

# Link homology and categorification

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**Abstract.** This is a short survey of algebro-combinatorial link homology theories which have the Jones polynomial and other link polynomials as their Euler characteristics.

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## 1. Introduction

The discovery of the Jones polynomial by V. Jones [J] and quantum groups by V. Drinfeld and M. Jimbo led to an explosive development of quantum topology. The newly found topological invariants were christened “quantum invariants”; for knots and links they often take the form of polynomials. By late 80s to early 90s it was realized that each complex simple Lie algebra  $\mathfrak{g}$  gives rise to a gaggle of quantum invariants. To a link  $L$  in  $\mathbb{R}^3$  with each component colored by an irreducible representation of  $\mathfrak{g}$  there is assigned an invariant  $P(L, \mathfrak{g})$  taking values in the ring of Laurent polynomials  $\mathbb{Z}[q, q^{-1}]$  (sometimes fractional powers of  $q$  are necessary). Polynomials  $P(L, \mathfrak{g})$  have a representation-theoretical description, via intertwiners between tensor products of irreducible representations of the quantum group  $U_q(\mathfrak{g})$ , the latter a Hopf algebra deformation of the universal enveloping algebra of  $\mathfrak{g}$ . These invariants by no means exhaust all quantum invariants of knots and links; various modifications and generalizations include finite type (Vassiliev) invariants, invariants associated with quantum deformations of Lie superalgebras, etc.

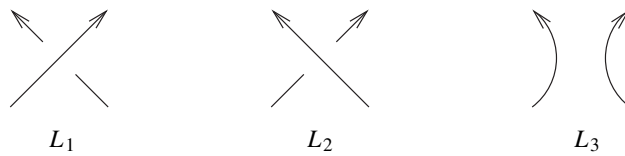
Quantum  $\mathfrak{sl}(n)$  link polynomials, when each component of  $L$  is colored by the fundamental  $n$ -dimensional representation, can be conveniently encapsulated into a single 2-variable polynomial  $P(L)$ , known as the HOMFLY or HOMFLY-PT polynomial [HOMFLY], [PT].

The skein relation

$$\lambda P(L_1) - \lambda^{-1} P(L_2) = (q - q^{-1}) P(L_3)$$

for any three links  $L_1, L_2, L_3$  that differ as shown below and the value of  $P$  on the unknot, uniquely determines the HOMFLY-PT invariant, which lies in the ring  $\mathbb{Z}[\lambda^{\pm 1}, (q - q^{-1})^{\pm 1}]$  (in the original papers a single variable was used instead of

$q - q^{-1}$ , making  $P$  a genuine Laurent polynomial in two variables).



Specializing  $\lambda = q^n$ , for  $n \geq 0$ , leads to a link polynomial invariant  $P_n(L) \in \mathbb{Z}[q, q^{-1}]$ , normalized so that  $P_n(\text{unknot}) = q^{n-1} + q^{n-3} + \dots + q^{1-n}$  for  $n > 0$  and  $P_0(\text{unknot}) = 1$ .

$P_0(L)$  and  $P_2(L)$  are the Alexander and Jones polynomials of  $L$ , respectively, while  $P_1(L)$  is a trivial invariant. For  $n > 0$ , the polynomial  $P_n(L)$  can be interpreted via the representation theory of quantum  $\mathfrak{sl}(n)$ , and  $P_0(L)$  – via that of the quantum Lie superalgebra  $U_q(\mathfrak{gl}(1|1))$ .

The miracle that emerged in the past few years is that these polynomials are Euler characteristics of link homology theories:

- The Jones polynomial  $P_2(L)$  is the Euler characteristic of a bigraded link homology theory  $\mathcal{H}(L)$ , discovered in [K1].
- The Alexander polynomial  $P_0(L)$  is the Euler characteristic of a bigraded knot homology theory, discovered by P. Ozsváth, Z. Szabó [OS1] and J. Rasmussen [R1].
- The polynomial  $P_3(L)$  is the Euler characteristic of a link homology theory  $H(L)$ , defined in [K2].
- For each  $n \geq 1$ , Lev Rozansky and the author constructed a bigraded link homology theory  $H_n(L)$  with  $P_n(L)$  as the Euler characteristic, see [KR1].
- The entire HOMFLY-PT polynomial is the Euler characteristic of a triply-graded link homology theory [KR2], [K6] (for a possible alternative approach via string theory see [GSV]).

Ideally, a link homology theory should be a monoidal functor  $\mathcal{F}$  from the category  $\text{LCob}$  of link cobordisms to a tensor triangulated category  $\mathbb{T}$  (for instance,  $\mathbb{T}$  could be the category of complexes of  $R$ -modules, up to chain homotopies, for a commutative ring  $R$ ). Objects of  $\text{LCob}$  are oriented links in  $\mathbb{R}^3$ , morphisms from  $L_0$  to  $L_1$  are isotopy classes (rel boundary) of oriented surfaces  $S$  smoothly and properly embedded in  $\mathbb{R}^3 \times [0, 1]$  such that  $L_0 \sqcup (-L_1)$  is the boundary of  $S$  and  $L_i \subset \mathbb{R}^3 \times \{i\}$ ,  $i = 0, 1$ . In many known examples,  $\mathcal{F}$  is a projective functor: the map  $\mathcal{F}(S) : \mathcal{F}(L_0) \rightarrow \mathcal{F}(L_1)$  is well defined up to overall multiplication by invertible central elements of  $\mathbb{T}$  (e.g. by  $\pm 1$  for homology theory  $\mathcal{H}$ ).

No a priori reason why quantum link invariants should lift to link homology theories is known and the general framework for lifting quantum invariants to homology

theories remains a mystery. We call such a lift a *categorification* of the invariant. The term categorification was coined by L. Crane and I. Frenkel [CF] in the context of lifting an  $n$ -dimensional TQFT to an  $(n + 1)$ -dimensional one ( $n = 2, 3$  are the main interesting cases).

Let us also point out that the Casson invariant (a degree two finite-type invariant of 3-manifolds) is the Euler characteristic of instanton Floer homology, that the Seiberg–Witten and Ozsváth–Szabó 3-manifold homology theories categorify degree one finite-type invariants of 3-manifolds (the order of  $H_1(M, \mathbb{Z})$  when the first homology of the 3-manifold  $M$  is finite and, more generally, the Alexander polynomial of  $M$ ), that equivariant knot signatures are Euler characteristics of  $\mathbb{Z}/4\mathbb{Z}$ -graded link homologies (O. Collin, B. Steer [CS], W. Li), and that there exist ideas on how to categorify the 2-variable Kauffman polynomial [GW], the colored Jones polynomial, and quantum invariants of links colored by arbitrary fundamental representations  $\Lambda^i V$  of  $\mathfrak{sl}(n)$  [KR1].

## 2. A categorification of the Jones polynomial

In the late nineties the author discovered a homology theory  $\mathcal{H}(L)$  of links which is bigraded,

$$\mathcal{H}(L) = \bigoplus_{i,j \in \mathbb{Z}} \mathcal{H}^{i,j}(L),$$

and has the Jones polynomial as the Euler characteristic,

$$P_2(L) = \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \text{rk}(\mathcal{H}^{i,j}(L)).$$

The construction of  $\mathcal{H}$  categorifies the Kauffman bracket description of the Jones polynomial. Starting from a plane projection  $D$  of  $L$  we build homology groups  $\mathcal{H}(D)$  inductively on the number of crossings of the projection via long exact sequences

$$\longrightarrow \mathcal{H} \left( \begin{array}{c} \text{ ) } \\ \text{ ( } \end{array} \right) \left( \begin{array}{c} \text{ ( } \\ \text{ ) } \end{array} \right) \longrightarrow \mathcal{H} \left( \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \right) \longrightarrow \mathcal{H} \left( \begin{array}{c} \text{ ) } \\ \text{ ) } \\ \text{ ( } \\ \text{ ( } \end{array} \right) \longrightarrow$$

and then check that  $\mathcal{H}(D)$  are invariants of  $L$  alone. Homology of the empty link is  $\mathbb{Z}$ , homology of the unknot is  $\mathcal{A} = \mathbb{Z}[X]/(X^2)$ , which should be thought of as the integral cohomology ring of the 2-sphere. Homology of the  $k$ -component unlink is  $\mathcal{A}^{\otimes k}$ . The obvious cobordisms between unlinks turn  $\mathcal{A}$  into a commutative Frobenius ring, with the trace map  $\text{tr}(1) = 0$ ,  $\text{tr}(X) = 1$  (in any full-fledged link homology theory homology of the unknot is a commutative Frobenius algebra over homology of the empty link).  $\mathcal{H}(D)$  is the homology of a complex  $\mathcal{C}(D)$  constructed in an elementary way from direct sums of tensor powers of  $\mathcal{A}$  and the structure maps of this Frobenius ring.

**Theorem 2.1.** *There exists a combinatorially defined bigraded homology theory  $\mathcal{H}(L)$  of oriented links in  $\mathbb{R}^3$ . The groups  $\mathcal{H}^{i,j}(L)$  are finitely-generated and their Euler characteristic is the Jones polynomial. The theory is functorial: to an oriented cobordism  $S$  between links  $L_0$  and  $L_1$  it assigns a homomorphism of groups*

$$\mathcal{H}(S): \mathcal{H}(L_0) \longrightarrow \mathcal{H}(L_1),$$

*well defined up to overall minus sign and of bidegree  $(0, -\chi(S))$ , where  $\chi(S)$  is the Euler characteristic of the surface  $S$ .*

That  $\pm\mathcal{H}(S)$  is well defined was proved in [Ja] and [K4] in two different ways.

The homology theory  $\mathcal{H}$  is manifestly combinatorial and programs computing it were written by D. Bar-Natan, A. Shumakovitch and J. Green. The earliest program [BN1] led to the conjecture that ranks of the homology groups of alternating links are determined by the Jones polynomial and the signature. This conjecture was proved by E.-S. Lee [L1]. For arbitrary knots and links, the structure of  $\mathcal{H}$  is more complicated than that of the Jones polynomial; right now we do not even have a guess at what the rational homology groups of arbitrary  $(n, m)$ -torus knots are.

We next list several interesting applications of  $\mathcal{H}$  and related developments.

1) J. Rasmussen used  $\mathcal{H}$  and its deformation studied by E. S. Lee [L2] to give a combinatorial proof of the Milnor conjecture that the slice genus of the  $(p, q)$ -torus knot is  $\frac{(p-1)(q-1)}{2}$  and of its generalization to all positive knots [R2]. This can also be used to show that certain knots are topologically but not smoothly slice without having to invoke Donaldson or Seiberg–Witten gauge theories. Originally, the Milnor conjecture was proved by P. Kronheimer and T. Mrowka via the Donaldson theory [KM].

2) Lenhard Ng [N] obtained an upper bound on the Thurston–Bennequin number of a Legendrian link from its homology  $\mathcal{H}(L)$ . This bound is sharp on alternating knots and on all but one or two knots with at most 10 crossings.

3) A. Shumakovitch [S] showed that over the 2-element field homology decomposes:  $\mathcal{H}(L, \mathbb{F}_2) \cong \tilde{\mathcal{H}}(L, \mathbb{F}_2) \otimes \mathbb{F}_2[X]/(X^2)$ , where  $\tilde{\mathcal{H}}(L, \mathbb{F}_2)$  is the reduced homology of  $L$  with coefficients in  $\mathbb{F}_2$ . P. Ozsváth and Z. Szabó [OS2] discovered a spectral sequence with the  $E^2$ -term  $\tilde{\mathcal{H}}(L, \mathbb{F}_2)$  that converges to the Ozsváth–Szabó homology of the double branched cover of  $L^1$ .

4) P. Seidel and I. Smith defined a  $\mathbb{Z}$ -graded homology theory of links via Lagrangian intersection Floer homology of a certain quiver variety [SS]. Their theory is similar to  $\mathcal{H}$  in many respects, and, conjecturally, isomorphic to  $\mathcal{H}$  after the bigrading in the latter is collapsed to a single grading.

### 3. Extensions to tangles

The quantum group  $U_q(\mathfrak{sl}(2))$  controls the extension of the Jones polynomial to an invariant of tangles, the latter a functor from the category of tangles to the category

of  $U_q(\mathfrak{sl}(2))$  representations. To a tangle  $T$  with  $n$  bottom and  $m$  top endpoints (an  $(m, n)$ -tangle) there is assigned an intertwiner

$$f(T): V^{\otimes n} \longrightarrow V^{\otimes m}$$

between tensor powers of the fundamental representation  $V$  of  $U_q(\mathfrak{sl}(2))$ .

A categorification of the invariant  $f(T)$  was suggested in [BFK]. We considered the category

$$\mathcal{O}^n = \bigoplus_{0 \leq k \leq n} \mathcal{O}^{k, n-k},$$

the direct sum of parabolic subcategories  $\mathcal{O}^{k, n-k}$  of a regular block of the highest weight category for  $\mathfrak{sl}(n)$ . The category  $\mathcal{O}^{k, n-k}$  is equivalent to the category of perverse sheaves on the Grassmannian of  $k$ -planes in  $\mathbb{C}^n$ , smooth with respect to the Schubert stratification. The Grothendieck group of  $\mathcal{O}^n$  is naturally isomorphic (after tensoring with  $\mathbb{C}$ ) to  $V^{\otimes n}$ , considered as a representation of  $U_{q=1}(\mathfrak{sl}(2))$ , and derived Zuckerman functors in  $D^b(\mathcal{O}^n)$  lift the action of  $\mathfrak{sl}(2)$  on  $V^{\otimes n}$ . We showed that projective functors in  $\mathcal{O}^n$  categorify the action of the Temperley–Lieb algebra on  $V^{\otimes n}$  and conjectured how to extend this to arbitrary tangles, by assigning to a tangle  $T$  a functor  $\mathcal{F}(T)$  between derived categories  $D^b(\mathcal{O}^n)$  and  $D^b(\mathcal{O}^m)$ .

Our conjectures were proved by C. Stroppel [St], who worked with the graded versions  $\mathcal{O}_{gr}^n$  of these categories, associated a functor  $\mathcal{F}(T)$  between derived categories  $D^b(\mathcal{O}_{gr}^n)$  and  $D^b(\mathcal{O}_{gr}^m)$  to each  $(m, n)$ -tangle  $T$  and a natural transformation  $\mathcal{F}(S): \mathcal{F}(T_0) \longrightarrow \mathcal{F}(T_1)$  to a tangle cobordism  $S$  between tangles  $T_0$  and  $T_1$ . The whole construction is a 2-functor from the 2-category of tangle cobordisms to the 2-category whose objects are  $\mathbb{C}$ -linear triangulated categories, 1-morphisms are exact functors and 2-morphisms are natural transformations of functors, up to rescalings by invertible complex numbers. When the tangle is a link  $L$ , this theory produces bigraded homology groups, conjecturally isomorphic to  $\mathcal{H}(L) \otimes \mathbb{C}$ .

The braid group action on  $V^{\otimes n}$  lifts to a braid group action on the derived category  $D^b(\mathcal{O}_{gr}^n)$ . Restricting to the subcategory  $D^b(\mathcal{O}_{gr}^{1, n-1})$  results in a categorification of the Burau representation, previously studied in [KS].

For a more economical extension of the Jones polynomial to tangles, we restrict to even tangles (when the number of endpoints on each of the two boundary planes is even) and to the subspace of  $U_q(\mathfrak{sl}(2))$ -invariants

$$\text{Inv}(n) = \text{Hom}_{U_q(\mathfrak{sl}(2))}(\mathbb{C}, V^{\otimes 2n})$$

in  $V^{\otimes 2n}$ . The invariant of a  $(2m, 2n)$ -tangle is a linear map

$$f_{\text{inv}}(T): \text{Inv}(n) \longrightarrow \text{Inv}(m)$$

between these subspaces.

A categorification of  $f_{\text{inv}}(T)$  was found in [K3], [K4]. We defined a graded ring  $H^n$  and established an isomorphism

$$K(H^n\text{-mod}) \otimes \mathbb{C} \cong \text{Inv}(n)$$

between the Grothendieck group (tensored with  $\mathbb{C}$ ) of the category of graded finitely-generated  $H^n$ -modules and the space of invariants in  $V^{\otimes 2n}$ . To an even tangle  $T$  we assigned an exact functor  $\mathcal{T}$  between the derived categories of  $H^n$ -mod (this functor induces the map  $f_{\text{inv}}(T)$  on the Grothendieck groups) and to a tangle cobordism – a natural transformation of functors. This results in a 2-functor from the 2-category of cobordisms between even tangles to the 2-category of natural transformation between exact functors in triangulated categories. Restricting to links, we recover homology groups  $\mathcal{H}(L)$ . This approach is more elementary than that via category  $\mathcal{O}$ , and should carry the same amount of information.

The space of invariants  $\text{Inv}(n)$  is a subspace of  $V^{\otimes 2n}(0)$ , the weight zero subspace of  $V^{\otimes 2n}$ . A categorification of this inclusion relates rings  $H^n$  and parabolic categories  $\mathcal{O}^{n,n}$ . The latter category is equivalent to the category of finite-dimensional modules over a  $\mathbb{C}$ -algebra  $A_{n,n}$ , explicitly described by T. Braden [B]. There exists an idempotent  $e$  in  $A_{n,n}$  such that  $eA_{n,n}e \cong H^n \otimes \mathbb{C}$ . This idempotent picks out all self-dual indecomposable projectives in  $A_{n,n}$ .

Rings  $H^n$  can also be used to categorify certain level two representations of  $U_q(\mathfrak{sl}(m))$ , see [HK].

For a more geometric and refined approach to invariants of tangles and tangle cobordisms we refer the reader to Bar-Natan [BN2]. Some of his generalizations of link homology can be thought of as  $G$ -equivariant versions of  $\mathcal{H}$ , for various compact subgroups  $G$  of  $\text{SU}(2)$ , see [K5] for speculations in this direction and for an interpretation of the Rasmussen invariant via the  $\text{SU}(2)$ -equivariant version of  $\mathcal{H}$ .

#### 4. $\mathfrak{sl}(n)$ link homology and matrix factorizations

**Theorem 4.1.** *For each  $n > 0$  there exists a homology theory which associates bigraded homology groups*

$$H_n(L) \cong \bigoplus_{i,j \in \mathbb{Z}} H_n^{i,j}(L)$$

to every oriented link in  $\mathbb{R}^3$ . The Euler characteristic of  $H_n$  is the polynomial invariant  $P_n$ ,

$$P_n(L) = \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim_{\mathbb{Q}}(H_n^{i,j}(L)).$$

The homology groups  $H_n^{i,j}(L)$  are finite-dimensional  $\mathbb{Q}$ -vector spaces, and, for a fixed  $L$ , only finitely many of them are non-zero. This homology is functorial: an oriented link cobordism  $S$  between  $L_0$  and  $L_1$  induces a homomorphism

$$H_n(S): H_n(L_0) \longrightarrow H_n(L_1),$$

well defined up to overall rescaling by nonzero rationals.

The groups  $H_n(L)$  are constructed in [KR1], where we start with a presentation for  $P_n(L)$  as an alternating sum

$$P_n(L) = \sum_{\Gamma} \pm q^{\alpha(\Gamma)} P_n(\Gamma). \tag{1}$$

Here we choose a generic plane projection  $D$  of  $L$  with  $m$  crossings, and form the sum over  $2^m$  planar trivalent graphs  $\Gamma$  which are given by replacing each crossing of  $D$  by one of the two planar pictures on the right



Each such planar graph  $\Gamma$  has a well-defined invariant  $P_n(\Gamma) \in \mathbb{Z}[q, q^{-1}]$ , with all the coefficients being nonnegative integers. Weights  $\alpha(\Gamma)$  are given by a simple rule. The edges are of two types: regular oriented edges and “wide” unoriented edges as on the rightmost picture above.

We then define single-graded homology groups  $H_n(\Gamma)$  which have the graded dimension  $P_n(\Gamma)$  and satisfy certain naturality conditions allowing us to build a complex out of  $H_n(\Gamma)$ , over all modifications  $\Gamma$  of the link diagram  $D$ . The complex is a categorification of the right hand side of the equation (1); its homology groups  $H_n(D)$  depend on  $L$  only and satisfy the properties listed in Theorem 4.1.

Our definition of  $H_n(\Gamma)$  is based on matrix factorizations. Let  $R = \mathbb{Q}[x_1, \dots, x_k]$ . A matrix factorization  $M$  of a polynomial  $f \in R$  consists of a pair of free  $R$ -modules and a pair of  $R$ -module maps

$$M^0 \xrightarrow{d} M^1 \xrightarrow{d} M^0$$

such that  $d^2 = f \cdot \text{Id}$ . The polynomial  $f$  is called the *potential* of  $M$ . A matrix factorization can be thought of as a two-periodic generalized complex; the square of the differential is not zero, but a fixed multiple of the identity operator. Matrix factorizations were introduced by D. Eisenbud [E] to study homological properties of hypersurface singularities, and later made an appearance in string theory, as boundary conditions in Landau–Ginzburg models [KL]. The tensor product  $M \otimes_R N$  of matrix factorizations with potentials  $f, g$  is a matrix factorization with potential  $f + g$ .

To each  $\Gamma$  we associate a collection of matrix factorizations  $M_1, \dots, M_m$ , one for each crossing of  $D$ , with potentials  $f_1, \dots, f_m$  that add up to zero:  $f_1 + f_2 + \dots + f_m = 0$ . The tensor product  $M_1 \otimes M_2 \otimes \dots \otimes M_m$  is a two-periodic complex (since the square of the differential is now zero). Finally,  $H_n(\Gamma)$  is defined as the cohomology of this complex; it inherits a natural  $\mathbb{Z}$ -grading from that of the polynomial algebra  $R$ .

The homology theory  $H_n$  is trivial when  $n = 1$ , while  $H_2(L) \cong \mathcal{H}(L) \otimes \mathbb{Q}$ . The theory  $H_3$  should be closely related to the homology theory constructed earlier in [K2]

(the two theories have the same Euler characteristic; the one in [K2] is defined over  $\mathbb{Z}$  and not just over  $\mathbb{Q}$ ).

J. Rasmussen [R3] determined homology groups  $H_n(L)$  for all 2-bridge knots  $L$  and a few other knots (with a mild technical restriction  $n > 4$ ). Little else is known about homology groups  $H_n(L)$  for  $n > 2$ .

A Lagrangian intersection Floer homology counterpart of  $H_n$  was discovered by C. Manolescu [M]. His theory  $\mathcal{H}_{(n)symp}(L)$  is singly-graded, but defined over  $\mathbb{Z}$ . Manolescu conjectured that  $\mathcal{H}_{(n)symp}$ , after tensoring with  $\mathbb{Q}$ , becomes isomorphic to  $H_n$ , with the bigrading of the latter folded into a single grading.

### 5. Triply-graded link homology and beyond

It turns out that the entire HOMFLY-PT polynomial, and not just its one-variable specializations, admits a categorification. The original construction via degenerate matrix factorizations with a parameter [KR2] was later recast in the language of Hochschild homology for bimodules over polynomial algebras [K6]. We represent a link  $L$  as the closure of a braid  $\sigma$  with  $m$  strands. To  $\sigma$  we assign a certain complex  $F(\sigma)$  of graded bimodules over the polynomial algebra  $R$  in  $m - 1$  generators. Taking the Hochschild homology over  $R$  of each term in the complex produces a complex of bigraded vector spaces

$$\dots \longrightarrow \text{HH}(R, F^j(\sigma)) \longrightarrow \text{HH}(R, F^{j+1}(\sigma)) \longrightarrow \dots$$

The cohomology groups  $H(\sigma)$  of this complex are triply-graded and depend on  $L$  only (a convenient grading normalization was pointed out by H. Wu [W]). The Euler characteristic of  $H(L)$  is the HOMFLY-PT polynomial  $P(L)$ , normalized so that  $P(\text{unknot}) = 1$ .

This homology theory suffers from two problems. First, the definition requires choosing a braid representative of a knot, rather than just a plane projection. Second, it is not possible to assign maps  $H(S)$  to all link cobordisms  $S$  so as to turn  $H$  into a functor from  $\text{LCob}$  to the category of (triply-graded) vector spaces (simply because homology of the unknot is one-dimensional, while that of an unlink is infinite-dimensional). We conjecture that the theory can be redefined on  $k$ -component links for all  $k > 1$  so as to assign finite-dimensional homology groups  $\tilde{H}(L)$  to all oriented links  $L$ , and not just to knots. The Euler characteristic of  $\tilde{H}(L)$  will still be the HOMFLY-PT polynomial, but rescaled so as to be a Laurent polynomial in  $\lambda$  and  $q$  rather than a rational function. The theory should extend to a projective functor from the category of *connected* link cobordisms to the category of triply-graded vector spaces.

Further extension of  $\tilde{H}$  to all link cobordisms should only require a minor modification, where one assigns the algebra  $\mathbb{Q}[a]$  to the empty link, the differential graded algebra

$$\mathbb{Q}\langle y_1, \dots, y_n \rangle \otimes \mathbb{Q}[a], \quad y_i y_j + y_j y_i = 0, \quad [y_i, a] = 0, \quad d(y_i) = a, \quad d(a) = 0$$

to a  $k$ -component unlink, and suitably resolves each  $\tilde{H}(L)$ , viewed as a  $\mathbb{Q}[a]$  module with the trivial action of  $a$ , into a complex of free  $\mathbb{Q}[a]$ -modules.

Understanding  $\tilde{H}$  could be an important step towards an algebraic description of knot Floer homology, since we expect  $\tilde{H}$  to degenerate (possibly via a spectral sequence) into knot Floer homology of Ozsváth–Szabó and Rasmussen [OS1], [R1], which categorifies the Alexander polynomial.

An algebraic description of knot and link Floer homology, if someday found and combined with the combinatorial construction [OS3] of Ozsváth–Szabó 3-manifold homology of surgeries on a knot from a filtered version of knot Floer homology (and a generalization of their construction to links), could lead to a combinatorial definition of Ozsváth–Szabó and Seiberg–Witten 3-manifold homology and, eventually, to an algebraic formulation of gauge-theoretical invariants of 4-manifolds.

In conclusion, we mention two other difficult open problems.

**I.** Categorify polynomial invariants  $P(L, \mathfrak{g})$  of knots and links associated to arbitrary complex simple Lie algebras  $\mathfrak{g}$  and their irreducible representations.

**II.** Categorify the Witten–Reshetikhin–Turaev invariants of 3-manifolds.

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