Finiteness of arithmetic Kleinian reflection groups

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Abstract. We prove that there are only finitely many arithmetic Kleinian maximal reflection groups.

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1. Introduction

A Kleinian reflection group Γ is a discrete group generated by reflections in the faces of hyperbolic polyhedron $P \subset \mathbb{H}^3$. We may assume that the dihedral angles of P are of the form π/n, $n \geq 2$, in which case P forms a fundamental domain for the action of Γ on $\mathbb{H}^3$. If P has finite volume, then $\mathbb{H}^3/\Gamma = \mathcal{O}$ is a hyperbolic orbifold of finite volume, which is obtained by "mirroring" the faces of P. Andreev gave a combinatorial characterization of hyperbolic reflection groups in 3-dimensions, in terms of the topological type of P and the dihedral angles assigned to the edges of P [1]. A reflection group Γ is maximal if there is no reflection group Γ' such that $\Gamma < \Gamma'$. We shall defer the definition of arithmetic groups until later, but a theorem of Margulis implies that Γ is arithmetic if and only if $[\mathrm{Comm}(\Gamma) : \Gamma] = \infty$, where $\mathrm{Comm}(\Gamma) = \{g \in \mathrm{Isom}(\mathbb{H}^3) \mid [\Gamma : g^{-1}\Gamma g \cap \Gamma] < \infty\}$ [14]. The main theorem of this paper is that there are only finitely many arithmetic Kleinian groups which are maximal reflection groups.

The argument generalizes an argument of Long–MacLachlan–Reid [12], which implies that there are only finitely many arithmetic minimal hyperbolic 2-orbifolds with bounded genus. Their argument is in fact a generalization of an argument of Zograf [26], who reproved that there are only finitely many congruence groups Γ commensurable with $\mathrm{PSL}(2,\mathbb{Z})$ such that $\mathbb{H}^2/\Gamma$ has genus 0 (this was proven originally by Dennin [4], [5], and was known as Rademacher's conjecture). The key ingredient of their argument is a theorem of Vigneras [20] (based on work of Gelbart–Jacquet [7] and Jacquet–Langlands [10]), which implies that a congruence arithmetic Fuchsian group has $\lambda_1 \geq \frac{3}{16}$ (and which has a generalization to higher dimensions [3]). The other key ingredient is an estimate of Zograf [27], which implies that for a hyperbolic 2-orbifold $\mathcal{O}$, $\lambda_1(\mathcal{O})\mathrm{Vol}(\mathcal{O})$ is bounded linearly by the genus of $\mathcal{O}$. Zograf's argument

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is based on a sequence of improvements of a result of Szëgo (who did this for planar domains)[18], by Hersch (for S^2) [9], Yang–Yau (for orientable surfaces) [25], and Li–Yau (for non-orientable surfaces and manifolds of a fixed conformal type) [11]. We observe that the Li–Yau argument (sharpened by El Soufi–Ilias [6]) generalizes to orbifolds, and we then apply this to arithmetic Kleinian reflection groups.

In the concluding section, we consider how one might generalize this result to higher dimensions to prove

Conjecture 1.1. There are only finitely many maximal arithmetic reflection groups in $\mathrm{Isom}(\mathbb{H}^n)$, $n > 1$.

2. Conformal volume of orbifolds

Conformal volume was first defined by Li–Yau, partially motivated by generalizing results on surfaces due to Yang–Yau, Hersch, and Szëgo. We generalize this notion to orbifolds. Let $(\mathcal{O}, g)$ be a compact Riemannian orbifold, possibly with boundary. Let $|\mathcal{O}|$ denote the underlying topological space. Denote the volume form by dv_g, and $\mathrm{Vol}(\mathcal{O}, g)$ its volume. Let $\mathrm{Möb}(\mathbb{S}^n)$ denote the conformal transformations of $\mathbb{S}^n$. It is well known that $\mathrm{Möb}(\mathbb{S}^n) = \mathrm{Isom}(\mathbb{H}^{n+1})$. $|\mathcal{O}|$ has a codimension zero dense open subset which is a Riemannian manifold. We will say that a map $\varphi\colon |\mathcal{O}_1| \to |\mathcal{O}_2|$ is *PC* if it is a continuous map which is piecewise conformal on the open submanifold of $|\mathcal{O}_1|$ which maps to the manifold part of $|\mathcal{O}_2|$. Clearly, if $\varphi\colon |\mathcal{O}| \to \mathbb{S}^n$ is PC, and $\mu \in \mathrm{Möb}(\mathbb{S}^n)$, then $\mu \circ \varphi$ is also PC. Let *can* denote the canonical round metric on $\mathbb{S}^n$.

Definition 2.1. For a piecewise smooth map $\varphi\colon |\mathcal{O}| \to (\mathbb{S}^n, can)$, define

$$V_c(n, \varphi) = \sup_{\mu \in \mathrm{Möb}(\mathbb{S}^n)} \mathrm{Vol}(\mathcal{O}, (\mu \circ \varphi)^*(can)).$$

If there exists a PC map $\varphi\colon |\mathcal{O}| \to \mathbb{S}^n$, then we also define

$$V_c(n, \mathcal{O}) = \inf_{\varphi\colon \mathcal{O} \to \mathbb{S}^n \text{ PC}} V_c(n, \varphi).$$

$V_c(n, \mathcal{O})$ is denoted the (n-dimensional) *conformal volume* of $\mathcal{O}$.

Remark. For our application, it would suffice to define a PC map to be Lipschitz and *a.e.* conformal. It seems likely that our definition of conformal volume coincides with that of Li–Yau for manifolds, but we have not checked this (it would suffice to show that a PC map can be approximated by conformal maps).

If there exists a piecewise isometric map $\varphi\colon (|\mathcal{O}|, g) \to \mathbb{E}^n$ for some n, then clearly $V_c(n, \mathcal{O})$ is well-defined, since $\mathbb{E}^n$ has a conformal embedding into $\mathbb{S}^n$. For an orbifold $(\mathcal{O}, g)$, to prove that $V_c(n, \mathcal{O})$ is well-defined, one need only check that $(|\mathcal{O}|, g)$ embeds piecewise isometrically into some compact Riemannian manifold, in which case the Nash isometric embedding theorem implies that $(|\mathcal{O}|, g)$ embeds piecewise isometrically into $\mathbb{E}^n$, for some n. We will only apply conformal volume to orbifolds which will obviously have a PC map to $\mathbb{S}^n$, for some n. We record basic facts about conformal volume which were recorded in [11], and which carry over to our notion of conformal volume for orbifolds.

Fact 1. If $\mathcal{O}$ admits a degree d PC map onto another orbifold $\mathcal{P}$, then

$$V_c(n, \mathcal{O}) \leq |d| V_c(n, \mathcal{P}).$$

Fact 2. Since $\mathbb{S}^n \hookrightarrow \mathbb{S}^{n+1}$ embeds isometrically, it is clear that $V_c(n, \mathcal{O}) \geq V_c(n+1, \mathcal{O})$. Define the conformal volume $V_c(\mathcal{O}) = \lim_{n\to\infty} V_c(n, \mathcal{O})$.

Fact 3. If $\mathcal{O}$ is of dimension m, and $\varphi\colon |\mathcal{O}| \to \mathbb{S}^n$ is a PC map, then

$$V_c(n, \varphi) \geq V_c(n, \mathbb{S}^m) = \mathrm{Vol}(\mathbb{S}^m).$$

The same argument as in Li–Yau works here: "blow up" about a smooth manifold point of $\varphi(|\mathcal{O}|)$ so that the image Hausdorff limits to a geodesic sphere of dimension m.

Fact 4. If $\mathcal{O}$ is an embedded suborbifold of the orbifold $\mathcal{P}$, and $\varphi\colon |\mathcal{P}| \to \mathbb{S}^n$ is PC, then $V_c(n, \varphi) \geq V_c(n, \varphi|_{|\mathcal{O}|})$. Thus, $V_c(n, \mathcal{P}) \geq V_c(n, \mathcal{O})$.

3. Finite subgroups of O(3)

Lemma 3.1. *Let $G < \mathrm{O}(3)$ be a finite subgroup. Then there exists a group G', $G \leq G' < \mathrm{O}(3)$, which is generated by reflections such that $[G' : G] \leq 4$.*

Proof. This follows from a case-by-case analysis of spherical 2-orbifolds. We will use Conway's notation for spherical orbifolds (see *e.g.* ch. 13 [19]).

Case $(*)$, $(*p, p)$ or $(*p, q, r)$: These orbifold groups are generated by reflections.

Case (p, p) or (p, q, r): These 2-fold cover $(*p, p)$ or $(*p, q, r)$ respectively.

Case $(n*)$: This 2-fold covers $(*n, 2, 2)$.

Case $(2 * m)$: This 2-fold covers $(*2m, 2, 2)$.

Case $(3 * 2)$: This 2-fold covers $(*4, 3, 2)$.

Case $(n|\circ)$: This 2-fold covers $(2n*) \xrightarrow{2:1} (*2n, 2, 2)$ (this includes the case $n = 1$, *i.e.* $(1|\circ) = \mathbb{RP}^2$).

This exhausts all possible spherical orbifolds, and we see that in each case the orbifold fundamental group is of index ≤ 4 in a reflection group (all but the last case have index ≤ 2). $\square$

Question. Given a dimension n, is there a constant $C(n)$ such that any finite subgroup of $\mathrm{O}(n)$ is of index $\leq C(n)$ in a reflection group? If so, then one should be able to prove Conjecture 7.1. We suspect the answer to this question is no, which is why we have not been able to generalize the proof in this section to higher dimensions.

4. Eigenvalue bounds

We observe that the argument of Thm. 2.2 [6] generalizes to our context of orbifolds (their theorem sharpens Cor. 3, Sect. 2 of [11]). If $(\mathcal{O}, g)$ is a Riemannian orbifold with piecewise smooth boundary, then $\lambda_1(\mathcal{O}, g)$ is the first non-zero eigenvalue of Δ_g on $\mathcal{O}$ with Neumann boundary conditions.

Theorem 4.1 ([6]). *Let $(\mathcal{O}, g)$ be a compact Riemannian orbifold of dimension m, possibly with boundary. If $\varphi\colon |\mathcal{O}| \to \mathbb{S}^n$ is a PC map,*

$$\lambda_1(\mathcal{O}, g)\mathrm{Vol}(\mathcal{O}, g)^{\frac{2}{m}} \leq m V_c(n, \varphi)^{\frac{2}{m}}.$$

Proof. Let $\boldsymbol{X} = (X_1, \ldots, X_{n+1})$ be the coordinate functions on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. Then $\sum_{i=1}^{n+1} X_i^2 = 1$, restricted to $\mathbb{S}^n$.

Lemma 4.2. *There exists $\mu \in \mathrm{M\ddot{o}b}(\mathbb{S}^n)$ such that $\int_{\mathcal{O}} \boldsymbol{X} \circ \mu \circ \varphi \, dv_g = \mathbf{0}$.*

Proof. Let $\boldsymbol{x} \in \mathbb{S}^n$. For $0 \leq t < 1$, $t\boldsymbol{x} \in \mathbb{H}^{n+1}$, let $\mu_{t\boldsymbol{x}} \in \mathrm{M\ddot{o}b}(\mathbb{S}^n) = \mathrm{Isom}(\mathbb{H}^{n+1})$ be the hyperbolic translation along the ray $\mathbb{R}\boldsymbol{x}$ taking $\mathbf{0}$ to $t\boldsymbol{x}$ (thus, $\mu_{0\mathbf{x}} = \mu_{\mathbf{0}} = \mathrm{Id}$). Let $H(t\boldsymbol{x}) = \frac{1}{\mathrm{Vol}(\mathcal{O},g)} \int_{\mathcal{O}} \boldsymbol{X} \circ \mu_{t\boldsymbol{x}} \circ \varphi \, dv_g$. We may think of H as defining a function $H\colon \mathbb{H}^{n+1} \to \mathbb{H}^{n+1}$, which gives the center of mass of the measure coming from $\varphi_* dv_g$, where we take the point $-t\boldsymbol{x}$ to be the origin of the sphere $\mathbb{S}^n = \partial\mathbb{H}^{n+1}$ by the conformal map $\mu_{t\boldsymbol{x}}$. As $t \to 1$, all of the mass of $\mu_{t\mathbf{x}} \circ \varphi(\mathcal{O})$ becomes concentrated at $\boldsymbol{x}$, and we see that H extends to a continuous function $H\colon B^{n+1} \to B^{n+1}$ (where $B^{n+1} = \mathbb{H}^{n+1} \cup \mathbb{S}^n$) such that $H_{|\mathbb{S}^n} = \mathrm{Id}_{|\mathbb{S}^n}$. Thus, H is onto, so there exists $\boldsymbol{y} \in \mathbb{H}^{n+1}$ such that $H(\boldsymbol{y}) = \mathbf{0}$, and we take $\mu = \mu_{\boldsymbol{y}}$. □

Now replace φ with $\mu \circ \varphi$, noting that this is still a PC map. Then $X_i \circ \varphi$ may be used as test functions in the Rayleigh–Ritz quotient, since they are Lipschitz functions which are L^2 orthogonal to the constant function. Thus,

$$\lambda_1(\mathcal{O}) \int_{\mathcal{O}} |X_i \circ \varphi|^2 \, dv_g \leq \int_{\mathcal{O}} |\nabla X_i \circ \varphi|^2 \, dv_g.$$

Summing, we see that

$$\begin{aligned}
\lambda_1(\mathcal{O}) \int_{\mathcal{O}} \sum_{i=1}^{n+1} |X_i \circ \varphi|^2 \, dv_g &= \lambda_1(\mathcal{O})\mathrm{Vol}(\mathcal{O}, g) \\
&\leq m \int_{\mathcal{O}} \frac{1}{m} \sum_{i=1}^{n+1} |\nabla X_i \circ \varphi|^2 \, dv_g \\
&\leq m \biggl(\int_{\mathcal{O}} \biggl(\frac{1}{m} \sum_{i=1}^{n+1} |\nabla X_i \circ \varphi|^2 \biggr)^{\frac{m}{2}} dv_g \biggr)^{\frac{2}{m}} \mathrm{Vol}(\mathcal{O}, g)^{1-\frac{2}{m}},
\end{aligned}$$

where the last inequality is Hölder's inequality. Now we use the fact that φ is PC to see that $\varphi^* can = \frac{1}{m}\sum_{i=1}^{n+1} |\nabla X_i \circ \varphi|^2$ *a.e.*, and thus

$$\int_{\mathcal{O}} \Big(\frac{1}{m}\sum_{i=1}^{n+1} |\nabla X_i \circ \varphi|^2\Big)^{\frac{m}{2}} dv_g \leq \mathrm{Vol}(\mathcal{O}, \varphi^* can).$$

Finally, we obtain the desired inequality putting these inequalities together. □

5. Congruence arithmetic hyperbolic 3-orbifolds

We need to know some properties of arithmetic hyperbolic 3-orbifolds. For background and notation, see Maclachlan–Reid [13].

Let $k \subset \mathbb{C}$ be a number field with only one complex place, and let A be a quaternion algebra over k. Let $\rho\colon A \to M(2,\mathbb{C})$ be a k-embedding, $P\colon \mathrm{GL}(2,\mathbb{C}) \to \mathrm{PGL}(2,\mathbb{C})$. Let $\mathcal{E} \subset A$ be either a maximal order or an Eichler order, and let $N(\mathcal{E}) \subset A^*$ be the normalizer of $\mathcal{E}$ in A. Let Γ be an arithmetic Kleinian group, such that $\Gamma = P\rho G$, $G \subset A^*$. Then there exists an order $\mathcal{E} \subset A$ which is either a maximal order or an Eichler order such that $G \subset N(\mathcal{E})$ (see thm. 11.4.3 [13]). In particular, if Γ is a maximal arithmetic Kleinian group, then $\Gamma = P\rho N(\mathcal{E})$, for some order $\mathcal{E}$.

An ideal I in A is a complete R_k-lattice. The left order of I is $\mathcal{O}_l(I) = \{a \in A \mid aI \subset I\}$, and the right order is $\mathcal{O}_r(I) = \{a \in A \mid Ia \subset I\}$. The ideal I is 2-*sided* if $\mathcal{O}_l(I) = \mathcal{O}_r(I)$. The ideal is *integral* if I lies in both $\mathcal{O}_l(I)$ and in $\mathcal{O}_r(I)$ (*i.e.* $I^2 \subseteq I$ is multiplicatively closed). If $\mathcal{O}$ is a maximal order in A, and I is a 2-sided integral ideal in $\mathcal{O}$ such that $\mathcal{O} = \mathcal{O}_l(I) = \mathcal{O}_r(I)$, then the *principal congruence subgroup* of $\mathcal{O}^1$ is

$$\mathcal{O}^1(I) = \mathcal{O}^1 \cap (1+I).$$

Thus, $\mathcal{O}^1(I)$ is the kernel of the map $\mathcal{O}^1 \to \mathcal{O}/I$, which is therefore of finite index in $\mathcal{O}^1$, since $\mathcal{O}/I$ is finite. A discrete group $G \subset A$ is *congruence* if it contains a principal congruence subgroup, and $\Gamma < \mathrm{PGL}(2,\mathbb{C})$ is congruence if $\Gamma = P\rho G$, for some $G < A$ congruence.

Lemma 5.1 (Long–MacLachlan–Reid [12]). *A maximal arithmetic Kleinian group is congruence.*

Proof. Let $\Gamma \subset \mathrm{PGL}(2,\mathbb{C})$ be a maximal arithmetic Kleinian group. Then $\Gamma = P\rho N(\mathcal{E})$, for some order $\mathcal{E}$ of square-free level. If $\mathcal{E}$ is a maximal order, then $\mathcal{E}^1$ is a congruence subgroup for the trivial ideal $I = \mathcal{E}$. If $\mathcal{E} = \mathcal{O}_1 \cap \mathcal{O}_2$, where $\mathcal{O}_i$ are maximal orders (so that $\mathcal{E}$ is an Eichler order), then choose $\alpha \in R_k - \{0\}$ such that $I = \alpha\mathcal{O}_1 \subset \mathcal{O}_2$. Then $\mathcal{O}_l(I) = \mathcal{O}_r(I) = \mathcal{O}_1$, and $I^2 = \alpha^2\mathcal{O}_1 \subset \alpha\mathcal{O}_1 = I$. Also, clearly $1 + I \subset \mathcal{O}_1 \cap \mathcal{O}_2$. Thus, $\mathcal{O}^1(I) = \mathcal{O}_1^1 \cap (1+I) \subset \mathcal{O}_1 \cap \mathcal{O}_2$. Thus, $\mathcal{O}^1(I) \subset \mathcal{E}^1$, and we see that $\mathcal{E}^1$ is a principal congruence subgroup. □

A fundamental theorem of Vigneras, making use of work of Jacquet–Langlands [10] and Gelbart–Jacquet [7], generalizes a result of Selberg for $\mathrm{PSL}(2, \mathbb{Z})$. For a hyperbolic orbifold $\mathcal{O}$, let $\lambda_1(\mathcal{O})$ be the minimal non-zero eigenvalue of the Laplacian Δ on $\mathcal{O}$.

Theorem 5.2 ([20], [3]). *Let $\mathcal{O} = \mathbb{H}^3/\Gamma$, where Γ is an arithmetic congruence subgroup. Then $\lambda_1(\mathcal{O}) \geq \frac{3}{4}$.*

It is conjectured that under the hypotheses of the above theorem, $\lambda_1(\mathcal{O}) \geq 1$, which is known as (a special case of) the *generalized Ramanujan conjecture* [3].

6. Finiteness of arithmetic Kleinian maximal reflection groups

We put together the results from the previous sections to prove our main theorem.

Theorem 6.1. *There are only finitely many arithmetic maximal reflection groups in* $\mathrm{Isom}(\mathbb{H}^3)$.

Proof. Suppose that Γ is an arithmetic maximal reflection group. That is, there is no group $\Gamma' < \mathrm{Isom}(\mathbb{H}^3)$, with $\Gamma < \Gamma'$, such that Γ' is generated by reflections. Then there exists $\Gamma \leq \Gamma_0 < \mathrm{Isom}(\mathbb{H}^3)$, such that Γ_0 is a maximal Kleinian group. Γ is generated by reflections in a finite volume polyhedron P.

Lemma 6.2 (Vinberg [21]). *Γ is a normal subgroup of Γ_0. Moreover, there is a finite subgroup $\Theta < \Gamma_0$ such that $\Theta \to \Gamma_0/\Gamma$ is an isomorphism, and Θ preserves the polyhedron P.*

Proof. This follows from the fact that the set of reflections in Γ_0 is conjugacy invariant, and therefore the group generated by reflections is normal in Γ_0. Since Γ is a maximal reflection group, this subgroup must be Γ. The polyhedron P is a fundamental domain of Γ, and if there is an element $\gamma \in \Gamma_0$ such that $\mathrm{int(P)} \cap \gamma(\mathrm{int(P)}) \neq \emptyset$, then $\gamma(P) = P$. Otherwise, there would be a geodesic plane V containing a face of P, such that $V \cap \mathrm{int(P)} \neq \emptyset$. The reflection $r_V \in \Gamma$ in the plane V would be conjugate to a reflection $r_{\gamma(V)} = \gamma r_V \gamma^{-1}$, which is not in Γ since $r_{\gamma(V)}(\mathrm{int(P)}) \cap \mathrm{int(P)} \neq \emptyset$, which implies that Γ is not a maximal reflection group, a contradiction. Let Θ be the subgroup of Γ_0 such that $\Theta(P) = P$. Clearly Θ is finite, since P is finite volume and has finitely many faces. If $\gamma_0 \in \Gamma_0$, let $\gamma \in \Gamma$ be such that $\gamma_0(\mathrm{int(P)}) \cap \gamma(\mathrm{int(P)}) \neq \emptyset$. Then $\gamma^{-1}\gamma_0(P) = P$, so $\gamma_0 \in \gamma\Theta$. Thus, $\Theta \to \Gamma_0/\Gamma$ is an isomorphism. □

Let $\mathcal{O} = \mathbb{H}^3/\Gamma_0$, and Θ is the finite group coming from the previous lemma.

Lemma 6.3. $\lambda_1(\mathcal{O}) = \lambda_1(P/\Theta)$.

Proof. Let f be an eigenfunction on P/Θ with eigenvalue $\lambda_1(P/\Theta)$. Since f has Neumann boundary conditions, its level sets in P/Θ are orthogonal to $\partial P/\Theta$. Let $\tilde{f}$

be the preimage of f under the map $P \to P/\Theta$, so that $\tilde{f}$ is invariant under the action of Θ. Extend $\tilde{f}$ to a function $\tilde{F}$ on $\mathbb{H}^3$, by the action of Γ (and therefore invariant under the action of Γ_0). By the reflection principle, $\tilde{F}$ is a smooth function, so it descends to an eigenfunction F of Δ on $\mathcal{O}$ with eigenvalue $\lambda_1(P/\Theta)$. Conversely, if F is an eigenfunction of Δ on $\mathcal{O}$ with eigenvalue $\lambda_1(\mathcal{O})$, then $F_{|P/\Theta}$ gives an eigenfunction on P/Θ with Neumann boundary conditions, since the level sets of $\tilde{F}$ must be invariant under reflections, and therefore perpendicular to the faces of P. □

Consider $\mathbb{H}^3 \subset \mathbb{S}^3$ embedded conformally as the upper half space of $\mathbb{S}^3$, so that $\mathrm{Isom}(\mathbb{H}^3)$ acts conformally on $\mathbb{S}^3$. Normalize so that Θ acts isometrically on $\mathbb{S}^3$, which we may do by a result of Wilker [24]. Clearly, $V_c(n, \mathcal{O}) = V_c(n, P/\Theta)$, since $|\mathcal{O}| = |P/\Theta|$. Then the orbifold $P/\Theta \subset \mathbb{S}^3/\Theta$ is a conformal embedding, so by Fact 4, $V_c(3, P/\Theta) \leq V_c(3, \mathbb{S}^3/\Theta)$. The group Θ embeds in a finite reflection group $\Theta' \subset \mathrm{O}(3)$ such that $[\Theta' : \Theta] \leq 4$. Then by Fact 1, $V_c(3, \mathbb{S}^3/\Theta) \leq 4V_c(3, \mathbb{S}^3/\Theta')$. Now, there is a polyhedron $Q \subset \mathbb{S}^3$ with geodesic faces which is the fundamental domain for Θ'. Clearly $V_c(3, Q) = V_c(3, \mathbb{S}^3/\Theta')$. By Facts 2 and 4, $V_c(3, Q) = \mathrm{Vol}(\mathbb{S}^3, can) = 2\pi^2$. Thus, we have $V_c(\mathcal{O}) \leq 8\pi^2$.

Now we apply the eigenvalue estimates

$$\frac{3}{4}\mathrm{Vol}(\mathcal{O})^{\frac{2}{3}} \leq \lambda_1(\mathcal{O})\mathrm{Vol}(\mathcal{O})^{\frac{2}{3}} \leq 3V_c(\mathbb{H}^3/\Gamma_0)^{\frac{2}{3}} \leq 3(8\pi^2)^{\frac{2}{3}}.$$

Thus we obtain $\mathrm{Vol}(\mathcal{O}) \leq 64\pi^2$. Since volumes of arithmetic hyperbolic orbifolds are discrete, and Γ_0 is uniquely determined by Γ, we conclude that there are only finitely many arithmetic maximal reflection groups. □

7. Conclusion

From the main theorem, we conclude that given an arithmetic reflection group $\Gamma < \mathrm{Isom}(\mathbb{H}^3)$, it must lie in one of finitely many maximal reflection groups (up to conjugacy). If Γ is a reflection group in a polyhedron P for which all the dihedral angles are $\pi/2$ or all are $\pi/3$, then there are infinitely many co-finite volume reflection subgroups of Γ. Thus we see that there are commensurability classes of arithmetic groups for which there are infinitely many reflection groups in the commensurability class, and thus in our finiteness result, the maximality assumption is crucial. It is an interesting project to try to identify all arithmetic maximal reflection groups, and to classify their reflection subgroups of finite covolume.

For the non-compact examples, one may apply volume formulae to estimate the maximal discriminant of a quadratic imaginary number field k for which $\mathrm{PGL}(2, k)$ contains a reflection group. Humbert first computed the covolumes of Bianchi groups, and a generalization due to Borel implies that for a non-compact arithmetic Kleinian group, the minimal covolume μ satisfies

$$\mu \geq \frac{|\Delta_k|^{\frac{3}{2}}\zeta_k(2)}{16\pi^2 h_k},$$

where h_k is the class number of the number field k. The Brauer–Siegel theorem gives an estimate of h_k for a number field and implies for a quadratic imaginary number field k that

$$|\Delta_k|\zeta_k(2) \geq \frac{h_k(2\pi)^2}{2w},$$

where w is the order of the group of roots of unity in k. If $k \neq \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$, then $w = 2$ (see the proof of theorem 11.7.2 [13]). Thus, we have

$$64\pi^2 \geq \mu \geq \frac{1}{16}|\Delta_k|^{\frac{1}{2}}.$$

Thus $|\Delta_k| < 2^{20}\pi^4 = 1.02 \times 10^8$. Hatcher has computed orbifold structures of some of the Bianchi groups of small discriminant, and from his pictures one may deduce that $\mathrm{PGL}(2, R_k)$ is commensurable with a reflection group for

$$\Delta_k = -3,\ -4,\ -7,\ -8,\ -11,\ -15,\ -19,\ -20,\\ -24,\ -39,\ -40,\ -52,\ -55,\ -56,\ -68,\ -84,$$

where k is a quadratic imaginary number field [8]. In principle, it ought to be possible to compute all arithmetic reflection groups, but clearly even in the non-compact case, the volume estimates we obtain do not make this computation feasible. To classify non-compact Kleinian arithmetic groups, it may require the infusion of some more number theory. Arithmetic restrictions have been found on reflection groups containing the Bianchi groups by Vinberg [23] and Blume-Nienhaus [2]. If these results could be extended to all maximal non-compact arithmetic groups, then one may be able to give a complete classification. The classification of compact arithmetic maximal reflection groups, although in principle decidable, is probably not feasible at this stage.

It is clear that for a finite volume polyhedron $P \subset \mathbb{H}^n$, $V_c(n, P) = \mathrm{Vol}(\mathbb{S}^n)$, so

$$\lambda_1(P)\mathrm{Vol}(P)^{\frac{2}{n}} < n\mathrm{Vol}(\mathbb{S}^n)^{\frac{2}{n}}.$$

Thus in n dimensions, there are finitely many reflection groups $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ which have a lower bound on $\lambda_1(\mathbb{H}^n/\Gamma)$. It would be interesting if one could generalize the arguments of the main theorem to higher dimensions. It is known by work of Prokhorov that there cannot be any reflection groups in dimension > 996 [17] (Vinberg showed that compact reflection groups cannot exist in dimension > 30 [22]). Vinberg also gave a characterization of arithmetic reflection groups, in terms of a totally real field of definition K and an integrality condition [21]. Nikulin has shown that there exists a constant N_0 such that the set of arithmetic groups in $\mathrm{Isom}(\mathbb{H}^n)$ generated by reflections with $n > 16$ and $[K : \mathbb{Q}] > N_0$ is empty [16]. It was proved in a previous paper by Nikulin that the set of maximal arithmetic groups generated by reflections in $\mathrm{Isom}(\mathbb{H}^n)$ with a fixed degree $[K : \mathbb{Q}]$ is finite [15]. Thus, to generalize the main argument of this paper, one would have to bound the conformal volume of a minimal arithmetic n-orbifold containing a reflection suborbifold up to dimensions $n \leq 16$. By the characterization of maximal arithmetic groups in $\mathrm{Isom}(\mathbb{H}^n)$, it should follow that they are congruence. By a result of Burger–Sarnak, if $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ is a congruence arithmetic reflection group with $n > 2$, then $\lambda_1(\mathbb{H}^n/\Gamma) > \frac{2n-3}{4}$ (Cor. 1.3 [3], and the fact that reflection groups are defined by quadratic forms). Every maximal arithmetic group covered by a reflection group will embed conformally in an elliptic n-orbifold. So to prove the following conjecture for $n \leq 16$:

Conjecture 7.1. There is a function $K(n)$, such that if $\mathcal{O}$ is an elliptic n-orbifold, then $V_c(\mathcal{O}) \leq K(n)$.

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