following Hitchin–Kobayashi correspondence for vector bundles:

Fact 2. *An indecomposable holomorphic vector bundle E over M is Takemoto–Mumford stable if and only if E admits a Hermitian–Einstein metric.*

This fact was established in 1980s by Donaldson [13], Kobayashi [22], Lübke [25], Uhlenbeck and Yau [47]. As a manifold analogue of this conjecture we can naturally ask whether the following conjecture (known as Yau’s conjecture) is true:

Conjecture. The polarization class of (M, L) admits a Kähler metric of constant scalar curvature (or more generally an extremal Kähler metric) if and only if (M, L) is asymptotically stable in a certain sense of GIT.

For the “only if” part of this conjecture, the first breakthrough was made by Tian [45]. By introducing the concept of K-stability, he gave an answer to the “only if” part for Kähler–Einstein manifolds, and showed that some Fano manifolds without nontrivial holomorphic vector fields admit no Kähler–Einstein metrics. A remarkable progress was made by Donaldson [14] who showed the Chow–Mumford stability for a polarized Kähler manifold (M, ω) of constant scalar curvature essentially when the connected linear algebraic part H of the group $\text{Aut}(M)$ of holomorphic automorphisms of M is semisimple. In the present paper we shall show how Donaldson’s work is generalized to extremal Kähler cases without any assumption on H (see also [31], [32]). The relationship between this generalization and a recent result by Chen and Tian [12] will be treated elsewhere.

4. The asymptotic Bergman kernel

For a polarized manifold (M, L) take a Hermitian metric h for L such that $\omega := c_1(L; h)$ is a Kähler form. Define a Hermitian pairing on $V_m := H^0(M, \mathcal{O}(L^m))$ by

$$\langle \sigma_1, \sigma_2 \rangle_{L^2} := \int_M (\sigma_1, \sigma_2)_h \omega^n, \quad \sigma_1, \sigma_2 \in V_m,$$

where $(\cdot, \cdot)_h$ denotes the pointwise Hermitian pairing by h^m for sections for L^m . For an orthonormal basis $\{\sigma_1, \sigma_2, \dots, \sigma_{N_m}\}$ of V_m we put

$$B_{m,\omega} := \frac{n!}{m^n} (|\sigma_1|_h^2 + |\sigma_2|_h^2 + \dots + |\sigma_{N_m}|_h^2), \quad (1)$$

where $|\sigma|_h^2 := (\sigma, \sigma)_h$ for $\sigma \in V_m$. This $B_{m,\omega}$ is called the m -th *Bergman kernel* for (M, ω) (cf. Tian [43], Zelditch [50], Catlin [4]), where we consider the asymptotic behavior of $B_{m,\omega}$ as $m \rightarrow \infty$. Note that $B_{m,\omega}$ depends only on (m, ω) and is independent of the choice of both h and the orthonormal basis for V_m . Next, for

$$D := \{\ell \in L^*; |\ell|_h < 1\}$$

the boundary $X := \partial D = \{\ell \in L^*; |\ell|_h = 1\}$ over M is an S^1 -bundle. Let $\text{pr}: X \rightarrow M$ be the natural projection. We now consider the Szegő kernel

$$S_\omega := S_\omega(x, y)$$

for the projection of $L^2(X)$ onto the Hardy space $L^2(X) \cap \Gamma(D, \mathcal{O})$ of boundary values of holomorphic functions on D . Then for each positive integer m the corresponding

m -th Bergman kernel $B_{m,\omega}$ for the Kähler manifold (M, ω) is characterized as the Fourier coefficient

$$\text{pr}^* B_{m,\omega} := \frac{n!}{m^n} \int_{S^1} e^{-im\theta} S_\omega(e^{i\theta}x, x) d\theta.$$

Now the Bergman kernel is defined not only for positive integers m but also for complex numbers ξ as follows. To see the situation, we first consider the case where M is a single point. Then in place of $S_\omega(e^{i\theta}x, x)$ consider a smooth function $S = S(\theta)$ on $S^1 := \mathbb{R}/2\pi\mathbb{Z}$ for simplicity. The associated Fourier coefficient B_m is

$$B_m = \int_{S^1} e^{-im\theta} S(\theta) d\theta$$

for each integer $m \neq 0$. Then for open intervals $I_1 := (-3\pi/4, 3\pi/4)$ and $I_2 = (\pi/4, 7\pi/4)$ in \mathbb{R} we choose the open cover

$$S^1 = U_1 \cup U_2$$

where $U_1 := I_1 \bmod 2\pi$ and $U_2 := I_2 \bmod 2\pi$. By choosing a partition of unity subordinate to this open cover we write

$$\rho_1(\theta) + \rho_2(\theta) = 1, \quad \theta \in S^1,$$

where $\rho_\alpha \in C^\infty(S^1)_{\mathbb{R}}$, $\alpha = 1, 2$, are functions ≥ 0 satisfying $\text{Supp}(\rho_\alpha) \subset U_\alpha$. For the coordinate $\tilde{\theta}$ for \mathbb{R} , writing $\tilde{\theta} \bmod 2\pi$ as θ , define $\tilde{\rho}_\alpha \in C^\infty(\mathbb{R})_{\mathbb{R}}$, $\alpha = 1, 2$, by

$$\tilde{\rho}_\alpha(\tilde{\theta}) = \begin{cases} \rho_\alpha(\theta), & \tilde{\theta} \in I_\alpha \\ 0, & \tilde{\theta} \notin I_\alpha. \end{cases}$$

Then the Fourier transform $\mathcal{F}(S) = \mathcal{F}(S)(\xi)$ of S is an entire function in $\xi \in \mathbb{C}$ defined as the integral

$$\mathcal{F}(S)(\xi) = \int_{\mathbb{R}} e^{-i\xi\tilde{\theta}} \{\rho_1(\tilde{\theta}) + \rho_2(\tilde{\theta})\} S(\tilde{\theta}) d\tilde{\theta}, \quad \xi \in \mathbb{C},$$

satisfying $\mathcal{F}(S)(m) = B_m$ for all integers m . Though $\mathcal{F}(S)$ may depend on the choice of the partition of unity, its restriction to \mathbb{Z} is unique. This situation is easily understood for instance by the fact the functions $\mathcal{F}(S)(\xi)$ and $\mathcal{F}(S)(\xi) + \sin(\pi\xi)$ in ξ coincide on \mathbb{Z} .

Now the above process of generalization from the Fourier series to the Fourier transform is valid also for the case where the base space M is nontrivial. Actually, we define an entire function $B_{\xi,\omega}$ in $\xi \in \mathbb{C}$ by

$$\text{pr}^* B_{\xi,\omega} := \frac{n!}{\xi^n} \int_{\mathbb{R}} e^{-i\xi\tilde{\theta}} \{\rho_1(\tilde{\theta}) + \rho_2(\tilde{\theta})\} S_\omega(e^{i\tilde{\theta}}x, x) d\tilde{\theta}.$$

By setting $q := \xi^{-1}$ we study the asymptotic behavior of $B_{\xi,\omega}$ as $q \rightarrow 0$ along the positive real line $\{q > 0\}$. Let σ_ω denote the scalar curvature of the Kähler manifold (M, ω) . Then as in discrete cases by Tian [43], Zelditch [50], Catlin [4], the asymptotic expansion of $B_{\xi,\omega}$ in q yields

$$B_{\xi,\omega} = 1 + a_1(\omega)q + a_2(\omega)q^2 + \dots, \quad 0 \leq q \ll 1, \tag{2}$$

where $a_1(\omega) = \sigma_\omega/2$ by a result of Lu [24]. For more details of the expansion in discrete cases see also Hirachi [21].

5. Balanced metrics

Choose a Hermitian metric h for L such that $\omega := c_1(L; h)$ is a Kähler form. Then ω is called an m -th *balanced metric* (cf. [51], [26]) for (M, L) if $B_{m,\omega}$ is a constant function ($= C_m$) on M . First put $q := 1/m$. By integrating (1) and (2) on the Kähler manifold (M, ω) we see that C_m is written as

$$C_m := \frac{n!}{m^n c_1(L)^n[M]} N_m = 1 + \frac{n}{2} \frac{c_1(M)c_1(L)^{n-1}[M]}{c_1(L)^n[M]} q + O(q^2), \quad 0 \leq q \ll 1,$$

where the left-hand side is the Hilbert polynomial $P(m)$ for (M, L) divided by $m^n c_1(L)^n[M]/n!$. Hence it is easy to define C_ξ by setting

$$C_\xi = \frac{n!P(\xi)}{\xi^n c_1(L)^n[M]}, \quad \xi \in \mathbb{C}^*. \tag{3}$$

Now put $q := 1/\xi$. Define the modified Bergman kernel $\beta_{q,\omega}$ by

$$\beta_{q,\omega} := 2\xi \left(1 + \frac{2q}{3} \Delta_\omega \right) (B_{\xi,\omega} - C_\xi) = \sigma_\omega - \bar{\sigma}_\omega + O(q), \tag{4}$$

where the average $\bar{\sigma}_\omega$ of the scalar curvature σ_ω is $nc_1(M)c_1(L)^{n-1}[M]/c_1(L)^n[M]$ independent of the choice of ω in $c_1(L)_\mathbb{R}$. Then, for $\xi = m$, ω is an m -th balanced metric for (M, L) if and only if $\beta_{q,\omega}$ vanishes everywhere on M . Recall the following result by Zhang [51] (cf. [26]; see also [30]):

Fact 3. *(M, L^m) is Chow–Mumford stable if and only if (M, L) admits an m -th balanced metric.*

Consider the maximal connected linear algebraic subgroup H of $\text{Aut}(M)$, so that the identity component of $\text{Aut}(M)/H$ is an Abelian variety. Let us now choose an algebraic torus $T \cong (\mathbb{C}^*)^r$ in the connected component $Z^\mathbb{C}$ of the center of the reductive part $R(H)$ for H in the Chevalley decomposition

$$H = R(H) \times H_u,$$

where H_u is the unipotent radical of H . Replacing L by its suitable positive multiple, we may assume that the H -action on M is lifted to a bundle action on L covering the H -action on M . For each character $\chi \in \text{Hom}(T, \mathbb{C}^*)$ we set

$$W\chi := \{\sigma \in V_m ; \sigma \cdot g = \chi(g)\sigma \text{ for all } g \in T\},$$

where $V_m \times T \ni (\sigma, g) \mapsto \sigma \cdot g$ is the right T -action on $V_m = H^0(M, \mathcal{O}(L^m))$ induced by the left H -action on L . Now we have characters $\chi_k \in \text{Hom}(T, \mathbb{C}^*)$, $k = 1, 2, \dots, r_m$, such that the vector space V_m is expressible as a direct sum

$$V_m = \bigoplus_{k=1}^{r_m} W\chi_k.$$

For the maximal compact torus T_c in T we may assume that both h and ω are T_c -invariant. Put $\mathbb{J}_m := \{1, 2, \dots, N_m\}$, where $N_m := \dim V_m$. Choose an orthonormal basis $\{\sigma_1, \sigma_2, \dots, \sigma_{N_m}\}$ for V_m such that all σ_j , $j \in \mathbb{J}_m$, belong to the union $\bigcup_{k=1}^{r_m} W\chi_k$. Hence there exists a map $\kappa : \mathbb{J}_m \rightarrow \{1, 2, \dots, r_m\}$ satisfying

$$\sigma_j \in W\chi_{\kappa(j)}, \quad j \in \mathbb{J}_m.$$

Put $\mathfrak{t}_{\mathbb{R}} := i\mathfrak{t}_c$ for the Lie algebra \mathfrak{t}_c of T_c where $i := \sqrt{-1}$. For each $\mathcal{Y} \in \mathfrak{t}_{\mathbb{R}}$, by setting $g := \exp(\mathcal{Y}/2)$, we put $h_g := h \cdot g$ for the natural T -action on the space of Hermitian metrics on L . Define the m -th weighted Bergman kernel $B_{m,\omega,\mathcal{Y}}$, twisted by \mathcal{Y} , for (M, ω) by setting

$$B_{m,\omega,\mathcal{Y}} := \frac{n!}{m^n} \sum_{j=1}^{N_m} |\sigma_j|_{h_g}^2 = g^* \left\{ \frac{n!}{m^n} \sum_{j=1}^{N_m} \frac{|\sigma_j|_h^2}{|\chi_{\kappa(j)}(g)|^2} \right\}.$$

Then ω is called an m -th T -balanced metric on (M, L) if $B_{m,\omega,\mathcal{Y}}$ is a constant function ($= C_{m,\mathcal{Y}}$) on M for some $\mathcal{Y} \in \mathfrak{t}$. Consider the natural action of the group

$$G_m := \bigoplus_{k=1}^{r_m} \text{SL}_{\mathbb{C}}(W\chi_k)$$

acting on $V_m = \bigoplus_{k=1}^{r_m} W\chi_k$ diagonally (factor by factor). We say that (M, L^m) is Chow–Mumford T -stable if the orbit $G_m \cdot M_m$ is closed in W_m^* . Note that (cf. [30])

Fact 4. *If M admits an m -th T -balanced metric on (M, L) , then (M, L^m) is Chow–Mumford T -stable.*

We shall now extend $\{B_{m,\omega,\mathcal{Y}} ; m = 1, 2, \dots\}$ to $\{B_{\xi,\omega,\mathcal{Y}} ; \xi \in \mathbb{C}\}$ in such a way that $B_{m,\omega,\mathcal{Y}}$ coincides with $B_{\xi,\omega,\mathcal{Y}}|_{\xi=m}$ for all positive integers m . By the definition of $B_{m,\omega,\mathcal{Y}}$ (see also [30], p. 578) the equality $B_{m,\omega,\mathcal{Y}} = B_{m,\omega}(h_g/h)^m$ always holds. Hence we put

$$B_{\xi,\omega,\mathcal{Y}} := B_{\xi,\omega} \cdot (h_g/h)^\xi = B_{\xi,\omega} \exp\{\xi \log(h_g/h)\}, \quad \xi \in \mathbb{C}. \tag{5}$$

Once $B_{\xi,\omega,\mathcal{Y}}$ is defined, we can also define $C_{\xi,\mathcal{Y}}$, $\xi \in \mathbb{C}$ in such a way that $C_{m,\mathcal{Y}}$ coincides with $C_{\xi,\mathcal{Y}}|_{\xi=m}$. Actually, we put

$$C_{\xi,\mathcal{Y}} := \int_M \frac{B_{\xi,\omega,\mathcal{Y}}}{c_1(L)^n[M]} g^* \omega^n. \tag{6}$$

6. A simple heuristic proof of Donaldson’s theorem

In this section we shall show that a heuristic application of the implicit function theorem simplifies the proof of Donaldson’s theorem [14]. Fix a Kähler metric ω_0 in $c_1(L)_{\mathbb{R}}$ of constant scalar curvature. Assume that the group H in the previous section is trivial. For each Kähler metric ω in $c_1(L)_{\mathbb{R}}$ we can associate a unique real-valued smooth function φ on M such that

$$\omega = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi$$

with normalization condition $\int_M \varphi \omega_0^n = 0$. For an arbitrary nonnegative integer k and a real number α satisfying $0 < \alpha < 1$, we more generally consider the case where $\varphi \in C^{k+4,\alpha}(M)_{\mathbb{R}}$, so that ω is a $C^{k+2,\alpha}$ Kähler metric on M . The Fréchet derivative $D_{\omega} \sigma_{\omega}$ at $\omega = \omega_0$ of the scalar curvature function $\omega \mapsto \sigma_{\omega}$ is given by

$$\{(D_{\omega} \sigma_{\omega})(\sqrt{-1} \partial \bar{\partial} \varphi)\}|_{\omega=\omega_0} = \lim_{\varepsilon \rightarrow 0} \frac{\sigma_{\omega_0 + \sqrt{-1} \varepsilon \partial \bar{\partial} \varphi} - \sigma_{\omega_0}}{\varepsilon} = L_{\omega_0} \varphi,$$

where $L_{\omega_0} : C^{k+4,\alpha}(M)_{\mathbb{R}} \rightarrow C^{k,\alpha}(M)_{\mathbb{R}}$ is the Lichnerowicz operator for the Kähler metric ω_0 (cf. [23], [14]). Then by (4) the Fréchet derivative $D_{\omega} \beta_{q,\omega}$ of $\beta_{q,\omega}$ with respect to ω at $(q, \omega) = (0, \omega_0)$ is written as

$$\{(D_{\omega} \beta_{q,\omega})(\sqrt{-1} \partial \bar{\partial} \varphi)\}|_{(q,\omega)=(0,\omega_0)} = \{(D_{\omega} \sigma_{\omega})(\sqrt{-1} \partial \bar{\partial} \varphi)\}|_{\omega=\omega_0} = L_{\omega_0} \varphi,$$

where L_{ω_0} is an invertible operator by the triviality of H . By setting $q := 1/\xi$, we move q in the half line $\mathbb{R}_{\geq 0} := \{0\} \cup \{1/\xi ; \xi > 0\}$. Replacing q by q^2 if necessary, we apply the implicit function theorem to the map $(q, \omega) \mapsto \beta_{q,\omega}$. (The required regularity for this map is rather delicate: By using [3], Theorem 1.5 and §2.c, we can write both $B_{\xi,\omega}$ and its ω -derivatives as integrals similar to (18) in [50]. Then the estimate of the remainder term in the asymptotic expansion for $B_{m,\omega}$ in [50], Theorem 1, is valid also for the asymptotic expansion of $B_{\xi,\omega}$ and its ω -derivatives. However, for continuity of $\beta_{q,\omega}$ and its ω -derivative, more delicate estimates are necessary. Related to Nash–Moser’s process, this will be treated elsewhere.) Then we have openness of the solutions for the one-parameter family of equations

$$\beta_{q,\omega} = 0, \quad q \geq 0, \tag{7}$$

i.e., there exists a one-parameter family of $C^{k+2,\alpha}$ solutions $\omega = \omega(q)$, $0 \leq q < \varepsilon$, for (7) with $\omega(0) = \omega_0$, where ε is sufficiently small. Hence by Fact 3 in Section 5, (M, L^m) is Chow–Mumford stable for all integers $m > 1/\varepsilon$. \square

7. The case where M admits symmetries

In this section we consider a polarized manifold (M, L) with an extremal Kähler metric ω_0 in $c_1(L)_{\mathbb{R}}$. Then following Section 6, a result on stability in [32] will be discussed. Let \mathcal{V} be the associated extremal Kähler vector field on M . Assume that H is possibly of positive dimension. Then the identity component K of the isometry group of (M, ω_0) is a maximal compact subgroup in H . Let \mathfrak{z} be the Lie algebra of the identity component Z of the center of K .

Step 1. We assume that the algebraic torus T in Section 5 is the complexification $Z^{\mathbb{C}}$ of Z in H , so that the Lie algebra \mathfrak{t} of T coincides with the Lie algebra $\mathfrak{z}^{\mathbb{C}}$ of $Z^{\mathbb{C}}$. Let $q \in \mathbb{R}_{\geq 0}$, where we set $q = 1/\xi$ for the part $0 < \xi \in \mathbb{R}$. Take an element \mathcal{W} in the Lie algebra \mathfrak{z} . Let ω be a Kähler metric in the class $c_1(L)_{\mathbb{R}}$. Then by setting $\mathcal{Y} := iq^2(\mathcal{V} + \mathcal{W})$ for $i := \sqrt{-1}$, we can now consider the following modified weighted Bergman kernel

$$\beta_{q,\omega,\mathcal{W}} := 2\xi \left(1 + \frac{2q}{3} \Delta_{\omega} \right) (B_{\xi,\omega,\mathcal{Y}} - C_{\xi,\mathcal{Y}}), \quad \xi \in \mathbb{C}^*,$$

where we used (5) and (6) in defining $B_{\xi,\omega,\mathcal{Y}}$ and $C_{\xi,\mathcal{Y}}$. For the Kähler manifold (M, ω) , consider the Hamiltonian function $\hat{\sigma}_{\omega} \in C^{\infty}(M)_{\mathbb{R}}$ for \mathcal{V} characterized by

$$\mathcal{V} \lrcorner \omega = \bar{\partial} \hat{\sigma}_{\omega} \quad \text{and} \quad \int_M \sigma_{\omega} \omega^n = \int_M \hat{\sigma}_{\omega} \omega^n.$$

For each $\mathcal{Z} \in \mathfrak{z}$ the associated Hamiltonian function $f_{\mathcal{Z},\omega} \in C^{\infty}(M)_{\mathbb{R}}$ is uniquely characterized by the identities $\mathcal{Z} \lrcorner \omega = \bar{\partial} f_{\mathcal{Z},\omega}$ and $\int_M f_{\mathcal{Z},\omega} \omega^n = 0$. Note that $\mathcal{V} \in \mathfrak{z}$. Then also in this case, as in (4), we obtain

$$\beta_{q,\omega,\mathcal{W}} = \sigma_{\omega} - \hat{\sigma}_{\omega} - f_{\mathcal{W},\omega} + O(q). \tag{8}$$

Choose an arbitrary nonnegative integer ℓ with a real number $0 < \alpha < 1$. For the space \mathcal{F}_{ℓ} of all K -invariant functions $f \in C^{\ell,\alpha}(M)_{\mathbb{R}}$ satisfying $\int_M f \omega_0^n = 0$, by setting $\mathcal{N} := \text{Ker } L_{\omega_0} \cap \mathcal{F}_{\ell}$, we have the identification

$$\theta: \mathfrak{z} \cong \mathcal{N}, \quad \mathcal{Z} \leftrightarrow \theta(\mathcal{Z}) := f_{\mathcal{Z},\omega_0},$$

where \mathcal{N} is independent of the choice of ℓ . Then the vector space \mathcal{F}_{ℓ} is written as a direct sum $\mathcal{N} \oplus \mathcal{N}_{\ell}^{\perp}$, where $\mathcal{N}_{\ell}^{\perp}$ is the space of all functions f in \mathcal{F}_{ℓ} such that $\int_M f \nu \omega_0^n = 0$ for all $\nu \in \mathcal{N}$. For an arbitrary integer $k \geq 0$, we make a small perturbation of ω_0 by varying ω in the space $\{\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi; \varphi \in \mathcal{N}_{k+4}^{\perp}\}$. Since ω_0 is an extremal Kähler metric, we see from (8) that $\beta_{q,\omega,\mathcal{W}}$ vanishes at $(q, \omega, \mathcal{W}) = (0, \omega_0, 0)$, i.e.,

$$\beta_{0,\omega_0,0} = 0 \quad \text{on } M.$$

Again from (8) we see that the Fréchet derivatives $D_\omega \beta_{q,\omega,\mathcal{W}}$ and $D_{\mathcal{W}} \beta_{q,\omega,\mathcal{W}}$ of $\beta_{q,\omega,\mathcal{W}}$ at $(q, \omega, \mathcal{Y}) = (0, \omega_0, 0)$ are

$$\begin{cases} \{(D_\omega \beta_{q,\omega,\mathcal{W}})(\sqrt{-1} \partial \bar{\partial} \varphi)\}_{|(q,\omega,\mathcal{W})=(0,\omega_0,0)} = L_{\omega_0} \varphi, & \varphi \in \mathcal{N}_{k+4}^\perp, \\ (D_{\mathcal{W}} \beta_{q,\omega,\mathcal{W}})_{|(q,\omega,\mathcal{W})=(0,\omega_0,0)} = -\theta, & \text{on } \mathcal{N}. \end{cases}$$

Since $L_{\omega_0}: \mathcal{N}_{k+4}^\perp \rightarrow \mathcal{N}_k^\perp$ is invertible and since θ is an isomorphism, the implicit function theorem is now applicable to the map: $(q, \omega, \mathcal{W}) \mapsto \beta_{q,\omega,\mathcal{W}}$. Then for some $0 < \varepsilon \ll 1$ we can write

$$\omega = \omega(q) \quad \text{and} \quad \mathcal{W} = \mathcal{W}(q), \quad 0 \leq q \leq \varepsilon,$$

solving the one-parameter family of equations

$$\beta_{q,\omega,\mathcal{W}} = 0, \quad q \geq 0,$$

in (ω, \mathcal{W}) . Hence by setting $\mathcal{Y}(q) := iq^2(\mathcal{V} + \mathcal{W}(q))$ for $q = 1/m$, the m -th weighted Bergman kernel $B_{m,\omega(q),\mathcal{Y}(q)}$ is constant on M for all $m > 1/\varepsilon$. Then by Fact 4 in Section 5, (M, L^m) is Chow–Mumford T -stable for all $m > 1/\varepsilon$.

Step 2. Though we assumed that T coincides with $Z^\mathbb{C}$ in the first step, it is better to choose T as small as possible. Then for a sufficiently small positive real number ε , the algebraic torus T in $Z^\mathbb{C}$ generated by

$$\bigcup_{1/\varepsilon < m \in \mathbb{Z}} \{\mathcal{V} + \mathcal{W}(q), i\mathcal{V} + i\mathcal{W}(q)\}$$

is a good choice, where $q = 1/m$ and $i := \sqrt{-1}$.

8. Concluding remarks

For the conjecture in Section 3 the “if” part is known to be a difficult problem and it is of particular interest. Assuming that a polarized manifold (M, L) is asymptotically Chow–Mumford stable, we are asked whether there exist Kähler metrics of constant scalar curvature in $c_1(L)_\mathbb{R}$. For simplicity, we consider the case where H is trivial. Then by Fact 3 the equation

$$\beta_{q,\omega} = 0 \tag{9}$$

has solutions $(q, \omega) = (1/m, \omega_m)$, $m \gg 1$, while each ω_m is an m -th balanced metric for (M, L) . Moreover, the triviality of H implies that ω_m is the only m -th balanced metric for (M, L) . Then we are led to study the graph

$$\mathcal{E} := \{(q, \omega); \beta_{q,\omega} = 0\}$$

in $\mathbb{C} \times \mathcal{K}$, where \mathcal{K} is the space of all Kähler metrics in the class $c_1(L)_{\mathbb{R}}$. Let \mathcal{H} , \mathcal{M} be the sets of all Hermitian metrics for L , V_m , respectively. Consider the Fréchet derivative $D_{\omega}\beta_{q,\omega}$ at $(q, \omega) = (1/m, \omega_m) \in \mathcal{E}$, where $m \gg 1$. With the same setting of differentiability as in Section 6, the Fréchet derivative will be shown to be invertible.

Let us give a rough idea how the invertibility can be proved. It suffices to show that the operator $D_{\omega}B_{m,\omega}$ is invertible at $\omega = \omega_m$. Choose a Hermitian metric h_m for L such that $c_1(L; h_m) = \omega_m$. Put $\Psi := \{\psi \in C^{\infty}(M)_{\mathbb{R}}; \int_M \psi \omega_m^n = 0\}$. Let $\varphi \in \Psi$. Then the \mathbb{R} -orbit in \mathcal{H} through h_m written in the form

$$h_{\varphi;t} := e^{-t\varphi} h_m, \quad t \in \mathbb{R}, \tag{10}$$

projects to the \mathbb{R} -orbit $\omega_{\varphi,t} := \omega_m + \sqrt{-1} t \partial\bar{\partial}\varphi$, $t \in \mathbb{R}$, in \mathcal{K} through ω_m . Note also that every Hermitian metric h for L induces a Hermitian pairing $\langle \cdot, \cdot \rangle_{L^2}$ which will be denoted by (V_m, \tilde{h}) (cf. Section 4). Choose an orthonormal basis $\{\sigma_1, \sigma_2, \dots, \sigma_{N_m}\}$ for (V_m, \tilde{h}_m) . Let Φ be the space of all $\varphi \in \Psi$ such that

$$(d/dt)(\tilde{h}_{\varphi,t})|_{t=0} = 0 \quad \text{on } V_m.$$

In other words, if $\varphi \in \Phi$ then the basis $\{\sigma_1, \sigma_2, \dots, \sigma_{N_m}\}$ infinitesimally remains to be an orthonormal basis for $(V_m, \tilde{h}_{\varphi,t})$ at $t = 0$ with t perturbed a little. Since

$$B_{m,\omega} := |\sigma_1|_h^2 + |\sigma_2|_h^2 + \dots + |\sigma_{N_m}|_h^2$$

is obtained from the contraction of $\Sigma := \sigma_1\bar{\sigma}_1 + \sigma_2\bar{\sigma}_2 + \dots + \sigma_{N_m}\bar{\sigma}_{N_m}$ by h^m , and since Σ is fixed by the infinitesimal action $(d/dt)(h_{\varphi,t})$ at $t = 0$ in (10), we obtain

$$(d/dt)(B_{m,\omega_{\varphi,t}})|_{t=0} = (d/dt)(h_{\varphi,t}^m)|_{t=0} = -m\varphi. \tag{11}$$

Let \mathbb{H} denote the set of all Hermitian metrics on the vector space V_m . Then we have a natural projection $\pi : \Psi \rightarrow T_{\tilde{h}_m} \mathbb{H}$ defined by

$$(d/dt)(h_{\varphi,t})|_{t=0} \mapsto (d/dt)(\tilde{h}_{\varphi,t})|_{t=0}.$$

In view of $\dim \mathbb{H} < +\infty$, it is now easy to see that this map is surjective. Since the kernel of this map is Φ , we obtain

$$\Psi/\Phi \cong T_{\tilde{h}_m} \mathbb{H}. \tag{12}$$

Finally by (11) and (12), the uniqueness of the m -th balanced metric (by $H = \{1\}$) implies the required invertibility of the Fréchet derivative (see also Donaldson [16]). \square

Now we can apply the implicit function theorem to obtain an open neighborhood U of $1/m$ in \mathbb{C} such that, for each $\xi \in U$, there exists a unique Kähler metric $\omega(\xi)$ in $c_1(L)_{\mathbb{R}}$ satisfying $\beta_{\xi,\omega(\xi)} = 0$.

Assuming non-existence of Kähler metrics of constant scalar curvature, we have some possibility that, by an argument as in Nadel, destabilizing objects are obtained by studying the behavior of the solutions along the boundary point of \mathcal{E} .

Finally, there are many interesting topics, which are related to ours, such as the geometry of Kähler potentials [10], [9], [38], [40] (see also [28]). The uniqueness of extremal Kähler metrics modulo the action of holomorphic automorphisms in compact cases is recently given in [12] (cf. [2], [14]). New obstructions to semistability of manifolds or to the existence of extremal metrics are done by [19], [29], [12]. The concept of K-stability, introduced by Tian [45] and reformulated by Donaldson [15], is deeply related to the Hilbert–Mumford stability criterion, and various kinds of works are actively done related to algebraic geometry.

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