Curve complexes, surfaces and 3-manifolds

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Abstract. A survey of the role of the complex of curves in recent work on 3-manifolds and mapping class groups.

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1. Disjoint curves in surfaces

The complex of curves, a combinatorial object associated to a surface, has been of interest recently in low-dimensional topology and geometry. In the study of mapping class groups of surfaces, it has shed some light on *relative hyperbolicity* properties, and more generally on the coarse geometry of these groups, and of the Teichmüller spaces (parameter spaces of Riemann surfaces) on which they act. In the setting of hyperbolic 3-manifolds, the complex of curves has been used in the classification theory, particularly the solution of Thurston's Ending Lamination Conjecture, and generally in the attempt to relate more concretely the topology of a 3-manifold to its geometric structure. This paper will attempt to survey some of these developments. Of necessity we will focus on those aspects with which the author is most familiar, thus leaving out a lot of interesting topology and geometry.

We will begin with a leisurely discussion of the natural ways in which simple closed curves (i.e. embedded circles), and particularly the relation of disjointness, occur in low-dimensional topology. After this, in § 2 we will lay out the basic definitions and theorems about curve complexes. In § 3 we will describe work with H. Masur on the inductive structure of curve complexes and its relation to the geometry of mapping class groups. In § 4 we will go into more detail about mapping class groups and outline some recent work with J. Behrstock on their asymptotic cones and the Brock–Farb rank conjecture. In § 5 we will describe work with J. Brock and R. Canary on Thurston's Ending Lamination Conjecture. In § 6 we will lay out some thoughts on the hyperbolic geometry of Heegaard splittings, an area in which our knowledge is still rather incomplete.

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Surfaces and mapping class groups. The first interesting thing one can do with a homotopically essential simple closed curve in a surface is to cut along it, and then glue back. If before gluing back by the identity, the complementary surface is given one full twist in a neighborhood of the curve, the resulting self-map is not homotopic to the identity, and is called a Dehn twist.

The group of orientation preserving homeomorphisms of an oriented surface S to itself, taken modulo isotopy, is called the *mapping class group* of S, or $\mathcal{MCG}(S)$. Dehn twists give rise to infinite cyclic subgroups of $\mathcal{MCG}(S)$, and two disjoint non-homotopic simple curves give rise to commuting Dehn twists.

More generally, let Δ denote a system of disjoint, essential, pairwise non-homotopic simple closed curves (necessarily a finite number, if S is compact). The stabilizer of Δ (up to homotopy) in $\mathcal{MCG}(S)$ is denoted $\mathrm{Stab}(\Delta)$, and we have a short exact sequence

$$0 \to \mathbb{Z}^n \to \mathrm{Stab}(\Delta) \to \mathcal{MCG}'(S \setminus \Delta) \to 0 \qquad (1)$$

where $\mathbb{Z}^n$ is the group of Dehn twists generated by elements of Δ and $\mathcal{MCG}'(S \setminus \Delta)$ is the finite-index subgroup of $\mathcal{MCG}(S \setminus \Delta)$ whose elements permute the boundary in such a way that it can still be glued back to obtain Δ in S.

A mapping class that preserves a system of disjoint essential simple closed curves is called *reducible*. Thurston classified the nontrivial conjugacy classes in $\mathcal{MCG}(S)$ as reducible, finite-order, and *pseudo Anosov* [49], [43]. A pseudo-Anosov mapping class does not preserve any finite set of closed curves. Instead, it preserves a pair of *measured geodesic laminations* (see §2), and any closed curve tends, under forward iteration, to one of these and under backward iteration to the other.

The kernel in (1) is an example of an abelian subgroup of $\mathcal{MCG}(S)$. Birman–Lubotzky–McCarthy [13] used Thurston's classification theorem to classify all abelian (and solvable) subgroups of $\mathcal{MCG}(S)$. In particular they showed that the maximal rank of an abelian subgroup is equal to the maximal cardinality of a disjoint system of curves Δ. The pure Dehn-twist group appearing in (1) is not the only way of obtaining a maximal rank abelian group, however – for example if Δ divides S into 3-holed spheres and 1-holed tori, and a Dehn twist is chosen on each component of Δ and a pseudo-Anosov is chosen on each 1-holed torus component, then these elements generate a maximal rank abelian group.

When studying the "coarse" geometry of $\mathcal{MCG}(S)$, i.e. its large-scale geometry when viewed as a metric space by means of its Cayley graph, these maximal abelian subgroups turn out to be quite important. They are quasi-isometrically embedded (see §4 for definitions) by Mosher [105] in the punctured case and Farb–Lubotzky–Minsky [48] in general (See also Theorem 4.1). On the other hand Brock and Farb asked whether these subgroups represent the largest n for which $\mathbb{Z}^n$ is quasi-isometrically embedded in $\mathcal{MCG}(S)$ (not necessarily as a subgroup). This was answered affirmatively by Hamenstädt [58] and by Behrstock–Minsky [8]. The techniques that go into the latter proof are part of what concerns us in this paper.

Hyperbolic geometry. In a complete hyperbolic surface S, each nontrivial homotopy class is represented by a unique geodesic (if it is not homotopic into a cusp). If Δ is a maximal set of disjoint essential simple closed curves, not homotopic to each other or the cusps, then they are realized as a disjoint set of simple geodesics which cut S into 3-holed spheres. It turns out that a hyperbolic 3-holed sphere is determined uniquely by its boundary lengths and all possible lengths can occur. Once lengths are selected for the components of Δ the hyperbolic metric of S is almost determined, except for the gluings of the boundaries back together, for which there is an additional real parameter for each one. This gives *Fenchel–Nielsen coordinates* for the Teichmüller space of S,

$$\mathcal{T}(S) \cong \mathbb{R}_+^\Delta \times \mathbb{R}^\Delta.$$

The Teichmüller space of S is the set of all hyperbolic metrics on S, up to isometries homotopic to the identity (see e.g. [131], [72], [51]).

On the other hand, the Collar Lemma (see e.g. [78], [38]) asserts that a closed geodesic in S has a regular neighborhood whose radius goes to infinity as the curve's length goes to 0. In particular any sufficiently short geodesic has no self-intersections, and two sufficiently short geodesics cannot cross each other. That suggests a division of $\mathcal{T}(S)$ into *thin regions*, where some curve is shorter than a certain threshold, and *thick regions* where no curves are very short. The intersection pattern of the thin regions is prescribed exactly by the disjointness relation among simple closed curves.

In three dimensional hyperbolic geometry, the generalization of the collar lemma is the "Margulis Lemma" (Kazhdan–Margulis [77]), or Jørgensen's inequality [76] (see also Brooks–Matelski [37], Thurston [131]). In particular sufficiently short closed geodesics have a standard solid torus neighborhood sometimes called a Margulis tube. If for example we consider a hyperbolic structure on $N = S \times \mathbb{R}$, work of Thurston and Bonahon [130], [19] showed that if a geodesic is sufficiently short it must be homotopic to a *simple curve* in S, and Otal [108], [109] showed that any number of sufficiently short curves must be *unknotted and unlinked* – that is, isotopic in N to a collection of disjoint simple level curves with respect to the product structure. See Souto [124] for a generalization of this to the setting of any embedded surface in a hyperbolic 3-manifold.

These sort of results are obtained using *pleated surfaces* (or sometimes *simplicial hyperbolic surfaces*). A pleated surface in a hyperbolic 3-manifold N is a map $f: S \to N$ where S is a surface, the pullback path-metric from N gives a complete hyperbolic metric on S, and moreover f is totally geodesic on the complement of a system of simple geodesic lines called a *lamination*. Thurston introduced these tools into hyperbolic geometry, and their great advantage is that (a) their presence greatly constrains the geometry of the 3-manifold in terms of the well-understood geometry of hyperbolic surfaces, and (b) they are plentiful. In fact if $f: S \to N$ is a π_1-injective map and Δ is a collection of disjoint, essential, nonhomotopic simple closed curves in S whose f-images are not homotopic to cusps, then there exists a pleated map homotopic to f which carries Δ to its geodesic representatives.

This again gives a geometric interpretation to the disjointness relation for simple curves. Indeed, the Gauss–Bonnet theorem implies that the diameter of a hyperbolic surface is bounded, outside of the thin collars, in terms of its topology. In our 3-manifold setting this means that the geodesic representatives of disjoint curves on S are "close modulo thin parts". Thus the pattern of Margulis tubes in a hyperbolic 3-manifold is closely related to patterns of disjoint curves.

Heegaard splittings. Any closed oriented 3-manifold can be expressed, in infinitely many ways, as a union of two handlebodies glued along their boundaries (a handlebody is a 3-ball with 1-handles attached, or a regular neighborhood of a 1-complex embedded in $\mathbb{R}^3$. For example, consider regular neighborhoods of the 1-skeletons of a triangulation and its dual). This is known as a Heegaard splitting. The non-uniqueness of Heegaard splittings means that it is difficult, although not impossible, to extract meaningful topological information from them. The literature on Heegaard splittings is by now extensive, and our discussion will not come close to being comprehensive. For more information the reader is directed to Birman [14], Scharlemann [116] and Zieschang [133].

Given a surface S, an identification of S with the boundary of a handlebody is determined (up to isotopy) by specifying a maximal set Δ of disjoint nonhomotopic simple curves which are to be the boundaries of essential disks, or *meridians*. (Actually it suffices to choose Δ so that its complement consists of a single genus 0 subsurface). The combination of two such sets for the two handlebodies is called a Heegaard diagram. This is a finite amount of information, but we are faced with the fact that there is no natural choice for this diagram among all possible ones for a given splitting.

A splitting is called *reducible* if there is a curve which is a meridian in both sides. The two disks thus bounded form a sphere, which either bounds a ball, in which case the genus of the splitting can be reduced, or is essential, in which case the manifold can be reduced to a connected sum of manifolds with lower-genus splittings. Haken [56] showed conversely that if a manifold is a connected sum and S is a Heegaard surface, then there exists a sphere that meets S in exactly one meridian. That is, a splitting of a reducible manifold is a reducible splitting. In an irreducible manifold, a splitting of minimal genus is irreducible. However irreducible splittings do not have to be of minimal genus.

Casson–Gordon [44] introduced the notion of *weak reducibility:* A splitting is weakly reducible if it contains two *disjoint* meridians (note that reducible implies weakly reducible, as two equal meridians can be made disjoint). They showed that a weakly reducible splitting is either reducible, or contains a 2-sided incompressible surface (an incompressible surface is a π_1-injectively embedded surface of positive genus; manifolds which admit 2-sided incompressible surfaces are called *Haken*, and are in many ways easier to study.) Thus an irreducible non-Haken manifold has a *strongly irreducible* (i.e. not weakly reducible) splitting. For Haken manifolds, extensions of the arguments of Casson–Gordon by Scharlemann–Thompson (see [117] and [115]) lead to a decomposition along incompressible surfaces into strongly irreducible Heegaard splittings-with-boundary, known as a *generalized Heegaard splitting*. This has become a widely applied and sophisticated theory.

Hempel [69] showed that a splitting of a manifold that is Seifert-fibred or contains an essential torus must have the *disjoint curve property*: There exist meridians m_1, m_2 from the two sides, and a curve γ so that both m_1 and m_2 are disjoint from γ. (Note that reducibility and weak-reducibility are special cases of this property).

At this point, the properties of splittings and their meridians come into contact with Thurston's geometrization conjecture. After decomposing along essential spheres and tori, a 3-manifold should fall into pieces admitting one of the eight 3-dimensional geometries – Euclidean, hyperbolic, spherical, or the five non-isotropic fibred geometries (see Scott [122]). Our discussion so far implies that manifolds that break up non-trivially have reducible splittings (by Haken) or splittings with the disjoint curve property (by Hempel).

Irreducible splittings for the non-hyperbolic pieces are now very well understood, through the work of Waldhausen [132], Bonahon–Otal [20], Moriah [103], Boileau–Rost–Zieschang and Boileau–Collins–Zieschang [17], [15], Frohman–Hass [50], Boileau–Otal [16], Moriah–Schultens [104] and Cooper–Scharlemann [46]. In all these cases splittings have the disjoint curve property.

A splitting without the disjoint curve property should therefore yield a hyperbolic manifold (modulo the geometrization conjecture, whose proof by Perelman is, as of this writing, edging ever closer to formal acceptance). Hempel brought the notions of reducibility, weak reducibility, and the disjoint curve property together as particular values of a *distance function on splittings*, which we will discuss further in §6. The main question that we will discuss in §6 is: how do we translate information in the Heegaard diagram into geometric data for a hyperbolic manifold?

2. Curve complexes

In view of the foregoing discussion, one is naturally led to consider a combinatorial object which captures the disjointness relation among simple closed curves (up to homotopy) on a surface. Actually when Harvey introduced the complex of curves in [63], [64] he was motivated by analogy, in the setting of $\mathcal{MCG}(S)$, with Bruhat–Tits buildings for Lie groups. We will now give some precise definitions, which are somewhat more tedious than one would hope because of the need to deal with a few special cases.

2.1. Definitions. Let $S = S_{g,b}$ be an orientable compact surface with genus g and b boundary components. An *essential simple curve* in S will, for us, be an embedded circle in S which is homotopically non-trivial, and not homotopic into ∂S (non-peripheral).

Let $\mathcal{C}(S)$ denote the simplicial complex whose vertices are homotopy classes of essential simple curves on S, and whose k-simplices are, except in a few sporadic cases described below, defined to be $(k+1)$-tuples of distinct vertices $[\alpha_0, \ldots, \alpha_k]$ whose representatives can be chosen to be pairwise disjoint. One may check that, when S has negative Euler characteristic, that there are at most $\xi(S) \equiv 3g - 3 + b$ such curves, and so $\dim \mathcal{C}(S) = \xi(S) - 1$. Although $\mathcal{C}(S)$ is finite dimensional, it is not locally finite, and this complexity accounts for much of the interest in studying and applying it. We let $\mathcal{C}_k(S)$ denote the k-skeleton of $\mathcal{C}(S)$.

Sporadic cases. For $S_{0,b}$ with $b \leq 3$ the complex is empty (although the annulus $S_{0,2}$ will later play an important role in a different way). For the tori $S_{1,0}$ and $S_{1,1}$, and for the 4-holed sphere $S_{0,4}$, the above definition gives a 0-dimensional complex. It turns out to be useful to add edges according to the rule that $[vw]$ is an edge if v and w have representatives that intersect once (in the case of the tori) or twice (in the case of the sphere). This forms our definition of $\mathcal{C}(S)$ for these cases.

Examples. For $S_{1,0}$ and $S_{1,1}$, simple curves are identified by their slopes in homology, i.e. $\mathcal{C}_0(S) \cong \mathbb{Q} \cup \{\infty\}$. Edges correspond to edges of the *Farey triangulation* of the disk, that is $\left[\frac{p}{q}, \frac{r}{s}\right]$ is an edge iff $|ps - qr| = 1$ (with fractions in lowest terms). See Figure 1. $\mathcal{C}(S_{0,4})$ is also isomorphic to $S_{1,1}$, with the isomorphism obtained from the fact that $S_{1,0}$ or $S_{1,1}$ modulo the hyperelliptic involution, and punctured at the fixed points, is $S_{0,4}$.

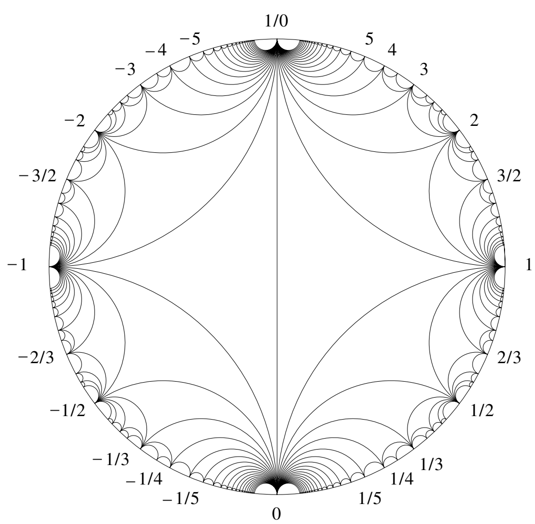

Figure 1. The Farey triangulation is the curve complex of $S_{1,0}$, $S_{1,1}$ and $S_{0,4}$.

For $S = S_{0,5}$ and $S_{1,2}$ (the case $\xi(S) = 2$), $\mathcal{C}(S)$ is again a graph, and in fact the two are isomorphic (again via the hyperelliptic involution). The link of a vertex $v \in \mathcal{C}(S_{0,5})$ is $\mathcal{C}_0(S_{0,4}) = \mathbb{Q} \cup \{\infty\}$.

Mapping class group action. $\mathcal{MCG}(S)$ acts naturally on $\mathcal{C}(S)$ and since homeomorphisms preserve disjointness it acts by simplicial automorphisms, and the quotient is easily seen to be finite (the set of vertices $[\alpha]$ of $\mathcal{C}(S)/\mathcal{MCG}(S)$, for example, is indexed by the topological type of $S \setminus \alpha$). The action is far from proper, as the large stabilizer in (1) indicates (we now recognize Δ as a simplex in $\mathcal{C}(S)$).

Nevertheless the simplicial structure of $\mathcal{C}(S)$ records all of the structure of $\mathcal{MCG}(S)$ in the following sense:

Theorem 2.1 (Ivanov [75], Luo [86], Korkmaz [82]). *The map $\mathcal{MCG}(S) \to \mathrm{Aut}(S)$ is an isomorphism in all cases except for $S_{1,2}$, where it is injective with index 2 image.*

(Ivanov proved this for genus at least 2. Luo and Korkmaz proved the remaining cases, with Luo's proof giving a unified argument.)

Hatcher–Thurston [67] defined a closely related complex and used it to give a presentation for $\mathcal{MCG}(S)$. Harer [61], [60] computed the homotopy type of $\mathcal{C}(S)$ and related complexes, and used it to study the homology of $\mathcal{MCG}(S)$. Hatcher [65] gave simplified proofs of contractibility for related complexes of arcs in a punctured surface.

2.2. Geometric structure. As the local structure of $\mathcal{C}(S)$ is rather intricate, it turns out that something can be gained by attempting to ignore it. First let us view $\mathcal{C}(S)$ as a metric space by making each simplex standard Euclidean with sidelength 1, say, and taking the path metric (a theorem of Bridson [24] assures us that this makes it into a complete geodesic metric space). It will sometimes be simpler to consider the path metric just on the 1-skeleton $\mathcal{C}_1(S)$. These two metric spaces are quasi-isometric, and we will be interested in coarse features such as quasi-geodesics, quasi-isometric type, etc.

Hyperbolicity. With Masur in [94], we proved the following theorem:

Theorem 2.2. *In all nontrivial cases $\mathcal{C}(S)$ is an infinite diameter δ-hyperbolic metric space.*

The usual definition of δ-hyperbolic, due to Rips, Gromov and Cannon, is equivalent for complete path metric spaces to the *δ-thin triangles property:* each leg of a geodesic triangle is contained in a δ-neighborhood of the other two legs.

Examples. Typical examples to keep in mind are metric trees, classical hyperbolic spaces, and Cayley graphs of fundamental groups of closed negatively curved manifolds. $\mathbb{R}^n$ and $\mathbb{Z}^n$ are not δ-hyperbolic when $n > 1$, and in fact any group that contains a $\mathbb{Z}^2$ subgroup does not have a δ-hyperbolic Cayley graph. Note, this includes $\mathcal{MCG}(S)$ for non-sporadic S.

The proof of Theorem 2.2 involved the construction of a family of paths in $\mathcal{C}(S)$ using geodesics in the Teichmüller space of S. If to each point along a Teichmüller geodesic we associate the set of shortest curves in the associated metric, we obtain a "quasi-path" in $\mathcal{C}(S)$. Properties of quadratic differentials and their foliations were used to construct a "quasi-retraction" from $\mathcal{C}(S)$ to such a path, which has certain strong contraction properties that imply that $\mathcal{C}(S)$ is hyperbolic, and the paths are quasi-geodesics.

See Bowditch [22] for a considerable simplification of this proof, which in particular provides explicit upper bounds for δ, logarithmic in $\xi(S)$.

Laminations and boundary. A δ-hyperbolic space has a natural boundary at infinity, whose points are equivalence classes of "fellow-traveling" quasigeodesic rays. Klarreich's theorem describes this boundary, and relates it to Thurston's theory of geodesic laminations on surfaces.

Fix a complete, finite-area hyperbolic metric on $\mathrm{int}(S)$. A geodesic lamination on a complete hyperbolic surface is a closed set which is a disjoint union of complete simple geodesics. For example, a closed geodesic loop is a lamination, and a sequence of such loops whose lengths increase without bound will have a subsequence that converges on compact sets to a geodesic lamination, in the Hausdorff topology. Thurston used this notion to complete the set of geodesic loops in S. (His construction involved additional structure – he considers laminations equipped with *transverse measures* – but we will ignore this point here).

We note also that different choices of hyperbolic structures give rise to canonically homeomorphic representations of the space of geodesic laminations, so we may consider this as an intrinsic topological object (see Hatcher [66]).

Of interest to us will be the set of *filling* laminations, which we denote $\mathcal{EL}(S)$. A lamination is filling if it intersects every simple closed geodesic. $\mathcal{EL}(S)$ comes with a natural topology (which is somewhat coarser than the topology of Hausdorff convergence, and involves the transverse measures).

Example. For $S_{1,1}$ and $S_{0,4}$ we have seen that $\mathcal{C}_0(S)$ can be identified with $\mathbb{Q} \cup \{\infty\}$. The set of all laminations is the circle $\mathbb{R} \cup \infty$, with irrational slopes corresponding to filling laminations.

Klarreich's theorem states the following:

Theorem 2.3 (Klarreich [79]). *The boundary of $\mathcal{C}(S)$ is naturally homeormorphic to $\mathcal{EL}(S)$.*

The proof uses the same Teichmüller-geodesic paths used in the proof of hyperbolicity. A Teichmüller ray has a "foliation at infinity" whose length shrinks exponentially as one travels along the ray. These foliations correspond to the laminations in $\mathcal{EL}(S)$ (see Levitt [85] for the general correspondence between foliations and laminations). An alternate proof was found by Hamenstädt [59], using the machinery of train-tracks.

The topological structure of $\mathcal{EL}(S)$ is still somewhat mysterious. It is not known for example if it is disconnected, except in the case $\xi = 1$ where it is the set of irrationals in $\mathbb{R}$ (this question was first raised by P. Storm).

3. Nested structure

The coarse information given by the hyperbolicity theorem is hard to use by itself. With Masur in [95], we refine the coarse approach by applying it to the link structure of $\mathcal{C}(S)$. The link of a simplex in $\mathcal{C}(S)$ is closely related to the curve complex of a subsurface of S, and hence the idea here is to use coarse geometry in an inductive way, looking successively at finer levels of structure.

Subsurfaces and partitions. Let W be an essential subsurface of S, and suppose first that $\xi(W) > 1$. It is evident that $\mathcal{C}(W)$ is a subcomplex of $\mathcal{C}(S)$, contained in the link of the simplex $[\partial W]$. The remaining cases, namely when W is an annulus, one-holed torus and four-holed sphere, cause a certain amount of extra trouble. For $W = S_{1,1}$ and $W = S_{0,4}$, for example, all the vertices of $\mathcal{C}(W)$ are vertices of $\mathcal{C}(S)$, but the edges are not edges of $\mathcal{C}(S)$, due to the special definition of $\mathcal{C}(W)$ in that case. When W is an annulus, $\mathcal{C}(W)$ is empty but we will want something that will correspond to the Dehn twist group around W. (If $W = S_{0,3}$ then $\mathcal{C}(W) = \emptyset$ also, and we will be happy with that).

So, let us make the following new definitions. Let W be a surface with nonempty boundary. Let $\mathcal{A}(W)$ be the complex whose vertices are essential curves (i.e. vertices of $\mathcal{C}(W)$) and properly embedded essential arcs. Except when W is an annulus, we consider these things up to isotopy rel boundary, and "essential" for arcs means not isotopic into the boundary. For an annulus we take the isotopies to have fixed endpoints. Simplices are sets of arcs and curves that have representatives with disjoint interiors. This makes $\mathcal{A}$ infinite dimensional for an annulus, but it is still quasi-isometric to $\mathbb{Z}$. For $\xi(W) > 1$, we note that $\mathcal{C}(W)$ embeds in $\mathcal{A}(W)$ and this embedding is a quasi-isometry (when $\xi(W) = 1$ there is a quasi-isometry which is not quite an embedding).

If Δ is a simplex of $\mathcal{C}(S)$, let $\sigma(\Delta)$ be the union of components of $S \setminus \Delta$ which are not 3-holed spheres, together with annuli whose cores are components of Δ. This is called the "partition" of Δ. We let $lk^*(\Delta)$ denote the join of $\mathcal{A}(W_i)$ over all components $W_i \in \sigma(\Delta)$.

We think of $lk^*(\Delta)$ as an "extended link" for Δ. The actual link of Δ is the join of the complexes for components of $S \setminus \Delta$ of $\xi > 1$, and the 0-skeletons of the complexes for $\xi = 1$ components. The annular factors are not visible in the link; they can be detected in the neighborhood of radius 2, however.

Basic geometric properties. There is a small number of geometric properties of $\mathcal{C}(S)$ which connect global geometry to geometry in the extended links lk^*, and are responsible for most of the rest of our analysis. In outline they are the following:

1. *Hyperbolicity at all levels.* By Theorem 2.2 $\mathcal{C}(S)$ is hyperbolic, and furthermore for every simplex Δ, $lk^*(\Delta)$ is the join of the hyperbolic complexes associated to the components of $\sigma(\Delta)$.

2. *Subsurface projection bounds.* There is a natural projection from $\mathcal{C}(S)$ minus a neighborhood of $[\partial W]$ to $\mathcal{A}(W)$, where W is an essential subsurface. A geodesic in $\mathcal{C}_1(S)$ which stays out of this neighborhood has uniformly bounded projection image.
3. *Hierarchy paths and rigidity.* There is a distinguished family of quasigeodesic paths in the graph of markings on S (see below) which are controlled in a strong way by the subsurface projections of their endpoints.

Subsurface projections. For any essential $W \subset S$ there is a useful map

$$\pi_W : \mathcal{C}(S) \to \mathcal{A}(W) \cup \{\emptyset\}.$$

Namely, given a point $x \in \mathcal{C}(S)$, let δ be the simplex whose interior contains x. There is a unique cover of S to which W lifts homeomorphically, and this cover has a natural compactification $\overline{W}$ that identifies it with W (inherited from the natural compactification of the universal cover of S). Each component of δ lifts to a system of curves and/or arcs in $\overline{W}$, and the essential ones define vertices of $\mathcal{A}(W)$. The union over all vertices of δ gives a (possibly empty) simplex in $\mathcal{A}(W)$, and barycentric coordinates of x in δ push forward to define a unique point in this simplex, which is $\pi_W(x)$. Note that $\pi_W(x) = \emptyset$ only if δ has no essential intersections with W. (See also Ivanov [74] for a version of π_W).

One can think of this map as analogous to visual projection from $X \setminus \{x\}$ to the unit sphere around x, for a reasonably nice space X.

Link projection bounds. In [95], we proved the following theorem:

Theorem 3.1. *Let $W \subset S$ be an essential surface. let g be a geodesic in the* 1*-skeleton of $\mathcal{C}(S)$ all of whose vertices intersect W essentially. Then*

$$\operatorname{diam}_{\mathcal{A}(W)}(g) \le B$$

where B depends only on the topological type of S.

A motivational analogy for this theorem may be found in the setting of CAT(0) complexes. (CAT(0) refers to non-positive curvature in the sense of comparison geometry, and in particular implies uniqueness of geodesics between points. See e.g. [25].) Let v be a point in a piecewise Euclidean CAT(0) complex X, and let $l(v)$ be its link, which we may identify with the unit "sphere" around v. There is a well-defined projection of $X \setminus \{v\}$ to $l(v)$, via geodesic segments from points in X to v, and the diameter $d_v(Y)$ of the projection of $Y \subset X \setminus \{v\}$ is the "visual diameter of Y". If g is a geodesic segment that avoids v then the cone of g onto v is an embedded triangle, and it follows that $d_{l(v)}(g) < \pi$. Conversely if $d_{l(v)}(g) \ge \pi$ it follows that g must pass through v.

The reason that this theorem holds in the curve complex, in spite of there not being a CAT(0) metric, is roughly the following: Let g be the geodesic and imagine that g

is extended to an infinite ray $\widehat{g}$, still with all vertices crossing W essentially (this is always possible in one direction or the other). Klarreich's theorem implies that $\widehat{g}$ converges to a filling lamination λ. The lift of λ to $\widehat{W}$ gives a point (or really simplex) in $\mathcal{A}(W)$, and since the vertices of $\widehat{g}$ converge to λ in a geometric sense, eventually their projections to $\mathcal{A}(W)$ are within bounded distance of the projection of λ. Thus $\mathrm{diam}_{\mathcal{A}(W)}(\widehat{g}) < \infty$.

Now to obtain the bound of Theorem 3.1, we need a uniform version of the above argument. This requires a more delicate analysis using some of the same machinery used to prove the hyperbolicity theorem.

Theorem 3.1 is the first step in a finer inductive study of the geometry of $\mathcal{MCG}(S)$. In order to describe this, we need to introduce some more terminology.

Markings and hierarchies. A convenient way to study the geometry of $\mathcal{MCG}(S)$ is to look at its action on the *marking graph* $\mathcal{M}(S)$. A marking μ of S is given by the following data: A maximal simplex of $\mathcal{C}(S)$, called base(μ), and a collection of *transversal curves* trans(μ), where each base curve α is equipped with one transversal curve t_α, which intersects α. t_α is disjoint from all other base curves, and α and t_α either intersect exactly once, or twice with opposite orientations (their regular neighborhood is then, respectively, $S_{1,1}$ or $S_{0,4}$). It is easy to see that $\mathcal{MCG}(S)$ acts on these markings with finite quotient and finite stabilizers. Moreover one can easily write down a simple finite list of "elementary moves" $\mu \to \nu$ such that the graph $\mathcal{M}(S)$ whose vertices are markings and whose edges are elementary moves is connected. Hence $\mathcal{M}(S)$, with the natural metric in which every edge has unit length, is quasi-isometric to $\mathcal{MCG}(S)$, which acts on it properly, cocompactly and isometrically.

In [95], we study a class of paths in $\mathcal{M}(S)$ which arise from an iterated construction in $\mathcal{C}(S)$ which we call a "hierarchy of geodesics".

Let us begin with an example (the same one treated extensively in [101]). If $S = S_{0,5}$ then $\mathcal{C}(S)$ is a graph, and a marking consists of two base curves and two transversals. Let μ and ν be two markings. Let $v_0, v_1, \ldots, v_N$ be the vertices of a geodesic in $\mathcal{C}_1(S)$ joining $v_0 \in \mathrm{base}(\mu)$ to $v_N \in \mathrm{base}(\nu)$. For $0 < i < N$ we note that v_i cuts S into $W_i \cong S_{0,4}$ and a three-holed sphere. Both v_{i-1} and v_{i+1} are vertices in $\mathcal{C}(W_i)$, and we may join them by a geodesic $v_{i-1} = u_0, \ldots, u_k = v_{i+1}$ in $\mathcal{C}_1(W_i)$. This gives us a sequence

$$[v_i, u_0], \ldots, [v_i, u_k]$$

of edges, or pants decompositions of S, with each step corresponding to a simple curve-replacement move. At the beginning we similarly join the second vertex of base(μ) to v_1 in W_0, and likewise at the end with base(ν). The result of this is a sequence of pants decompositions of S joining μ to ν, and separated by simple moves. Now consider an interior point u_j of the $\mathcal{C}(W_i)$-geodesic built over v_i. u_{j-1} and u_{j+1} intersect u_j, and they give two points in $\mathcal{A}(u_j)$, the annulus complex. We can join one to the other by a geodesic in $\mathcal{A}(u_j)$, which amounts to a sequence of Dehn twists. Something slightly subtle happens at the endpoints of the $\mathcal{C}(W_i)$-geodesics, which the reader is invited to investigate. This procedure extends all the pants decompositions we built into markings, which can be traversed in a sequence of elementary moves from μ to ν.

In general something similar happens – starting with a geodesic in $\mathcal{C}_1(S)$ we inductively add geodesics in the extended links of the simplices traversed. (Actually we work with sequences of simplices called "tight geodesics", but we will ignore this technicality here). The final output of this procedure is not exactly a sequence of markings; this is because, when the process fills in two or more disjoint subsurfaces, the geodesics in those subsurfaces can be traversed in either order. That is, we are dealing with the unavoidable appearance of product regions in $\mathcal{M}(S)$. However the structure can be resolved (non-uniquely) into a sequence of markings connected by elementary moves. We call these *hierarchy paths*. The following theorem summarizes the properties of hierarchies and their paths established in [95]:

Theorem 3.2. *For any pair of points $\mu, \nu \in \mathcal{M}(S)$ there is a hierarchy and a family of hierarchy paths with the following properties.*

- Efficiency. *Hierarchy paths are quasigeodesics in $\mathcal{M}(S)$, with uniform constants.*
- Monotonicity. *For any essential $W \subseteq S$ and a hierarchy path β, $\pi_W \circ \beta$ traverses quasi-monotonically a bounded neighborhood of a geodesic between $\pi_W(\mu)$ and $\pi_W(\nu)$.*
- Forced traversals. *There is a constant B depending only on the topology of S such that if, for $W \subset S$, we have $d_W(\mu, \nu) > B$, then W appears in any hierarchy for μ and ν, and in particular any hierarchy path must have a subpath in which all the markings contain $[\partial W]$.*
- Partial ordering and stability. *There is a partial order defined among subsurfaces which appear in a hierarchy, such that any two subsurfaces which intersect essentially are ordered. If $W < Z$ in this order for a hierarchy from μ to ν then ∂W appears before ∂Z in any hierarchy path from μ to ν. If also $d_W(\mu, \nu)$ and $d_Z(\mu, \nu)$ are larger than a certain a priori constant, then this partial ordering is consistent over all hierarchies from μ to ν, and moreover for any ν' for which Z appears in a hierarchy from μ to ν', W is forced to appear as well.*

(We have abbreviated $d_{\mathcal{A}(W)}(\pi_W(\mu), \pi_W(\nu))$ as $d_W(\mu, \nu)$, and will continue to do this.) The Forced Traversals property is essentially a generalization of Theorem 3.1.

Distance formula. The partial ordering and monotonicity properties mean that the length of a hierarchy path is, roughly, the sum of the projection distances of its endpoints in all subsurfaces which it traverses. Moreover the forced traversals property implies that those subsurfaces traversed account for all subsurfaces in which these projection distances are sufficiently large. Any competing path from μ to ν is forced to make up the same distances at some point, and this is the basic reason for the quasi-geodesic property of hierarchy paths. This argument also gives rise to the following Distance Formula, which plays an important role later on.

Theorem 3.3. *If* $\mu, \nu \in \mathcal{M}(S)$,

$$d_{\mathcal{M}(S)}(\mu, \nu) \approx \sum_{Y \subseteq S} \{d_Y(\mu, \nu)\}_K . \tag{2}$$

Some explanations are in order here: We define the expression $\{N\}_K$ to be N if $N > K$ and 0 otherwise – hence K functions as a "threshold" below which contributions are ignored. The constant K used depends only on the topological type of S, and the expression $f \approx g$ means $g/a - b < f < ag + b$ where $a > 1, b > 0$ are also a priori constants. Note that, if $W \subset S$ is an essential surface and $\mu \in \mathcal{M}(S)$ is a marking, then $\pi_W(\mu) \in \mathcal{A}(W)$ is always defined (up to finitely many choices), since some part of μ, possibly only a transversal curve, must intersect W essentially.

So this theorem is saying that, after throwing away the low-level "noise", only finitely many subsurfaces $W \subseteq S$ remain, and the projection distances in these account for the distance between μ and ν. Note also that the sum includes a term for $Y = S$, i.e. distance in the curve complex itself.

Geodesics in Teichmüller space. Just as Teichmüller geodesics play a role in the proof of hyperbolicity for $\mathcal{C}(S)$, the geometry of $\mathcal{C}(S)$ and its nested structure gives us some added understanding of Teichmüller geodesics. Rafi [112], [111] analyzed the extent to which the long-term behavior of a Teichmüller geodesic mirrors the combinatorial structure of a hierarchy, and also develops in [110] a distance formula in $\mathcal{T}(S)$ analogous to Theorem 3.3. See also Rees [113] for a related but independent study of Teichmüller geodesics.

4. Coarse geometry of $\mathcal{MCG}(S)$

The study of coarse geometric properties of abstract groups, by means of their Cayley graphs (or equivalently their word metric), can be traced back to the theorem of Milnor and Švarc [98], [128] on growth rates of groups, to Gromov's work [53] on groups of polynomial growth, and the introduction by Gromov [54] and Cannon [41], [42] of hyperbolic groups. This field is now enormous and we will not attempt to survey it. Some good general references are Gromov [55], [54] and Bridson–Haefliger [25].

We will be interested in examining phenomena of hyperbolicity, undistorted (quasi-isometrically embedded) subgroups, geometric rank, and asymptotic cones, as they relate to $\mathcal{MCG}(S)$. $\mathcal{MCG}(S)$ is not hyperbolic, but as we have already seen it is hyperbolic in a relative sense, through its action on the hyperbolic space $\mathcal{C}(S)$. Before we proceed let us record some of the usual definitions.

A map $f\colon X \to Y$ between metric spaces is *coarse Lipschitz* if a uniform inequality holds of the form

$$d(f(x), f(x')) \le ad(x, x') + b$$

for all $x, x' \in X$. It is *coarsely bilipschitz*, or a *quasi-isometric embedding*, if the opposite inequality

$$d(x, x') \le ad(f(x), f(x')) + b$$

holds as well. Finally we say that $f\colon X \to Y$ is a *quasi-isometry* if it is coarsely bilipschitz, and in addition there is an upper bound on $d(y, f(X))$ for all $y \in Y$.

We assume from now on that generators for $\mathcal{MCG}(S)$ are fixed, giving $\mathcal{MCG}(S)$ a word metric, which (as remarked in § 3) is quasi-isometric to $\mathcal{M}(S)$.

Quasi-isometrically embedded subgroups. One almost immediate consequence of the Distance Formula is this theorem:

Theorem 4.1. *Let Δ be a simplex in $\mathcal{C}(S)$. The subgroup* Stab(Δ) *is quasi-isometrically embedded in $\mathcal{MCG}(S)$. Moreover, there is a coarse-Lipschitz retraction $\mathcal{MCG}(S) \to$* Stab(Δ). *Finally,* Stab(Δ) *is quasi-isometric to a product*

$$\prod_{W \in \sigma(\Delta)} \mathcal{MCG}(W)$$

where $\sigma(\Delta)$ refers to the partition of S defined by Δ, as in § 3.

(For annular components of $\sigma(\Delta)$, we interpret $\mathcal{MCG}(W)$ to be the Dehn twist group of W).

The constants of the quasi-isometry and retraction depend on the choice of Δ. A uniform statement is obtained by fixing Δ and considering all left-cosets of Stab(Δ). More geometrically, and more in keeping with our approach, we consider the marking graph $\mathcal{M}(S)$, and within it the subsets $Q(\Delta)$ which consist of all markings whose base contains the simplex Δ. This subset is clearly quasi-isometric to Stab(Δ) and its left cosets. With this notation, what we produce is a coarse-Lipschitz retraction $\mathcal{M}(S) \to Q(\Delta)$, and a quasi-isometry

$$Q(\Delta) \cong \prod_{W \in \sigma(\Delta)} \mathcal{M}(W). \tag{3}$$

Where now the constants depend only on the topological type of S, and not on the choice of Δ. For an annulus W, $\mathcal{M}(W)$ is just $\mathcal{A}(W)$, which we recall is quasiisometric to $\mathbb{Z}$.

Example. If $S = S_{0,5}$ and $\Delta = [\delta_0, \delta_1]$ is an edge, then $Q(\Delta)$ is quasi-isometric to $\mathbb{Z}^2$ (and stabilized by the $\mathbb{Z}^2$ subgroup of Dehn twists about δ_0 and δ_1). If Δ is a vertex then $Q(\Delta)$ is quasi-isometric to $\mathbb{Z} \times \mathrm{SL}_2(\mathbb{Z})$, with the second factor corresponding to $\mathcal{M}(S_{0,4})$. Compare to the short exact sequence (1).

The quasi-isometric embedding part of the statement is a reflection of the fact that in the distance formula for two elements of $Q(\Delta)$, the only contributions come from terms that have no essential intersection with Δ, i.e. those that contribute also to the product over $\sigma(\Delta)$. See also Hamenstädt [57] for a proof of a similar statement using train-track technology.

The coarse retraction to $Q(\Delta)$ is built, inductively, using the subsurface projections described in the previous section. See Behrstock [7] for details. We call this retraction $\pi_{Q(\Delta)}$. The composition of $\pi_{Q(\Delta)}$ with projection to any of the factors $\mathcal{M}(W)$ of the product in (3) is called $\pi_{\mathcal{M}(W)}$, and it is well defined up to bounded ambiguity.

Asymptotic cones. Another way to quantify our coarse understanding of the group is to consider its *asymptotic cones*, which are rescaling limits of the group, in a certain sense. They were used implicitly by Gromov in [53], and introduced explicitly in Van den Dries–Wilkie [47]. In order to define the limit one resorts to the mechanism of *ultrafilters*, which can be briefly described as follows.

Let $X = (X, d)$ be a metric space, fix a sequence $s_n \to \infty$, and consider the metric spaces $(X_n, d_n) \equiv (X, \frac{1}{s_n} d)$, namely X with metric scaled down by $1/s_n$. We want, essentially, to consider the set of all "limiting pictures" of X_n, as $n \to \infty$. An *ultrafilter* is a way of organizing the natural numbers so as to pick out a convergent subsequence for every sequence of points in a compact space. More precisely, let ω be a *finitely additive probability measure* on $\mathbb{N}$, which is defined on every subset, and takes on only values of 0 and 1. Further, we assume ω is *non-principal*, meaning it is 0 on finite subsets. Existence of such measures is a nice exercise in using Zorn's lemma. If p_n is a sequence in a Hausdorff space T, we declare the ω-limit to be $\lim_\omega p_n = p$ if, for every neighborhood U of p, $\omega\{j : x_j \in U\} = 1$.

With this definition we find that *every* sequence in a compact space has a unique ω-limit. Now returning to X_n, we consider all sequences $\boldsymbol{x} = (x_n \in X_n)$. A pseudo-distance can be defined by

$$d_\omega(\boldsymbol{x}, \boldsymbol{y}) = \lim_\omega d_n(x_n, y_n)$$

which gives a point in $[0, \infty]$. Fixing a basepoint sequence $\boldsymbol{x}_0$ we can restrict to the subset $\{\boldsymbol{x} : d_\omega(\boldsymbol{x}_0, \boldsymbol{x}) < \infty\}$ and identify pairs $\boldsymbol{x} \sim \boldsymbol{y}$ whenever $d_\omega(\boldsymbol{x}, \boldsymbol{y}) = 0$. This gives a metric space, known as an asymptotic cone for X.

Note that this construction depends on the choice of scaling constants, ultrafilter ω, and basepoints. In our setting X will be $\mathcal{M}(S)$ (or equivalently $\mathcal{MCG}(S)$), and the basepoint will not matter because the space is (coarsely) homogeneous. The scaling constants and ultrafilter will be assumed fixed. We denote the asymptotic cone of $\mathcal{M}(S)$ by $\mathcal{M}^\omega(S)$.

Examples. The asymptotic cone of $\mathbb{Z}^n$ is always $\mathbb{R}^n$. The symptotic cone of a δ-hyperbolic space is an $\mathbb{R}$-tree, that is, a geodesic metric space in which any two points are joined by a unique embedded path. $\mathcal{MCG}(S_{1,1})$ and $\mathcal{MCG}(S_{0,4})$ are both commensurable with $\mathrm{SL}_2(\mathbb{Z})$, and hence with the free group F_2 whose Cayley graph is a tree. Hence their asymptotic cones are $\mathbb{R}$-trees as well. (Note that these $\mathbb{R}$-trees have dense, and uncountable, branching).

A quasi-isometric embedding gives rise, after taking asymptotic cones, to a bilipschitz embedding. This allows us to replace coarse statements with topological statements, which is part of what makes asymptotic cones useful.

A sequence of subsets $Q(\Delta_i)$ of $\mathcal{M}(S)$ gives rise to an ω-limit which we denote by $\mathcal{Q}^\omega(\mathbf{\Delta}) \subset \mathcal{M}^\omega(S)$ (where $\mathbf{\Delta}$ denotes the sequence (Δ_i)). Theorem 4.1, or rather the quasi-isometry (3), immediately tells us that $\mathcal{Q}^\omega(\mathbf{\Delta})$ is a bilipschitz-embedded product of lower-complexity asymptotic cones, admitting a Lipschitz retraction from $\mathcal{M}^\omega(S)$.

This retraction and some related constructions are instrumental in studying separation and dimension properties of the asymptotic cone, as is done in Behrstock [7] and Behrstock–Minsky [8]. The main theorem of [8] is the following:

Theorem 4.2. $\widehat{\dim}\, \mathcal{M}^\omega(S) = \xi(S)$.

Here $\widehat{\dim}$ denotes the maximal topological dimension over all locally compact subsets of $\mathcal{M}^\omega(S)$.

A direct consequence of this is a proof of the "rank conjecture"of Brock–Farb [30], which states that $\xi(S)$ is the maximal rank of a quasi-isometrically embedded flat in $\mathcal{M}^\omega(S)$. Independently, Hamenstädt [58] proved this theorem by somewhat different means, establishing in particular a homological version of the dimension theorem. The main idea of our proof is to study certain families of separating sets, which we describe in more detail below.

Separation properties of the asymptotic cone. In Behrstock [7] it was shown that every point of $\mathcal{M}^\omega(S)$ is a cut point. An extension and refinement of Behrstock's construction leads to the following statement. First, let $r(W) = \xi(W)$ if W is a connected non-annular essential subsurface of S, $r(W) = 1$ if W is an essential annulus, and define it over disjoint unions to be additive. It is not hard to see that Theorem 4.2 for essential (but not necessarily connected) subsurfaces W in S should state that $\widehat{\dim}\, \mathcal{M}^\omega(W) = r(W)$, where $\mathcal{M}^\omega(W)$ is the product of $\mathcal{M}^\omega(W_i)$ over the components. (When W is an annulus in particular, note that $\mathcal{M}^\omega(W)$ is $\mathcal{A}^\omega(W) = \mathbb{R}$, hence the definition $r = 1$ in that case).

Theorem 4.3. *There is a family $\mathcal{L}$ of closed subsets of $\mathcal{M}^\omega(S)$ with the following properties:*

- *Each $L \in \mathcal{L}$ is either a single point, or is bilipschitz equivalent to $\mathcal{M}^\omega(W)$ for a (possibly disconnected) subsurface $W \subset S$ with $r(W) < r(S)$.*
- *For any $x, y \in \mathcal{M}^\omega(S)$ there exists $L \in \mathcal{L}$ which separates x from y.*

We call $\mathcal{L}$ "separators" of $\mathcal{M}^\omega(S)$. The case that $L \in \mathcal{L}$ is a single point is Behrstock's cut point theorem, and the theorem on topological dimension can be obtained by induction (in a locally compact space, the existence of a family of separators like this with dimensions at most $n-1$ implies a dimension upper bound of n for the whole space. The lower bound is easy).

The idea of the proof is to find "rank 1 directions" in the cone, and establish the separators as "transverse sets" for these directions. Let us consider the special case of $S = S_{0,5}$, let $\boldsymbol{\delta} = (\delta_n)$ be a sequence of simple closed curves in S, and let $\mathcal{Q}^\omega(\boldsymbol{\delta})$ be the ω-limit of $Q_n(\delta_n)$. We know from Theorem 4.1 that $\mathcal{Q}^\omega(\boldsymbol{\delta})$ can be identified with $\mathbb{R}\times T$, where $\mathbb{R}$ is $\mathcal{M}^\omega(\delta)$, the asymptotic cone of the sequence of twist complexes $\mathcal{A}(\delta_n)$, and T is $\mathcal{M}^\omega(S\setminus\boldsymbol{\delta})$, the asymptotic cone of $\mathcal{M}(S\setminus\delta_n)$, which is an $\mathbb{R}$-tree since $S\setminus\delta_n \cong S_{0,4}$.

A separator associated to this picture is a subset of $\mathcal{Q}^\omega(\boldsymbol{\delta})$ of the form $\{s\}\times T$, which certainly separates $\mathcal{Q}^\omega(\boldsymbol{\delta})$, and has $\widehat{\dim} = 1$ since it is an $\mathbb{R}$-tree. To show that it has global separation properties we consider the map

$$\pi_{\boldsymbol{\delta}}\colon \mathcal{M}^\omega(S) \to \mathcal{M}^\omega(\boldsymbol{\delta}) = \mathbb{R}$$

which is the rescaled ω-limit of the projection maps from $\mathcal{M}(S)$ to $\mathcal{A}(\delta_n)$. This is certainly a Lipschitz map, and restricted to $\mathcal{Q}^\omega(\boldsymbol{\delta})$ becomes projection to the first factor. Globally, it has the following useful property:

Lemma 4.4. *$\pi_{\boldsymbol{\delta}}$ is locally constant in the complement of $\mathcal{Q}^\omega(\boldsymbol{\delta})$.*

To understand why this holds, consider a point $\boldsymbol{x}\in\mathcal{M}^\omega(S)\setminus\mathcal{Q}^\omega(\boldsymbol{\delta})$. $\boldsymbol{x}$ is a sequence (or rather an equivalence class of sequences) of markings (x_n) whose distance from $Q(\delta_n)$ is growing linearly (with respect to the scale constants s_n). Using the distance formula (Theorem 3.3), one can show that $d(x_n, Q(\delta_n))$ is estimated by

$$\sum_{Y\cap\delta_n\neq\emptyset}\{d_{\mathcal{A}(Y)}(x_n,\delta_n)\}_K \tag{4}$$

which is therefore growing linearly as well. The domains in the sum (4) are partially ordered by their appearance in any hierarchy from δ_n to x_n (see Theorem 3.2). Now if y_n is given where $d_{\mathcal{M}(S)}(x_n,y_n)$ is a (sufficiently small) fraction of s_n, then the stability property in Theorem 3.2 implies that most of these domains are forced to appear in the hierarchy from δ_n to y_n as well.

In particular, the boundaries of these domains cross δ_n, and with this one can then show that $\pi_{\delta_n}(x_n)$ and $\pi_{\delta_n}(y_n)$ are a bounded distance apart. In the rescaling limit, we conclude that there is a neighborhood of $\boldsymbol{x}$ on which $\pi_{\boldsymbol{\delta}}$ is constant.

Once this lemma is established, we see immediately that the complement of $L_s = \{s\}\times T$ in $\mathcal{M}^\omega(S)$ is a union of three disjoint open sets: $\pi_{\boldsymbol{\delta}}^{-1}((-\infty,s))$, $\pi_{\boldsymbol{\delta}}^{-1}((s,\infty))$, and $\pi_{\boldsymbol{\delta}}^{-1}(s)\setminus L_s$. This gives the desired separation property.

In the general setting, the tricky question is what should take the place of the $\mathbb{R}$ factor in $\mathcal{Q}^\omega(\boldsymbol{\delta})$. One might for example take a sequence $\boldsymbol{W} = (W_n)$ and look at the product decomposition

$$\mathcal{Q}^\omega(\partial\boldsymbol{W}) \cong \mathcal{M}^\omega(\boldsymbol{W})\times\mathcal{M}^\omega(\boldsymbol{W}^c)$$

(where $\boldsymbol{W}^c = (W^c_n)$, and W^c_n denotes all the pieces except W_n in the partition $\sigma(\partial W_n)$.) There is a projection $\mathcal{M}^\omega(S) \to \mathcal{M}^\omega(\boldsymbol{W})$, but it does *not* have the required locally constant properties.

Instead we identify a certain decomposition of $\mathcal{M}^\omega(\boldsymbol{W})$ into $\mathbb{R}$-trees, and for each such tree F we consider the set

$$\mathcal{P}_F = F \times \mathcal{M}^\omega(\boldsymbol{W}^c) \subset \mathcal{Q}^\omega(\partial \boldsymbol{W}).$$

F now plays the role of the $\mathbb{R}$ factor in the example – there is a map $\mathcal{M}^\omega(S) \to F$ which is locally constant outside $\mathcal{P}_F$, and is projection to the first factor within $\mathcal{P}_F$. The sets $\{s\} \times \mathcal{M}^\omega(\boldsymbol{W}^c)$ are our separators.

This family of product regions and retractions gives a tractable structure for analyzing $\mathcal{M}^\omega(S)$. There is a reasonable hope that these techniques can lead to a good understanding of bilipschitz flats in $\mathcal{M}^\omega(S)$ and hence quasiflats in $\mathcal{M}(S)$, and more generally to a global understanding of the topology of the asymptotic cone. In particular we are hopeful, at the time of this writing, that this should yield another approach to showing the *quasi-isometric rigidity of* $\mathcal{MCG}(S)$. (This is the property that the group of quasi-isometries of $\mathcal{MCG}(S)$ is, up to bounded error, the same as the left-action of $\mathcal{MCG}(S)$ on itself). We note that Hamenstädt has announced a proof of quasi-isometric rigidity using her technique of analyzing train-track splitting sequences, which has a somewhat different flavor.

5. Hyperbolic geometry and ending laminations

The complex of curves, and the machinery of hierarchies, play an important role in the solution of Thurston's Ending Lamination Conjecture, a classification conjecture for the deformation space of a Kleinian group. We will give a brief description of this conjecture and its connection to the complex of curves; for more details see the expository article [102].

If G is a torsion-free, finitely generated group we may consider the set $AH(G)$ of (conjugacy classes of) discrete, faithful representations of G into $\mathrm{PSL}_2(\mathbb{C})$, the isometry group of hyperbolic 3-space $\mathbb{H}^3$. Equivalently $AH(G)$ is the set of marked hyperbolic 3-manifolds with fundamental group G. Let $N_\rho = \mathbb{H}^3/\rho(G)$ for $\rho \in AH(G)$.

Mostow/Prasad rigidity states that, if $AH(G)$ contains an element with finite volume, then in fact $AH(G)$ is a singleton – there is a unique hyperbolic manifold in the corresponding homotopy class. In the infinite volume case, there is a rich deformation theory, in which the dominant theme is that the geometry of a hyperbolic 3-manifold is controlled by its *ends*, which are in turn described by deformation theory of surfaces.

Let us sketch this structure briefly in rough historical order, focusing first on the case that G is $\pi_1(S)$, for a closed surface S (more about the general case below).

Fuchsian groups. If ρ takes values in $\mathrm{PSL}_2(\mathbb{R})$ then it leaves invariant the circle $\mathbb{R} \cup \infty$ in the Riemann sphere, and its convex hull the hyperbolic plane $\mathbb{H}^2 \subset \mathbb{H}^3$. The quotient is diffeomorphic to $S \times \mathbb{R}$, with a well-known warped product metric.

Kleinian groups. Poincaré perturbed Fuchsian groups in $\mathrm{PSL}_2(\mathbb{C})$ to obtain groups that are still topologically (in fact quasiconformally) conjugate to Fuchsian groups. In particular their quotients are still $S \times \mathbb{R}$. The circle becomes a Jordan curve, now defined as the *limit set* of ρ.

QC deformation theory. Bers and Ahlfors [2], [9] showed how to parameterize the entire quasiconformal deformation space of a Fuchsian group as $\mathcal{T}(S) \times \mathcal{T}(S)$. The idea is that the two sides of the limit set, modulo the group, give two Riemann surfaces, and conversely (the hard part) these Riemann surfaces can be arbitrarily prescribed using the Measureable Riemann Mapping Theorem.

Degenerations. Bers, Greenberg, Kra, Maskit, Marden and Sullivan [10], [12], [52, 83], [91], [89], [87], [88], [127] studied spaces of quasiconformal deformations, and particularly limits in which cusps are formed, and more exotic "degenerate groups" occur. The action on the Riemann sphere is no longer conjugate to the Fuchsian action (in particular the limit set is no longer a Jordan curve) but the quotient of $\mathbb{H}^3$ is still $S \times \mathbb{R}$ (this was initially known only in the case where the degeneration only involved cusp formation, i.e. the *geometrically finite* case).

Geometric tameness. Thurston proposed the notion of a geometrically tame group, whose convex core is swept out by pleated surfaces (see § 1). The convex core is the quotient of the hyperbolic convex hull of the limit set, and is also the smallest convex submanifold of N_ρ carrying the fundamental group. Geometrically finite manifolds have convex hulls of finite volume. He showed that if ρ is geometrically tame then N_ρ is still homeomorphic to $S \times \mathbb{R}$, and established tameness for certain limits of quasiconformal deformations. He defined the notion of *ending laminations* which, for the degenerate ends, take the place of the Riemann surfaces of Ahlfors–Bers.

Tameness for surface groups. Bonahon [19] established tameness for *all* surface groups $\rho \in AH(\pi_1(S))$, and more generally for all groups that have no free-product decomposition.

So far the description of $\rho \in AH(\pi_1(S))$ is that N_ρ is diffeomorphic to $S \times \mathbb{R}$, has two ends (corresponding to the two ends of $\mathbb{R}$), each of which has either a "Riemann surface at infinity" which describes its asymptotic structure, or an "ending lamination" which we describe now. (We are oversimplifying – when cusps are present we should cut along them and obtain a larger collection of ends).

In language amenable to our curve-complex discussion, we can define the ending laminations as follows: If one end of N_ρ is degenerate, it is filled by a sequence of pleated surfaces homotopic to $S \times \{0\}$. Each such surface contains simple curves of uniformly bounded length, and these must go to infinity in $\mathcal{C}(S)$. They converge to a point in $\partial\mathcal{C}(S) = \mathcal{EL}(S)$ (this is by no means obvious and is a major ingredient in the work of both Thurston and Bonahon).

Thurston's Ending Lamination Conjecture is then the following statement:

Theorem 5.1 (Brock–Canary–Minsky [99], [29], [28]). *A hyperbolic* 3*-manifold with finitely-generated fundamental group is uniquely determined by its topological type and its list of end invariants.*

This theorem gives a complete classification of the deformation space of a Kleinian group, and this has a number of consequences. For example, it is an ingredient of the proof of the Bers–Sullivan–Thurston Density Conjecture, which states that the geometrically finite (or equivalently structurally stable) representations are dense in the deformation space of any group. (This proof, whose outline was in place since the work of Thurston and Bonahon, required for its completion also the work of Kleineidam–Souto [80], Lecuire [84], Kim–Lecuire–Ohshika [71], and Namazi–Souto. An alternate treatment was given by Rees [114]. A completely different, rather surprising proof which works at least in the incompressible-boundary case was given by Bromberg [33] and Brock–Bromberg [27], before Theorem 5.1 was established.)

On the other hand, the Ending Lamination Conjecture does not give us a complete understanding of the geometry and topology of Kleinian deformation spaces. In fact this structure turns out to be considerably more complex than originally suspected, and perhaps too complex for any neat description. See e.g. Anderson–Canary–McCullough [3], Bromberg [34], Bromberg–Holt [36], Ito [73], McMullen [97].

A little about the proof. We begin by attempting to understand the *bounded-length curves* in N_ρ – that is to locate the homotopy classes in S of those curves whose geodesic representatives in N_ρ satisfy some fixed length bound. As above, many such curves exist because of the presence of pleated surfaces.

The structure of $\mathcal{C}(S)$ comes in naturally here because, as observed in the introduction, if $[vw]$ is an edge in $\mathcal{C}(S)$ then there is a pleated surface mapping v and w simultaneously to geodesics. Thus we study the *length function*

$$\ell_\rho \colon v \mapsto \ell(\rho(v))$$

on $\mathcal{C}(S)$, associating to each vertex the length of the corresponding closed geodesic in N_ρ. The ending laminations are accumulation points, on the boundary at infinity $\partial\mathcal{C}(S) \cong \mathcal{EL}(S)$, of the set $\{\ell_\rho \le L_0\}$ where L_0 is a fixed constant depending on the topology of S. The initial geometric result linking the geometry of N_ρ with that of $\mathcal{C}(S)$ is

Theorem 5.2 ([100]). *The sublevel set*

$$\{v \in \mathcal{C}(S) : \ell_\rho(v) \le L\}$$

is quasiconvex for $L \ge L_0$*.*

This gives some kind of very rough control, but as in the discussion of the mapping class group, we need to somehow refine this by looking at the link structure of the complex. This is accomplished in [99], where we show that, for any essential subsurface $W \subset S$, the projection

$$\pi_{\mathcal{A}(W)}(\{v : \ell_\rho(v) \le L\}) \tag{5}$$

is also quasiconvex. In the end, this refined control leads to a theorem about hierarchies. We extend the notion of a finite hierarchy to an infinite one whose "endpoints" are filling laminations, and establish a statement of this type:

Theorem 5.3 ([99]). *Given an end-invariant pair (ν_-, ν_+) in a surface S, there exists a hierarchy of geodesics H in $\mathcal{C}(S)$ connecting ν_- to ν_+, such that, if $\rho \in AH(\pi_1(S))$ has end invariants $\nu_\pm$ then*

1. *All the vertices that appear in H have uniformly bounded ρ-length.*
2. *All sufficiently ρ-short curves do appear in H, and the geometry of their Margulis tubes can be estimated from the data in H.*
3. *The order in which the Margulis tubes are arranged in N_ρ is consistent with the order of hierarchy paths of H.*

By "sufficiently" and "uniformly", we refer to bounds that depend only on the topological type of S, and not on ρ or $(\nu_\pm)$.

In order to prove this theorem, we study the map $\Pi\colon \mathcal{M}(S) \to \mathcal{M}(S)$ which maps a marking μ to a marking of minimal length in a pleated surface that maps $\operatorname{base}(\mu)$ to its geodesic representative in N_ρ. In fact this map plays an important role in the proof of Theorem 5.2 and its generalization (5). Its "shadow" in the complex of curves, for example, is a map from $\mathcal{C}(S)$ to $\{v \in \mathcal{C}(S) : \ell_\rho(v) \le L\}$ which we show is coarsely Lipschitz and coarsely the identity on its target – i.e. a coarse Lipschitz retraction. In a δ-hyperbolic space, the image of such a map is quasi-convex, and furthermore has the stability property that any geodesic with endpoints in $\{\ell_\rho(v) \le L\}$ must be mapped uniformly close to itself.

A generalization of this argument shows, for any μ in a hierarchy path associated to H, that

$$d_{\mathcal{A}(W)}(\mu, \Pi(\mu))$$

is uniformly bounded, for *any* essential W. The distance formula of Theorem 3.3 now shows that μ and $\Pi(\mu)$ are a uniform distance apart in $\mathcal{M}(S)$, and in particular it follows that the length of $\operatorname{base}(\mu)$ in N_ρ is uniformly bounded. These *a priori* bounds are the main step, and make the proof of the rest of the theorem possible.

Theorem 5.3 enables us to build a *combinatorial model* M_ν for the geometry of N_ρ, which depends only on the end invariants themselves, and show (in [29]) that M_ν and N_ρ are bilipschitz homeomorphic. Hence two N_ρ's with the same end invariants are bilipschitz equivalent to each other, and Sullivan's rigidity theorem [126] then implies that they are isometric.

The structure of the model, very roughly, is this: M_ν, which we identify topologically with $S \times \mathbb{R}$, contains a union $\mathcal{U}$ of level solid tori, i.e. regular neighborhoods of curves of the form $\gamma \times \{t\}$. Each of these solid tori is given the geometry of a Margulis tube. The complement of $\mathcal{U}$ is broken up into "blocks", each of which falls into a finite number of topological types $F \times [-1, 1]$ and a compact set of geometric shapes. The tubes and the boundaries of the block surfaces F all correspond to vertices in the hierarchy. (We are oversimplifying a bit by not describing the cases where there are cusps, or where the convex core is not the whole manifold). See [101] for a detailed exposition of this construction when $S = S_{0,5}$.

The general case. Most of the geometry discussed above applies to any π_1-injectively immersed surface in a hyperbolic 3-manifold, by considering the appropriate cover. In particular, a general hyperbolic N with finitely-generated fundamental group has a *compact core* $K \subset N$ (Scott [121], McCullough [96]) whose inclusion is a homotopy-equivalence, and when ∂K is incompressible we can apply the same technique to understand the *ends* of N, i.e. the components of $N \setminus K$.

When K has compressible boundary the situation is more delicate, but a collection of, by now, well-known techniques, together with the resolution by Agol [1] and Calegari–Gabai [39] of Marden's Tameness Conjecture, allow us again to extend the arguments.

The Tameness Conjecture (now theorem) states that N is homeomorphic to the interior of K, or equivalently K can be chosen so that $N \setminus K \cong \partial K \times \mathbb{R}$. (This was not obvious in the incompressible boundary case either, but was established by the work of Thurston and Bonahon [130], [19]).

Canary [40] showed that the end of a tame hyperbolic 3-manifold is isometric to an *incompressible* end of a 3-manifold with *pinched negative curvature*. This is done by means of lifting to an appropriately chosen branched cover. Working in this branched cover we can apply the techniques that worked in the incompressible boundary case to obtain again a bilipschitz model for the ends.

One disadvantage of this technique is that *uniformity* of the model no longer holds. Whereas in the case of $S \times \mathbb{R}$ the quality of our estimates depended only on the topological type of S, in general even the topology of the branched cover construction depends on the geometry, via the choice of branching locus. Outside some compact set our model does have uniform quality, but there is no uniform control on the size of this compact set, and hence no uniform overall model. This issue plays a role in the next section as well.

Our proof in the compressible-boundary case [28] has still, alas, not appeared. However, accounts (with alternative proofs) are available by Bowditch [23] and Rees [114]. Namazi [106] has also written down some of the tools needed to attack the compressible case.

6. Heegaard splittings

By Mostow's theorem, the geometry of a closed hyperbolic 3-manifold is uniquely determined by its topology. There does not, however, presently exist an effective general way of describing the geometry from topological data. Let us examine a specific version of this question, namely how the data of a Heegaard splitting give us geometric (and for that matter, topological) information.

The genus, of course, is the first interesting piece of data, and in particular Heegaard splittings of minimal genus (called the *Heegaard genus*) should be of interest. The genus clearly bounds from above the rank of the fundamental group, but no general bound exists in the opposite direction. Boileau–Zieschang [18] first gave examples with rank 2 and genus 3. Irreducible examples giving arbitrarily large difference between genus and rank have been found by Schultens–Weidmann [120] – these examples are all graph manifolds. There are no hyperbolic examples with rank smaller than genus, and one may speculate that they are equal, or more conservatively that rank gives some explicit upper bound on genus in the hyperbolic setting. Some partial results in this direction are due to Namazi-Souto [107] and Souto [125]. Bachman–Cooper–White [4] have shown that the Heegaard genus gives an upper bound for the injectivity radius at any point in a hyperbolic 3-manifold. Their methods involve "sweepouts" a la Pitts–Rubinstein, and should have further applications (see also Souto [125]). There are many related questions, e.g. behavior of tunnel number for knot complements, which we will not touch here. Instead we will concentrate on fixing a genus and obtaining geometric information from the complexity of the gluing map between the two handlebodies.

Let H_+ and H_- denote the pair of handlebodies of a Heegard splitting of a 3-manifold M, with $S = \partial H_+ = \partial H_-$. This defines two *meridian sets*, $\mathcal{D}_+$ and $\mathcal{D}_-$ in $\mathcal{C}(S)$, namely those simple curves that are the boundaries of essential disks in H_+ and H_- respectively. In turn, $\mathcal{D}_\pm$ determine the pair (M, S) uniquely. Note that $\mathcal{D}_\pm$ contain all possible Heegaard diagrams for the splitting.

Hempel distance. Hempel [68] pointed out that the quantity

$$d(M, S) = d_{\mathcal{C}(S)}(\mathcal{D}_+, \mathcal{D}_-),$$

called "Heegaard distance" or "Hempel distance" (where $d_{\mathcal{C}(S)}$ denotes minimal distance between the two sets), is a useful indication of complexity which generalizes the notions of weak and strong reducibility we mentioned in the introduction. Indeed it is not hard to see that the definitions from the introduction translate this way:

- $d(M, S) = 0$ iff the splitting is reducible;
- $d(M, S) \leq 1$ iff the splitting is weakly reducible;
- $d(M, S) \leq 2$ iff the splitting has the disjoint curve property;
- $d(M, S) \geq 2$ iff the splitting is strongly irreducible.

Moreover Hempel showed that there are splittings with arbitrarily large distance. As mentioned in the introduction (see also Thompson [129]), Hempel showed that if M is Seifert fibred or has an essential torus then $d(M, S) \leq 2$. One can also check that $d(M, S) \leq 2$ for all remaining geometrizable non-hyperbolic cases, and Hempel therefore conjectured that

$$d(M, S) \geq 3 \implies M \text{ is hyperbolic.}$$

Perelman's work on Thurston's geometrization conjecture, once accepted, implies that this conjecture holds.

Schleimer [119] proved that, if M is fixed, then there are at most finitely many (isotopy classes of) Heegaard surfaces with $d(M, S) > 2$. This is a sort of rigidity property for high-distance splittings, and suggests that they should convey topological information.

Hartshorn [62] obtained upper bounds on Hempel distance in the presence of incompressible surfaces, using ideas of Kobayashi [81]. Bachman–Schleimer [5], [6] obtain related results for knots and surface bundles. Scharlemann–Tomova [118] have obtained distance bounds from the interaction of pairs of splittings in a manifold.

However, one soon observes that $d(M, S)$ by itself is far from being an accurate measure of complexity. For example, for fixed g one can find hyperbolic manifolds, of arbitrarily high volume, which have a (minimal) genus g splitting of distance 1 (see Souto [123]).

One should really look at a more refined comparison of $\mathcal{D}_+$ and $\mathcal{D}_-$, much as in §5 we looked at hierarchies connecting end invariants rather than plain geodesics in $\mathcal{C}(S)$ connecting them.

If $\mathcal{P}_+$ is the set of pants decompositions of S composed of meridian curves in H_+ (and similarly $\mathcal{P}_-$) we could for example consider the combinatorial distance $d(\mathcal{P}_+, \mathcal{P}_-)$, by which we mean the distance in the "graph of pants decompositions" as in [26]. This distance, by a triangulation argument, gives an upper bound for the hyperbolic volume. Brock–Souto [31] have formulated a slightly different combinatorial distance (enlarging $\mathcal{P}_\pm$ to include more pants decompositions) which gives both upper and lower bounds.

Geometric models for handlebodies. One is naturally led from Heegaard splittings into the related problem of describing hyperbolic structures on a *single* handlebody. If one were to have a recipe for handlebodies one could attempt to glue them together and obtain models for Heegaard splittings. Namazi carried this out in [106] for a special family of splittings.

A partial answer was provided by the Ending Lamination construction, namely a geometric model for the end of the manifold. As mentioned above, this model was lacking in uniformity. In other words the work of [28] partitions a hyperbolic handlebody N into a compact handlebody K and an end E homeomorphic to $\partial K \times (0, \infty)$.

We provide a bilipschitz model for E of uniform quality, but give no information at all about the geometry of K.

What sort of geometric pictures do we expect to find? Let us describe a motivating list of examples, drawn from work of Namazi and Brock–Souto. When we say a subset of a manifold has *bounded geometry* here, we mean that, within the family of examples in question, it is drawn from a compact set of possibilities.

Capped-off products. One can construct hyperbolic handlebodies H for which there is a decomposition $H = E \cup K$, with K a compact handlebody with bounded geometry and E bilipschitz-equivalent to an end of an $S \times \mathbb{R}$. Indeed, Namazi [106] exhibited a family of such manifolds whose end invariants satisfy a *bounded combinatorics* condition, and which furthermore satisfy a lower bound on injectivity radius.

Small 1*-handles.* There are hyperbolic handlebodies H which admit a decomposition $H = E \cup B \cup K_1 \cup K_2$, where K_1 and K_2 are handlebodies of lower genus, each K_i is bilipschitz equivalent to the convex hull of a "capped off product" as above, B is a 1-handle of bounded geometry joining K_1 to K_2, and E is bilipschitz equivalent to an end of an $S \times \mathbb{R}$, and is attached along its bottom boundary to the handlebody $K_1 \cup B \cup K_2$.

Brock–Souto show that this (with variations in the number and arrangement of the pieces and 1-handles) is the structure of a general hyperbolic handlebody with a uniform lower bound on injectivity radius

Incompressible I-bundles. After representing a (closed) handlebody as $F \times [-1, 1]$ where F is a compact surface with boundary, let γ be a copy of $\partial F \times \{0\}$ pushed slightly into the interior H. One can find a hyperbolic structure on $H \setminus \gamma$ which makes the components of γ into cusps, and then after a geometric step known as "Dehn filling" obtain a hyperbolic structure on H where γ is extremely short. This can be done so that the geometry of H, outside the Margulis tubes of γ, is essentially that of a surface group with cusps based on F, and hence is described by a model manifold of the type used for the Ending Lamination Conjecture.

Tubes with 1*-handles* One can start with a finite number of Margulis tubes (solid tori), and join them with 1-handles of bounded geometry.

These examples can be combined – for example the tubes with 1-handles can serve as a core to which a standard product end is added, and the product structure example can be attached to something else using a 1-handle, or a peripheral annulus. This can be done geometrically via the Klein–Maskit combination theorem [90], the Ahlfors–Bers deformation theory [11], and variations on Thurston's Dehn filling construction (see Bonahon–Otal [21], Bromberg [32], [35], Comar [45], Hodgson–Kerckhoff [70]).

We expect that any handlebody will have a decomposition into pieces of this type. An interesting challenge is to *predict* from the topological data what pieces actually occur. In other words, starting with a pair $(\mathcal{D}, \mu)$ where $\mathcal{D}$ is a meridian set and μ an end invariant on S, we would like to give such a decomposition, whose structure and the shape of the pieces (e.g. boundary structure of the Margulis tubes, and end invariants of the I-bundle pieces) can be read off from the input data.

Geometry of $\mathcal{D}$. A first step toward the goal of providing uniform geometric models is to understand the structure of $\mathcal{D}$ and how it embeds in $\mathcal{C}(S)$. With Masur in [92] we proved

Theorem 6.1. *$\mathcal{D}$ is a quasi-convex subset of $\mathcal{C}(S)$.*

This is analogous to Theorem 5.2 on quasiconvexity of the bounded-curve set in the incompressible setting.

The next step might be to analyze projections of $\mathcal{D}$ into subsurfaces. Very roughly, one expects surfaces $W \subset S$ in which $d_W(\mathcal{D}, \mu)$ is very large to play a role in the hyperbolic geometry associated to the end invariant μ – much as subsurfaces where $d_W(\nu_+, \nu_+)$ is large played a role in the geometry of surface groups with end invariants ν_+ and ν_-. For the purpose of understanding when this happens, those W for which $\operatorname{diam}_W(\mathcal{D})$ is bounded are somewhat easier to analyze. Masur–Schleimer have studied the structure of $\mathcal{D}$ quite extensively in [93], and have shown in particular that

Theorem 6.2 (Masur–Schleimer). *Let S be the boundary of a handlebody H. If W is an essential subsurface of S then* $\operatorname{diam}_W(\mathcal{D})$ *is bounded by a number depending on the genus of S, unless*

1. *there is a meridian in the complement of W,*
2. *there is a meridian in W but not in its complement,*
3. *there is an essential I-bundle B in H with W a component of its horizontal boundary, and at least one vertical annulus of B lying in S.*

In cases 1 *and* 3*, $\pi_W(\mathcal{D})$ is all of $\mathcal{A}(W)$ up to bounded gaps. In case* 2*, $\pi_W(\mathcal{D})$ is within a bounded neighborhood of $\mathcal{D} \cap \mathcal{C}(W)$.*

This is part of a larger analysis in which, in particular, they show that the Hempel distance of a splitting can be estimated algorithmically.

There is an interplay between this classification of subsurfaces and the geometric examples we listed above. To understand case 3, for example, note that in a product $F \times [-1, 1]$, where F is a surface with boundary, there are essential disks of the form $a \times [-1, 1]$ where a is any essential arc in F. Hence (supposing $F \times [-1, 1]$ to be embedded in the handlebody H as in part (3)) we see that $\pi_{F\times\{1\}}(\mathcal{D})$ gives all arcs of $\mathcal{A}(F)$. This corresponds to the "incompressible I-bundle" case in the list of geometric examples, and we expect in this case that the projections of μ to $F \times \{\pm 1\}$ should act as end invariants for this incompressible surface group within H.

With these results as a starting point, one can hope that there is a model based, in a way analogous to hierarchies, on the shortest path from a marking μ to the meridian set $\mathcal{D}$. The parts of the path far from $\mathcal{D}$ should give a portion of the model analogous to the compressible products in the example list, whereas near $\mathcal{D}$ the construction would need to give rise to a system of 1-handles and incompressible I-bundles. A successful version of this would give us uniform bilipschitz models for handlebodies, and via some deformation and gluing could enable us to build models for closed hyperbolic 3-manifolds as well. One would also like to apply our understanding of $\mathcal{D}$ directly to the combinatorics of a Heegaard splitting, that is to the relative positioning of a pair $(\mathcal{D}_+, \mathcal{D}_-)$.

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