Properly embedded minimal surfaces with finite topology

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Abstract. We present a synthesis of the situation as it now stands about the various moduli spaces of properly embedded minimal surfaces of finite topology in flat 3-manifolds. This family includes the case of minimal surfaces with finite total curvature in $\mathbb{R}^3$ as well as singly, doubly and triply periodic minimal surfaces.

Mathematics Subject Classification (2000). Primary 53A10; Secondary 53C42.

Keywords. Minimal surfaces, flux, least area.

1. Introduction

In these notes we will consider minimal surfaces Σ of finite topology which are properly embedded in $\mathbb{R}^3$ or in a complete flat 3-manifold $M = \mathbb{R}^3/\mathcal{G}$. For most of purposes, up to passing to a finite covering, we can assume that Σ is orientable and $\mathcal{G}$ is a cyclic group of rank 1, 2 or 3, which correspond to the singly, doubly and triply periodic cases, respectively. In the singly periodic case, $\mathcal{G}$ is generated by a screw motion (which in particular could be a translation). In the other cases, $\mathcal{G}$ consisting only on translations and M is either a flat 2-torus times $\mathbb{R}$, $T^2 \times \mathbb{R}$, or a flat 3-torus T^3.

We will focus on uniqueness and classification results. We will also emphasize those ideas and techniques which are (or could be) useful to understand the structure of moduli spaces of minimal surfaces. Although we have a large number of results in this area, several important questions about these moduli spaces remain unanswered.

There is also an interesting theory for the family of properly embedded minimal surfaces of finite genus and infinitely many ends, see for instance Meeks, Pérez and Ros [33] for structure results and Hauswirth and Pacard [13] for some recent examples. However we will not consider this situation in this paper. It is worth noticing that Colding and Minicozzi have proved that complete minimal surfaces of finite topology embedded in $\mathbb{R}^3$ are necessarily proper [3]. The same result holds in flat 3-manifolds, see [51].

Most of the surfaces we will consider have finite total curvature. They form an important and natural subclass. We refer the reader to the texts Pérez and Ros [50] and Hoffman and Karcher [15] and references therein, for more details about these surfaces.

*Partially supported by MCYT-FEDER research project MTM2004-02746.

Proceedings of the International Congress
of Mathematicians, Madrid, Spain, 2006

2. Geometry of the ends

An important achievement of the last years has been the complete understanding of the asymptotic geometry of these surfaces: the ends approximate simple model surfaces like the plane, the Catenoid or the Helicoid. Hoffman and Meeks [17] and Collin [5] showed that each end of a properly embedded minimal surface in Σ in $\mathbb{R}^3$ with finite topology and more than one end is asymptotic to either a plane or a Catenoid and the ends are all parallel (henceforth we will assume that these ends are horizontal). If Σ has just one end, then Meeks and Rosenberg [38] prove that Σ approaches to a Helicoid. The proof depends on recent results concerning limits of embedded minimal surfaces without curvature bounds, see Colding and Minicozzi [4], [40] and references therein.

In the periodic case, the geometry of the surface at infinity has been determined by Meeks and Rosenberg [35], [36], see Table 1 (note that when M is a 3-torus, we are just considering compact minimal surfaces). Assuming $M = \mathbb{R}^3/S_\theta$, where S_θ denotes a screw motion of angle θ and vertical axis, each end is asymptotic either to a plane (when $\theta \neq 0$, the plane must be horizontal), or to a flat vertical annulus like in the Scherk surface, see Figure 1 (this kind of end occurs only if θ is rational), or to the end of a vertical Helicoid.

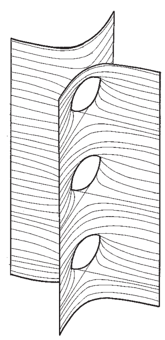

Figure 1. The singly periodic Scherk surface (seeing as a surface in the quotient space) has genus zero and four ends asymptotic to half-cylinders. It is a 1-parameter family of surfaces (the parameter is given by the angle between two consecutive wings).

If $M = T^2 \times \mathbb{R}$, then Σ has an even number of top (resp. bottom) ends and all of them are of Scherk type. The top ends are all parallel and the same holds for the bottom ones. Moreover, in case the top ends are not parallel to the bottom ends, then all the ends are vertical.

The above description of the ends has strong consequences on the geometry and the conformal structure of the minimal surface: Σ is conformally equivalent to a closed Riemann surface $\overline{\Sigma}$ with a finite number of punctures and the surface Σ can be described, by means of the Weierstrass representation, in terms of meromorphic data on the compactified surface $\overline{\Sigma}$.

Table 1. The admissible behaviour of the ends of a nonflat finite topology minimal surface. All the cases, but the first one, have finite total curvature. In the singly periodic case, all the ends of a given surface must be of the same type. The non-periodic helicoidal end is asymptotic to the Helicoid in $\mathbb{R}^3$, while the singly periodic one is asymptotic to that surface in $\mathbb{R}^3/S_\theta$.

periodicity	kind of ends
non-periodic	one helicoidal end
non-periodic	planar or catenoidal (more than one end)
singly-periodic	planar, helicoidal or Scherk ends
doubly-periodic	Scherk type

3. Minimal surfaces with finite topology in $\mathbb{R}^3$

Given integers $k \geq 0$ and $r \geq 1$, let $\mathcal{M}(k,r)$ be the moduli space of minimal surfaces $\Sigma \subset \mathbb{R}^3$ of genus k and r horizontal ends, properly embedded in $\mathbb{R}^3$. The simplest examples in this family can be characterized in terms of its topology.

Theorem 3.1. *The only properly embedded minimal surfaces of finite topology and genus zero in $\mathbb{R}^3$ are the Plane, the Catenoid and the Helicoid.*

If the surface has more than one end, the above result was proved by López and Ros [26]. In the one ended case this is a recent result of Meeks and Rosenberg [38]. The uniqueness of the Helicoid was a long standing problem which has been solved by using results of Colding and Minicozzi [4], [40].

Theorem 3.2. *The unique properly embedded minimal surface of finite topology in $\mathbb{R}^3$ with two ends is the Catenoid.*

Observe that in this characterization we prescribe only the number of ends, but not the genus. It was proved by Schoen [60] using the Alexandrov reflexion technique. A result in the same spirit has been obtained recently by Meeks and Wolf [39]: they prove that the singly periodic Scherk surface is characterized as the unique properly embedded minimal surface in $\mathbb{R}^3/T$ with four ends of Scherk type.

The first examples of higher genus where obtained by Costa [7] and Hoffman and Meeks [16] in the eighties. They constructed surfaces $\Sigma(k)$ of genus $k \geq 2$, two catenoidal ends and a middle planar end. The picture of the surface can be described as follows: each horizontal plane, other that $x_3 = 0$, meets the surface in a Jordan curve and $\Sigma \cap \{x_3 = 0\}$ consists on an equiangular system of $k+1$ straight lines, see Figure 2.

These examples can be characterized as the ones of maximal symmetry in term of the genus of the surface.

Theorem 3.3. *Let Σ be a properly embedded minimal surface $\mathbb{R}^3$ with finite topology and positive genus. Then the symmetry group of Σ satisfies* $|\,\mathrm{Sym}(\Sigma)| \leq$

4(genus(Σ)+1). *Moreover, the equality holds if and only if Σ is one of the three-ended surfaces $\Sigma(k)$ constructed by Costa, Hoffman and Meeks.*

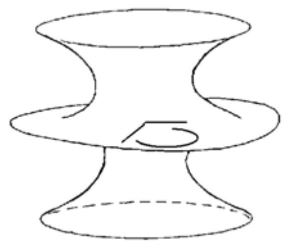

Figure 2. The Costa surface has genus one and three ends: the middle one is planar and the other are of catenoidal type. The surface has two vertical mirror planes and two horizontal reflection axes contained in the surface, but not in the mirror planes.

In the case Σ has three ends, the uniqueness in Theorem 3.3 was proved by Meeks and Hoffman [16]: the assumptions allows us to determine the symmetry group $\mathcal{G}$, the conformal structure of the surface, its picture in $\mathbb{R}^3$, up to a $\mathcal{G}$-invariant isotopy, and, finally, its Weierstrass data. This analysis has been used in several situations to produce new examples or to classify minimal surfaces with prescribed symmetry, see §1 in [1] for a general description of the method and concrete applications (in the final step we meet the so called *periods problem* which can be completely solved only in some cases). If the number of ends r is larger than 3, then the same kind of analysis shows that $r = 4$, $x_3 = 0$ is a mirror plane of the surface, $\Sigma \cap \{x_3 = 0\}$ consists of $k + 1$ Jordan curves with pairwise disjoint interior and $\Sigma \cap \{x_3 \geq 0\}$ is a surface of genus zero, 2 ends and $k + 1$ boundary components like in Figure 3, see Ros [54]. Finally we can prove that this surface does not exists by using the *vertical*

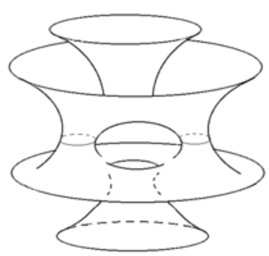

Figure 3. In $\mathbb{R}^3$, there are not properly embedded minimal surfaces of genus one, finitely many horizontal ends and an horizontal mirror plane. This can be shown by using the vertical flux deformation argument.

flux deformation argument, see [54] and Section 5 below. In fact, the surfaces exist as immersed surfaces, but they are not embedded. If $r = 1$, then the end is helicoidal and, therefore, the order of the symmetry group is at most 4. So this case is discarded.

The surface $\Sigma(k)$ admits a 1-parameter deformation by means of embedded minimal surfaces $\Sigma(k, t)$, $t \in \mathbb{R}$, with $\Sigma(k, 0) = \Sigma(k)$ and three catenoidal ends for $t \neq 0$. Its symmetry group is generated by k vertical reflexion planes in equiangular position, see Hoffman and Karcher [15].

Costa [8] classified minimal tori with three punctures. This is the only positive genus case where the moduli space is completely known. The proof depends on the properties of elliptic functions.

Theorem 3.4. *Any properly embedded minimal surface in $\mathbb{R}^3$ with genus 1 and three ends lies in the family above, $\mathcal{M}(1, 3) = \{\Sigma(1, t)\}$.*

For genus larger than 1, the above surfaces can be characterized in terms of its symmetries. The following classification theorem by Martin and Weber [30] extends previous results in [15], [25].

Theorem 3.5. *Let Σ be a properly embedded minimal surface in $\mathbb{R}^3$ with three ends and genus $k \geq 2$. If $|\operatorname{Sym}(\Sigma)| \geq 2(k + 1)$, then Σ is one the of surfaces $\Sigma(k, t)$.*

A central and basic open problem concerning finite topology minimal surfaces in $\mathbb{R}^3$ is the following one, see Hoffman and Meeks [16].

Conjecture. The moduli space $\mathcal{M}(k, r)$ is empty for $r > k + 2$.

In the case $k = 0$, this is has been proved in [26]. For higher genus, Meeks, Pérez and Ros [34] have proved that, given k, $\mathcal{M}(k, r)$ is empty for r large enough.

In the one-ended case, Hoffman, Weber and Wolf [18] have constructed a properly embedded minimal surface of genus one and one helicoidal end. No characterization of this example is known at the present. Meeks and Rosenberg [38] have proposed the following question (which they proved for $k = 0$).

Conjecture. For each $k = 0, 1, \ldots$, there is a unique properly embedded minimal surface in $\mathbb{R}^3$ of genus k and one end.

There is a large number of examples constructed by *desingularization*, see for instance Kapouleas [21] and Traizet [61] for the non-periodic case and Traizet and Weber [62], [64] for the periodic one.

There is another group of ideas (depending on conformal geometry tools as flat structures, Teichmüller theory, extremal length, …) which is well-adapted to prove existence, nonexistence and deformability results for minimal surfaces with prescribed symmetric-isotopy class, when the fundamental region of the surface is a disc bounded by mirror lines. The method has been developed by Weber and Wolf [65], [66], [67]. For instance, it can be shown by this method that a genus three surface, with the same symmetries than the Costa surface, and in the symmetric isotopy class of the surface in Figure 4 cannot be realized by a minimal surface, see [30].

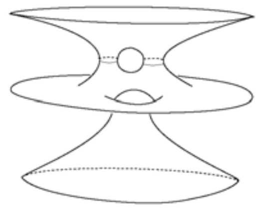

Figure 4. There are no properly embedded minimal surfaces with genus three, three ends, the same symmetries than the Costa surface and belonging to the symmetric-isotopy class of this figure. This can be shown by using flat structures and extremal length arguments.

4. The periodic case

There are a large number of examples of periodic minimal surfaces. As we are interested in those surfaces which have been characterized in some way, we will mention only a few of them.

Meeks [31] has classified the moduli space of minimal Klein bottles with a handle in flat 3-tori. These surfaces have (absolute) total curvature equal to 4π and correspond to the largest Euler characteristic among nonflat closed minimal surfaces in tori.

Theorem 4.1. *The space of closed minimal surfaces* Σ *in flat* 3*-tori with total curvature* 4π *(or, equivalently nonorientable surfaces with* $\chi(\Sigma) = -2$*) is parametrized by the family of antipodally invariant sets* X *in the* 2*-sphere with* $\#(X) = 8$. *Each* X *produces two surfaces, one and its conjugate. The surfaces* Σ *are all embedded.*

The next simplest situation to be understood is the genus 3 case, which contains in particular the orientable double covering of the Meeks surfaces above. There are genus 3 minimal surfaces, like the Schoen Gyroid [59], which cannot be obtained in this way. The following is a basic open question.

Problem. Give an explicit description of the moduli space of closed minimal surfaces of genus 3 in flat 3-tori.

Classical triply periodic minimal surfaces have a large symmetry group. Interestingly, several important examples have been constructed by crystallographers, see for instance Schoen [59], Fischer and Koch [11] and Lord and Mackay [27]. Many of these examples deserve a more exhaustive mathematical treatment, see Karcher [23] and Huff [19] for some results in this direction. To fix that idea, we focus now in a concrete question. Fischer and Koch [11] (see also Kawasaki [24]) have classified all the closed spatial polygons $\Gamma \subset \mathbb{R}^3$ which produce embedded triply periodic minimal surfaces by means of the following routine:

1) Construct a discoidal patch Δ by solving the Plateau problem for the boundary Γ.
2) After reflecting the patch Δ with respect to the edges of Γ, and so on, we get an embedded triply periodic minimal surface.

Most of the polygons Γ project monotonically onto a convex polygon in a plane. As a well-known consequence of the maximum principle, each of these Γ bound a unique minimal surface Δ. Therefore we deduce a number of uniqueness results for triply periodic minimal surfaces in terms of symmetry and topology. Eleven families of surfaces can be characterized in this way, see Fisher and Koch [11]. However there are four of these polygons, the ones named S, Y, $C(S)$ and $C(Y)$, which do not satisfy the convexity condition above, and so, the existence and uniqueness question remains to be clarified. In fact the surfaces Y and $C(S)$ can be realized, at least, by the D and P Schwarz surfaces, respectively (to do that we need to take on D and P patches larger than the usual ones) but other realizations cannot not be discarded at the moment.

Among noncompact periodic minimal surfaces which can be characterized in terms of its topology and symmetry, we have the following theorem, concerning a singly periodic version of Costa, Hoffman and Meeks surfaces, which combines existence results of Callahan, Hoffman and Meeks [1] with a uniqueness property by Martín and Rodriguez [29], [28].

Theorem 4.2. *Let $\Sigma \subset \mathbb{R}^3/S_\theta$ be a properly embedded minimal surface of genus $k \geq 2$ and two planar ends. Then* $|\operatorname{Sym}(\Sigma)| \leq 4(k+1)$. *Moreover, if the equality holds, then k is odd, $(k+1)\theta \in 4\pi\,\mathbb{Z}$ and Σ is one of the* (*translation invariant*) *surfaces constructed by Callahan, Hoffman and Meeks.*

It would be interesting and very useful to have a complete list of properly embedded minimal surfaces in flat three manifolds with small total curvature, or more generally, minimal surfaces with small total curvature modulo symmetries. As a first step, we propose the following more concrete question.

Problem. Classify properly embedded minimal surfaces in flat 3-manifolds with (absolute) total curvature smaller than or equal to 4π.

We have ten different types of compact flat 3-manifolds (some of them admit minimal surfaces of total curvature 2π).

5. Vertical flux

According to the *Weierstrass representation*, an orientable minimal surface in $\mathbb{R}^3$ can be represented by the data (Σ, g, ω), where Σ is a Riemann surface, g is a meromorphic map on Σ which corresponds (up to stereographic projection) to the Gauss map of the surface and ω is an holomorphic 1-form vanishing just at the poles of g (and the order of the zero being double of the order of the pole). The minimal surface is recovered as the immersion $\psi\colon \Sigma \to \mathbb{R}^3$ given by

$$\psi = \Re \int \left(\frac{1}{2}(1-g^2), \frac{i}{2}(1+g^2), g\right)\omega. \qquad (1)$$

In order the immersion to be globally well-defined, we require that the real part of the periods of the above integral vanish. More generally, if the surface is periodic, the real part of the periods of (1) must be compatible with the prescribed periodicity. Several aspects of the geometry of the minimal surface are reflected in its Weierstrass representation.

If C is a closed curve on Σ, the *flux along the curve* is defined as the integral of the unit conormal vector, see Figure 5. This corresponds with the imaginary part of the periods of (1).

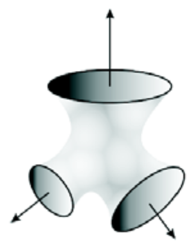

Figure 5. Flux of a minimal surface along its boundary components.

Among the simplest deformations of a minimal surface by minimal surfaces we have the *associated family*, given by $\Sigma_\theta = (\Sigma, g, e^{i\theta}\omega)$, $0 \leq \theta < 2\pi$, and the *vertical flux deformation* $\Sigma_\lambda = (\Sigma, \lambda g, \frac{1}{\lambda}\omega)$, $\lambda > 0$. In the first one the Gauss map and the induced metric are preserved. This deformation is globally well-defined if and only if all fluxes vanish. The second deformation fixes the third coordinate of the immersion and transforms the normal direction in a simple conformal way. It gives globally well-defined immersions if and only if the flux of any curve on Σ is vertical. An important difference between both deformations is that the first one consists on a compact family of surfaces while the second one is noncompact. After reparametrization and change of scale in $\mathbb{R}^3$, the part of the surface around a point where g has a zero converges, when λ goes to infinity, to the surface given by the Weierstrass data $(\mathbb{C}, z^n, a\,dz)$, $a \in \mathbb{C}^*$, which is not embedded. In a neighborhood of an end of the surface given by a punctured disc $0 < |z| < \varepsilon$ with meromorphic g and ω and $g(0) = 0$, Σ_λ converges to the minimal surface $(\mathbb{C}^*, z^n, a\,z^m dz)$, $a \in \mathbb{C}^*$. It can be checked easily that the only surfaces of this type which are embedded are the (vertical) Catenoid and the Helicoid. These surfaces correspond to the cases $n = 1, m = -2, a = 1$ and $m = n - 1, a = i$, respectively. Planar, catenoidal and helicoidal ends with vertical normal directions transform by the λ-deformation into ends of the same type (note that the flux at this ends is always vertical). Under suitable global assumptions this deformation gives strong restrictions on the geometry and the topology of minimal surfaces all of whose fluxes are vertical, see works of López, Pérez and Ros [26], [47], [54], [44]. In particular we have the following result.

Theorem 5.1. *A nonflat properly embedded minimal surface $\Sigma \subset \mathbb{R}^3$ of finite topology, horizontal ends and vertical flux is either a Catenoid or an Helicoid.*

As a minimal surface in the hypothesis of Theorem 3.1 has necessarily vertical flux, this theorem follows from the result above.

Now we explain briefly the proof of Theorem 5.1. First observe that the deformation Σ_λ, $\lambda > 0$, is well defined. It follows easily from the maximum principle for minimal surfaces that, if we start with an embedded nonflat triply-periodic minimal surface and we deform it continuously by triply-periodic minimal surfaces, embeddedness is preserved along the deformation. For general properly embedded minimal surfaces this fact is not generally true and it depends of the behaviour of the deformation at infinity. The maximum principle at infinity [37] and the behaviour of the vertical flux deformation at the ends allows to conclude that the surfaces Σ_λ are all embedded. Taking λ going to zero and infinity we conclude that no point on Σ has vertical normal vector. In the same way, we deduce that Σ has no planar ends and we obtain directly that Σ is homeomorphic either to a plane or a annulus. Now the theorem follows from [43], [38].

Some uniqueness results for singly periodic embedded minimal surfaces of finite topology, can be obtained by the arguments above:

i) there are not genus one minimal surfaces in $\mathbb{R}^3/S_\theta$, $\theta \neq 0$, with finitely many horizontal planar ends, see [47], and

ii) the only genus zero minimal surface in $\mathbb{R}^3/T$ with finitely many helicoidal ends is the Helicoid, [44].

Table 2. The surfaces in the first column are characterized as the unique embedded minimal surfaces satisfying the restrictions in the other columns. The last three results follow from the vertical flux deformation argument. The genus must be computed in the quotient surface.

surface	periodicity	genus	ends
Helicoid	none	0	one end
Catenoid	none	whatever	two ends
Catenoid	none	0	more than one
Helicoid	translation	0	helicoidal
none	screw $\theta \neq 0$	1	planar

6. Compactness and limit configurations

Consider a sequence $\{\Sigma_n\} \subset \mathcal{M}(k, r)$, where $\Sigma_n \subset \mathbb{R}^3$ is a properly embedded minimal surface of genus k and $r \geq 2$ horizontal ends. As the Gauss map $g_n : \overline{\Sigma}_n \to \overline{\mathbb{C}}$ is a meromorphic map of fixed degree, it converges up to a subsequence, to a family of nonconstant meromorphic maps $g_{\infty,1} : \overline{\Sigma}_{\infty,1} \to \overline{\mathbb{C}}, \dots, g_{\infty,m} : \overline{\Sigma}_{\infty,m} \to \overline{\mathbb{C}}$, defined over closed Riemann surfaces with degree$(g_{\infty,1})+\cdots+$degree$(g_{\infty,m})$ = degree(g_n). For large n, one can see inside Σ_n large pieces of the surfaces $\overline{\Sigma}_{\infty,s}$ joined by regions with almost constant g_n.

A more careful analysis, see Ros [53], shows that each one of these meromorphic maps $g_{\infty,s}$ is the Gauss map of a properly embedded minimal surface $\Sigma_{\infty,s} \subset \mathbb{R}^3$ with horizontal ends, and that suitably chosen homothetic images of Σ_n converge to the different $\Sigma_{\infty,s}$. Moreover the regions joining these surfaces consist of *unbounded pieces* Ω_i, $i = 1, \dots, r$ satisfying the following properties:

a) Each Ω_i contains exactly one end of Σ_n. So, the unbounded pieces are naturally ordered by their levels in $\mathbb{R}^3$,

b) the projection of Ω_i over the plane $\{x_3 = 0\}$ is one-to-one, and

c) the boundary of Ω_i consists of several convex Jordan curves in horizontal planes.

Therefore, for large n, the surface Σ_n looks like the one in Figure 6. As an example, the three-ended surfaces $\Sigma(k, t)$ described in §3 converge, when t goes to infinity, to a Catenoid between the level 1 and 2 and $k + 1$ Catenoids between levels 2 and 3.

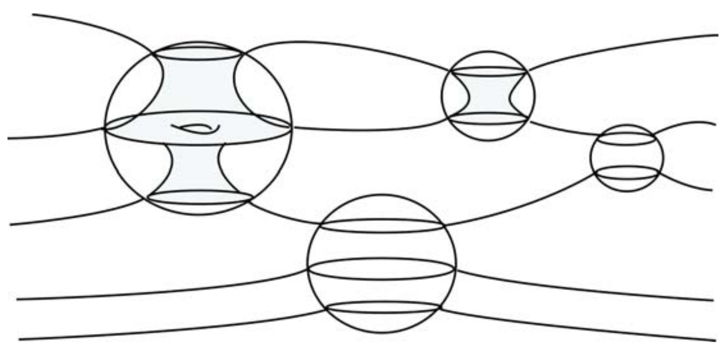

Figure 6. A sequence of minimal surfaces with fixed finite topology converges, up to a subsequence, to a union of surfaces with smaller topology joined by unbounded domains at different levels.

Working at the right scale, the limit can be seen as a horizontal plane of finite multiplicity with a finite number of marked points at different levels. Traizet [63] associates to this figure a number of horizontal forces (which correspond to rescaled limits of the fluxes of Σ_n). This *limit configuration* must be balanced in a natural sense.

We say that the moduli space $\mathcal{M}(k, r)$ is *compact (in the strong sense)* if any sequence $\{\Sigma_n\} \subset \mathcal{M}(k, r)$ converges (up to a subsequence) to a limit which consists of a single surface $\Sigma_\infty \in \mathcal{M}(k, r)$. In particular the moduli spaces $\mathcal{M}(k, 3)$, $k \geq 1$, are noncompact. The following theorem have been obtained by Ros [53] (for $r \geq 5$) and Traizet [63].

Theorem 6.1. *The moduli spaces* $\mathcal{M}(1, r)$, $r \geq 4$ *are compact* (*in the strong sense*).

In fact the compactness result can be extended (in a conditional but useful way) to higher genus. The space $\mathcal{M}(k, r)$ is compact for $r \geq g + 3$, assuming that the Hoffman–Meeks conjecture is true for genus smaller than k. This compactness can be seen as a first step in the proof of this conjecture. Thus Hoffman–Meeks conjecture will follow from the following one.

Conjecture. The moduli space $\mathcal{M}(k, r)$ is either empty or noncompact.

The compactness result in Ros [53] depends of the non existence of the piece described in Figure 7 in a limit of surfaces Σ_n with fixed topology. This piece consists

Figure 7. A sequence of minimal surfaces with fixed finite topology in $\mathbb{R}^3$ cannot converge to a limit which contains that piece, because the vertical flux argument gives a contradiction.

on an unbounded domain with just two catenoidal ends forming on it, one of positive and the other of negative logarithmic growth. The nonexistence of this piece is shown by using the vertical flux deformation argument.

The compactness theorem of Traizet [63] follows from the nonexistence in the limit of the surfaces Σ_n of a subsurface like the one in Figure 8. It is formed by

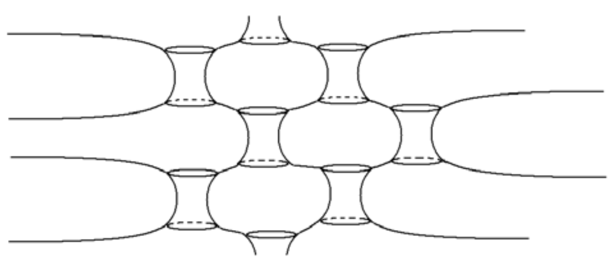

Figure 8. This picture cannot appear in a limit of a sequence of minimal surfaces in $\mathcal{M}(k, r)$. Otherwise, the limit configuration will be unbalanced.

several unbounded pieces at consecutive levels. Each one of this pieces is connected with the nearest ones by exactly two Catenoids forming. The top and the bottom unbounded pieces connect with the remaining part of the surface just by a catenoidal end forming. The reason why this piece cannot exist is because it is unbalanced.

The above ideas can be extended to the periodic case although the results have been explicitly stated only in some cases. Consider, for instance, a sequence $\{\Sigma_n\}$ of genus one minimal surfaces with r horizontal planar ends, properly embedded in the flat three manifold $\mathbb{R}^3/T_n$, where T_n is a non-horizontal vector. It can be shown that the third coordinate has no critical points on $\overline{\Sigma}_n$ and so, each horizontal level curve is a Jordan curve (which might pass through one of the ends) and the Gauss map omits the vertical directions.

Up to a subsequence and suitable choice of scaling, $\{\Sigma_n\}$ converges either to a minimal surface $\Sigma_\infty \subset \mathbb{R}^3/T_\infty$ with the same topology than Σ_n or to a union of genus zero surfaces. Using the uniqueness results stated in Section 3 and further analysis, we can see that Σ_n approach to either r vertical Catenoids forming or to 2 vertical Helicoids forming, [32], see Figures 9 and 10. The second option is a simple example of the so called *parking garage* structure.

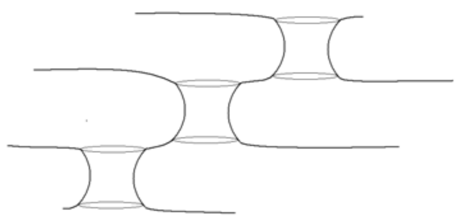

Figure 9. The Catenoid forming limit.

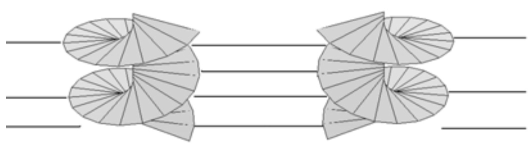

Figure 10. The Helicoid forming figure.

7. Smoothness of moduli spaces

Now we want to understand the structure of the moduli space $\mathcal{M}(k,r)$, $r \geq 2$, of finite topology minimal surfaces in $\mathbb{R}^3$ at its nonsingular points.

By expository reasons, we will consider in this section not $\mathcal{M}(k,r)$ but the subspace $\mathcal{M}'(k,r)$ of surfaces whose end logarithmic growths are all different (thus embeddedness is preserved by natural deformations in this space). Infinitesimal deformations of Σ in $\mathcal{M}'(k,r)$ are represented by *Jacobi functions*. These are smooth solutions $u\colon \Sigma \to \mathbb{R}$ of the equation $\Delta u + |\sigma|^2 u = 0$, where Δ is the Laplacian of the induced metric and $|\sigma|^2$ is the square length of the second fundamental form of the immersion. The functions u have at most logarithmic singularities at the ends of Σ, which correspond to the fact that the growing of the catenoidal ends changes along the deformation. Denote by $\mathcal{J}(\Sigma)$ the space of these Jacobi functions. Using linear elliptic theory, it can be shown that $\dim \mathcal{J}(\Sigma) \geq r+3$, Pérez and Ros [48]. Moreover the subspace of nondegenerate surfaces $\mathcal{M}^*(k,r) = \{\Sigma \in \mathcal{M}'(k,r) : \dim \mathcal{J}(\Sigma) = r+3\}$ is a real analytic manifold, whose tangent space at a point Σ coincides with $\mathcal{J}(\Sigma)$. On this manifold we have the following additional structure: let $f : \mathcal{M}(k,r) \to \mathbb{R}^{2r}$ be the map which associates to a surface Σ the logarithmic growth of the asymptotic Catenoid and the *height of its neck* for each one of its ends. Then f induces a Lagrangian immersion of $\mathcal{M}^*(k,r)$ (modulo horizontal translations) in $\mathbb{R}^{2r-2}$, see [48]. We have been also able to compute the second fundamental form of this immersion, see Pérez and Ros [49].

If $\mathcal{B}(\Sigma)$ denotes the space of bounded Jacobi functions on Σ, then $\mathcal{B}(\Sigma)$ contains, at least the linear functions of the Gauss map (which correspond to the infinitesimal translations) and the function $\det(N, p, e_3)$, N, p and e_3 being the normal vector, the position vector and the vertical direction, respectively. This Jacobi function corresponds to the infinitesimal rotation of the surface around the vertical axis. It can be shown that if the above functions are the unique functions in $\mathcal{B}(\Sigma)$, then the surface is nondegenerate.

An interesting, and somewhat intriguing, fact is that bounded Jacobi functions in $\mathcal{B}(\Sigma)$ can be represented by branched conformal minimal immersions from $\overline{\Sigma} - B$ into $\mathbb{R}^3$, B being the ramification divisor of the Gauss map of Σ, whose Gauss map is the same than the one of Σ and whose ends have a bounded coordinate function, see Montiel and Ros [41] and Ejiri and Kotani [10]. Using this representation, Nayatani [42] has shown that the Costa, Hoffman and Meeks surfaces $\Sigma(k)$ are nondegenerate, for $k \leq 37$.

The local structure around a nondegenerate surface has been also considered in the periodic case and either planar or Scherk type ends [45], [14]. It would be interesting to clarify the nondegeneration condition and the smoothness properties of the moduli space of minimal surfaces in the case of helicoidal ends, both periodic and non-periodic.

Let $\mathcal{M}$ be the space of genus 3 embedded minimal surfaces in flat 3-tori. Then we can prove that there are some degenerate surfaces in $\mathcal{M}$, arguing as follows: assuming that any surface is nondegenerate, the subset $\mathcal{M}'$ of surfaces in $\mathcal{M}$ obtained as two sheeted coverings of non-orientable minimal surfaces of Euler characteristic -2 described in Theorem 4.1 is a union of connected components of $\mathcal{M}$. However, this is impossible as it is known that the Schwarz minimal surface $P \in \mathcal{M}'$, can be deformed to the Schoen Gyroid $G \in \mathcal{M} - \mathcal{M}'$.

An important open problem is to decide if a (generic) surface in $\mathcal{M}'(k,r)$ is nondegenerate. A related question is to give practical criteria which guarantee that a surface is nondegenerate. As an example, Montiel and Ros [41] proved that if all the branch values of the Gauss map (on the compactified surface $\overline{\Sigma}$) lie on a great circle, then the only bounded Jacobi functions are the linear functions of the Gauss map. In this way, we can prove the nondegeneracion of some surfaces, like finite coverings of singly and doubly periodic Scherk surfaces, Riemann examples and Saddle towers constructed by Karcher [22].

8. Classification results

In this section we describe a strategy which has been used several times to classify surfaces in a moduli space. The first result of this type was proved by Meeks, Pérez and Ros [32] who classified genus zero properly embedded minimal surfaces in $\mathbb{R}^3$ with infinite symmetries. One of the key results in this classification is contained in the following theorem, see Figure 11.

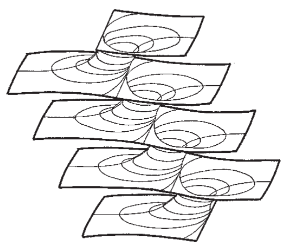

Figure 11. The Riemann minimal example.

Theorem 8.1. *A minimal surface* Σ *of genus* 1 *and r planar horizontal ends properly embedded in* $\mathbb{R}^3/T$, T *being a non horizontal vector, is a finite covering of one of the Riemann minimal examples.*

We will use this result to explain briefly how the method works. Consider the moduli space $\mathcal{M}$ of surfaces described in the theorem (r is fixed). As described in Section 6, any horizontal section C (whose level does not coincide with the level of an end) of a surface in $\mathcal{M}$ is a Jordan curve. The flux along C cannot be vertical, as the vertical flux deformation give a contradiction. Normalize the surfaces so that the third coordinate of the flux vector of C is 2π and define *the horizontal flux map* $F\colon \mathcal{M} \to \mathbb{R}^2 - \{0\}$ as the horizontal component of the flux along C (note that the flux of a curve around a planar end is zero and therefore the flux vector does not depend of the level we use to compute it). The following property follows from the results in Section 6.

I) *The map* F *is proper.*

In fact, the Catenoid forming picture has almost vertical flux which means that $F(\Sigma)$ converges to 0. In the Helicoid forming case the flux vector is almost horizontal and so $F(\Sigma)$ goes to ∞.

Denote by $\mathcal{R}$ the subspace of $\mathcal{M}$ formed by the coverings of the Riemann surfaces. The results in Section 7 imply that $\mathcal{R}$ is a smooth 2-dimensional manifold and moreover,

II) *$\mathcal{R}$ is an open an closed subset of $\mathcal{M}$, and $F: \mathcal{R} \to \mathbb{R}^2 - \{0\}$ is a diffeomorphism.*

We want to prove that F is an open map. This fact would follow, via the implicit function theorem, if we know that the surfaces in $\mathcal{M}$ are nondegenerate. However we do not have this property a priori. Instead we use the following complex variable theorem.

Theorem 8.2. *Let $f: \{z \in \mathbb{C}^m : |z| < \varepsilon\} \to \mathbb{C}^m$ be a holomorphic map with $f(0) = 0$. If 0 is an isolated point in $f^{-1}(0)$, then f is open around the origin.*

In our context we consider the space of *Weierstrass data* $\mathcal{W}$ which is a m-dimensional complex manifold. We also consider the *Period map* which consists on, both, the real and the imaginary parts of the periods which appear in the Weierstrass representation along a certain basis of the homology of the Riemann surface. A crucial point is that in this way we find a holomorphic map $P: \mathcal{W} \to \mathbb{C}^m$ between manifolds of the same dimension. This is a strong restriction which limitates the range of application of the whole method to problems where the involved surfaces are essentially (may be modulo symmetries) of genus zero.

In order a point Σ in $\mathcal{W}$ to define an immersed minimal surface, the real part of $P(\Sigma)$ must be zero. The imaginary part corresponds to the fluxes along the homology base and in our case this fluxes reduce to $F(\Sigma)$. Given a surface $\Sigma_0 \in \mathcal{M}$, the analytic subset $\mathcal{S} = \{\Sigma \in \mathcal{W} : P(\Sigma) = P(\Sigma_0)\}$ coincides, locally around Σ_0, with $\{\Sigma \in \mathcal{M} : F(\Sigma) = F(\Sigma_0)\}$ which is compact. After that we can deduce that Σ_0 is an isolated surface in $\mathcal{S}$, and then, using Theorem 8.2, we get that

III) *The map F is open.*

The final step is to prove that for some value of $\mathbb{R}^2 - \{0\}$ the pullback image of F consists only of Riemann examples. To do that we use an implicit function argument at the degenerate point of $\mathcal{W}$ given by the r Catenoids forming limit. Is can be shown that P extends holomorphically at that boundary point and has nonzero Jacobian. This proves that

IV) *Given $\Sigma \in \mathcal{M}$, if the length of $F(\Sigma)$ is small enough, then Σ is one of the Riemann examples.*

In Table 3 we have collected the different situations where the above strategy has been successfully applied. The first column contains the surfaces whose uniqueness have been shown. Lazard–Holly and Meeks [20] proved that the doubly periodic Scherk surface is the only genus zero minimal surface in $T^2 \times \mathbb{R}$. Pérez and Traizet [51] have proved recently that the Saddle towers constructed by Karcher [22] are the only examples of genus in $\mathbb{R}^3/T$ other than the Helicoid, and Rodriguez, Pérez and Traizet [46] characterized a 3-dimensional family of standard examples constructed by Karcher [22] and Meeks and Rosenberg [35], as the unique double periodic minimal surfaces of genus one and parallel Scherk type ends. Finally Meeks and Wolf [39]

Table 3. The (families of) surfaces in the first column have been characterized as the unique among surfaces satisfying the restrictions of the other columns. The topology we consider is the one of the quotient surface.

surfaces	periodicity	genus	ends
Riemann	singly	1	planar
Scherk	doubly	0	whatever
Saddle towers	single translation	0	Scherk
Standard examples	doubly	1	parallel Scherk
Scherk	single translation	whatever	four Scherk

have proved that the singly periodic Scherk surface is the unique surface in $\mathbb{R}^3/T$ with four Scherk ends. The proofs of these results follow formally the same steps than the one of the Riemann examples case, but the concrete arguments are more involved and several additional problems appear. In particular, the proof of the last result depends on a different group of ideas developed around the notion of orthodisks in Riemann surface theory. The following is a very natural problem which contains several of the results above.

Problem ([51]). Classify complete genus zero surfaces (of finite topology) embedded in complete flat 3-manifolds.

We remark that there exists a large family of genus zero surfaces with helicoidal ends in $\mathbb{R}^3/S_\theta$. These are called *twisted saddle towers* and were constructed by Karcher [22]. These surfaces are obtained as twisted deformations of the surfaces characterized in [51]. Other flat 3-manifolds may have genus zero minimal surfaces with ends of Scherk type.

9. Least area surfaces

Of course, the most natural class of minimal surfaces is the one of area minimizing surfaces. Although the plane is the unique least area surface in $\mathbb{R}^3$, there are nonplanar least area surfaces if we prescribe suitable symmetries. The complete description of area minimizing surfaces among surfaces satisfying a natural constraint, like to belong to a given symmetric isotopy class, is natural question. As a first goal, we propose the following more precise problem.

Problem. Classify area minimizing surfaces, modulo 2, in complete flat 3-manifolds.

Table 4. $\mathbb{Z}_2$-least area surfaces in the flat 3-manifolds obtained as a quotient of $\mathbb{R}^3$ by a translations group.

3-manifold	least area surfaces (mod 2)
$\mathbb{R}^3$	plane
$\mathbb{R}^3/T$	planar and 2π-Helicoid
$T^2 \times \mathbb{R}$, T^2 rectangular	planar and 2π-Scherk
$T^2 \times \mathbb{R}$, $T^2 \neq$ rectangular	planar
T^3	2-tori and $\chi(\Sigma) = -2$ (?)

It can be seen that any solution of this problem is either (a quotient of a) plane or an one-sided surface. In Table 4 we have listed the solutions of the problem for the quotients of $\mathbb{R}^3$ by translation groups. The result has been proved by Ros [55] and follows from the classification of complete stable surfaces in these 3-manifolds. The 2-sided case was solved by do Carmo and Peng [9], Fisher-Colbrie and Schoen [12] and Pogorelov [52], but the 1-sided remained open; for previous related results see Ross and Schoen [57], [58]. We prove that a complete stable minimal surface in $\mathbb{R}^3/T$ (resp. $T^2 \times \mathbb{R}$) is either planar or the nonorientable Helicoid (resp. Scherk surface) of total curvature 2π. We also prove that these surfaces are, both, area minimizing (mod 2). In the case of flat 3-tori, we prove that any stable nonflat closed minimal surface is a nonorientable surface with Euler characteristic equal to -2. Some surfaces of this topology are stable, like P and D Schwarz surfaces, see [56], but some other are unstable. However it is natural to hope that area minimizing surfaces in a flat 3-torus are flat 2-tori.

References

[1] Callahan, M., Hoffman, D., and Meeks, W. H., III, Embedded minimal surfaces with an infinite number of ends. *Invent. Math.* **96** (1989), 459–505.

[2] —, The structure of singly-periodic minimal surfaces. *Invent. Math.* **99** (1990), 455–481.

[3] Colding, T. H., and Minicozzi, W. P., II, The Calabi-Yau conjectures for embedded surfaces. math.DG/0404197.

[4] —, What are the shapes of embedded minimal surfaces and why? math.DG/0511740.

[5] Collin, P., Topologie et courbure des surfaces minimales proprement plongées de $\mathbf{R}^3$. *Ann. of Math.* (2) **145** (1997), 1–31.

[6] Cosín, C., and Ros, A., A Plateau problem at infinity for properly immersed minimal surfaces with finite total curvature. *Indiana Univ. Math. J.* **50** (2001), 847–878.

[7] Costa, C. J., Example of a complete minimal immersion in $\mathbb{R}^3$ of genus oneand three embedded ends. *Bull. Soc. Brasil. Math.* **15** (1984), 47–54.

[8] —, Classification of complete minimal surfaces in $\mathbb{R}^3$ with total curvature 12π. *Invent. Math.* **105** (1991) 273–303.

[9] do Carmo, M., and Peng, C. K., Stable complete minimal surfaces in $\mathbb{R}^3$ are planes. *Bull. Amer. Math. Soc.* (N.S.) **1** (1979), 903–906.

[10] Ejiri, N., and Kotani, M., Index and flat ends of minimal surfaces. *Tokyo J. Math.* **16** (1993), 37–48.

[11] Fischer, W., and Koch, E., Spanning minimal surfaces. *Philos. Trans. Roy. Soc. London Ser. A* **354** (1996), 2105–2142.

[12] Fischer-Colbrie, D., and Schoen, R., The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature. *Comm. Pure Appl. Math.* **33** (1980), 199–211.

[13] Hauswirth, L., and Pacard, F., Minimal surfaces of finite genus with two limit ends. Preprint.

[14] Hauswirth, L., and Traizet, M., The space of embedded doubly-periodic minimal surfaces. *Indiana Univ. Math. J.* **51** (2002), 1041–1080.

[15] Hoffman, D., and Karcher, H., Complete embedded minimal surfaces of finite total curvature. In *Geometry V: Minimal surfaces*, Encyclopaedia Math. Sci. 90, Springer-Verlag, Berlin 1997, 5–93.

[16] Hoffman, D., and Meeks, W. H., III, Embedded minimal surfaces of finite topology. *Ann. of Math.* **131** (1990), 1–34.

[17] —, The asymptotic behavior of properly embedded minimal surfaces of finite topology. *J. Amer. Math. Soc.* **2** (1989), 667–682.

[18] Hoffman, D., Weber, M., and Wolf, M., An Embedded Genus-One Helicoid. *Proc. Nat. Acad. Sci. U.S.A.* **102** (46) (2005), 16566–16568.

[19] Huff, R., Existence proofs of the C(H) and tC(P) surfaces. Preprint.

[20] Lazard-Holly, H., and Meeks, W. H., III, Classification of doubly-periodic minimal surfaces of genus zero. *Invent. Math.* **143** (2001), 1–27.

[21] Kapouleas, N., Complete embedded minimal surfaces of finite total curvature. *J. Differential Geom.* **47** (1997), 95–169.

[22] Karcher, H., Embedded minimal surfaces derived from Scherk's examples. *Manuscripta Math.* **62** (1988), 83–114.

[23] —, The triply periodic minimal surfaces of Alan Schoen and their constant mean curvature companions. *Manuscripta Math.* **64** (1989), 291–357.

[24] Kawasaki, T., Classification of spatial polygons that could possibly generate embedded triply periodic minimal surfaces. *Tokyo J. Math.* **26** (2003), 23–53.

[25] López, F. J., and Martin, F., Complete minimal surfaces in $\mathbb{R}^3$. *Publ. Mat.* **43** (1999), 341–449.

[26] López, F. J., and Ros, A., On embedded minimal surfaces of genus zero. *J. Differential Geom.* **33** (1991), 293–300.

[27] Lord, E. A., and Mackay, A. L., Periodic minimal surfaces of cubic symmetry. *Current Sci.* **85** (2003), 346–362.

[28] Martín, F., and Rodríguez, D., A characterization of the periodic Callahan-Hoffman-Meeks surfaces in terms of their symmetries. *Duke Math. J.* **89** (1997), 445–463.

[29] Martín, F., A note on the uniqueness of the periodic Callahan-Hoffman-Meeks surfaces in terms of their symmetries. *Geom. Dedicata* **86** (2001), 185—190.

[30] Martín, F., and Weber, M., Properly Embedded Minimal Surfaces with Three Ends. *Duke Math. J.* **107** (2001), 533–560.

[31] Meeks, W. H., III, The theory of triply periodic minimal surfaces. *Indiana Univ. Math. J.* **39** (1990), 877–936.

[32] Meeks, W. H., III, Pérez, J., and Ros, A., Uniqueness of the Riemann minimal examples. *Invent. Math.* **131** (1998), 107–132.

[33] —, The Geometry of Minimal Surfaces of Finite Genus I: Curvature Estimates and Quasiperiodicity. *J. Differential Geom.* **66** (2003), 1–45.

[34] —, The geometry of minimal surfaces of finite genus III: bounds on the topology and index of classical minimal surfaces. Preprint.

[35] Meeks, W. H., III, and Rosenberg, H., The global theory of doubly periodic minimal surfaces. *Invent. Math.* **97** (1989), 351–379.

[36] —, The geometry of periodic minimal surfaces. *Comment. Math. Helv.* **68** (1993), 538–578.

[37] —, The maximum principle at infinity for minimal surfaces in flat three manifolds. *Comment. Math. Helv.* **66** (1991), 263–278.

[38] —, The uniqueness of the helicoid and the asymptotic geometry of properly embedded minimal surfaces with finite topology. *Ann. of Math.*, to appear.

[39] Meeks, W. H., III, and Wolf, M., Minimal surfaces with the area growth of two planes; the case of infinite symmetry. Preprint

[40] Minicozzi, W. P., II, Embedded minimal surfaces. In *Proceedings of the International Congress of Mathematicians* (Madrid, 2006), Volume II, EMS Publishing House, Zürich 2006, 853–877.

[41] Montiel, S., and Ros, A., Schrödinger operators associated to a holomorphic map. In *Global differential geometry and global analysis* (ed. by D. Ferus, U. Pinkall, U. Simon and B. Wegner), Lecture Notes in Math. 1481, Springer-Verlag, Berlin 1991, 147–174.

[42] Nayatani, S., Morse index and Gauss map of complete minimal surfaces in Euclidean 3-space. *Comment. Math. Helv.* **68** (1993), 511–537.

[43] Osserman, R., *A survey of minimal surfaces*. 2nd ed., Dover, New York 1986.

[44] Pérez, J., A rigidity theorem for periodic minimal surfaces. *Comm. Anal. Geom.* **7** (1999), 95–104.

[45] —, On singly periodic minimal surfaces with planar ends. *Trans. Amer. Math. Soc.* **349** (1997), 2371–2389.

[46] Pérez, J., Rodríguez, M., and Traizet, M., The classification of doubly periodic minimal tori with parallel ends. *J. Differential Geom.* **69** (2005), 523–577.

[47] Pérez, J., and Ros, A., Some uniqueness and nonexistence theorems for embedded minimal surfaces. *Math. Ann.* **295** (1993), 513–525.

[48] —, The space of properly embedded minimal surfaces with finite total curvature. *Indiana Univ. Math. J.* **45** (1996), 177–204.

[49] —, The space of complete minimal surfaces with finite total curvature as lagrangian submanifold. *Trans. Amer. Math. Soc.* **351** (1999), 3935–3952.

[50] —, Properly embedded minimal surfaces with finite total curvature. In *The global theory of minimal surfaces in flat spaces* (ed. by G. P. Pirola), Lecture Notes in Math. 1775, Springer-Verlag, Berlin 2002, 15–66.

[51] Pérez, J., and Traizet, M., The classification of singly periodic minimal surfaces with genus zero and Scherk type ends. *Trans. Amer. Math. Soc.*, to appear.

[52] Pogorelov, A. V., On the stability of minimal surfaces. *Dokl. Akad. Nauk SSSR* **260** (1981), 293–295.

[53] Ros, A., Compactness of spaces of properly embedded minimal surfaces with finite total curvature. *Indiana Univ. Math. J.* **44** (1995), 139–152.

[54] —, Embedded minimal surfaces : forces, topology and symmetries. *Calc. Var. Partial Differential Equations* **4** (1996), 469–496.

[55] —, One-sided complete stable minimal surfaces. *J. Differential Geom.*, to appear.

[56] Ross, M., Schwarz' P and D surfaces are stable. *Differential Geom. Appl.* **2** (1992), 179–195.

[57] —, Complete nonorientable minimal surfaces in $\mathbf{R}^3$. *Comment. Math. Helv.* **67** (1992), 64–76.

[58] Ross, M., and Schoen, C., Stable quotients of periodic minimal surfaces. *Comm. Anal. Geom.* **2** (1994), 451–459.

[59] Schoen, A. H., Infinite periodic minimal surfaces without self-intersections. *NASA Technical Note No.* TN D-5541, 1970.

[60] Schoen, R., Uniqueness, symmetry and embeddedness of minimal surfaces. *J. Differential Geom.* **18** (1983), 791–809.

[61] Traizet, M., An embedded minimal surface with no symmetries. *J. Differential Geom.* **60** (2002), 103–153.

[62] —, Weierstrass representation of some simply-periodic minimal surfaces. *Ann. Global Anal. Geom.* **20** (2001), 77–101.

[63] —, A balancing condition for weak limits of minimal surfaces. *Comment. Math. Helv.* **79** (2004), 798–825.

[64] Traizet, M., and Weber, M., Hermite polynomials and helicoidal minimal surfaces. *Invent. Math.* **161** (2005), 113–149.

[65] Weber, M., On singly periodic minimal surfaces invariant under a translation. *Manuscripta Math.* **101** (2000), 125–142.

[66] Weber, M., and Wolf, M., Teichmüller theory and handle addition for minimal surfaces. *Ann. of Math.* **156** (2002), 713—795.

[67] Wolf, M., Flat structures, Teichmüller theory and handle addition for minimal surfaces. In *Global theory of minimal surfaces*, Clay Math. Proc. 2, Amer. Math. Soc., Providence, RI, 2005, 211–241.

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