$\mathbb{A}^1$-algebraic topology

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Abstract. We present some recent results in $\mathbb{A}^1$-*algebraic topology*, which means both in $\mathbb{A}^1$-homotopy theory of schemes and its relationship with algebraic geometry. This refers to the classical relationship between homotopy theory and (differential) topology. We explain several examples of "motivic" versions of classical results: the theory of the Brouwer degree, the classification of $\mathbb{A}^1$-coverings through the $\mathbb{A}^1$-fundamental group, the Hurewicz Theorem and the $\mathbb{A}^1$-homotopy of algebraic spheres, and the $\mathbb{A}^1$-homotopy classification of vector bundles. We also give some applications and perspectives.

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1. The Brouwer degree

Let $n \geq 1$ be an integer and let X be a pointed topological space. We shall denote by $\pi_n(X)$ the n-th homotopy group of X. A basic fact in homotopy theory is:

Theorem 1.1. *Let $n \geq 1$, $d \geq 1$ be integers and denote by S^n the n-dimensional sphere.*

1) *If $d < n$ then $\pi_d(S^n) = 0$;*

2) *If $d = n$ then $\pi_n(S^n) = \mathbb{Z}$.*

A classical proof uses the Hurewicz Theorem and the computation of the integral singular homology of the sphere. Half of this paper is devoted to explain the analogue of these results in $\mathbb{A}^1$-homotopy theory [54], [38].

For our purpose we also recall a more geometric proof of 2) inspired by the definition of Brouwer's degree. Any continuous map $S^n \to S^n$ is homotopic to a $\mathcal{C}^\infty$-differentiable map $f\colon S^n \to S^n$. By Sard's theorem, f has at least one regular value $x \in S^n$, so that $f^{-1}(x)$ is a finite set of points in S^n and for each $y \in f^{-1}(x)$, the differential $df_y\colon T_y(S^n) \to T_x(S^n)$ of f at y is an isomorphism. The "sign" $\varepsilon_y(f)$ at y is $+1$ if df_y preserves the orientation and -1 else. The integer $\delta(f) := \sum_{y \mapsto x} \varepsilon_y(f)$ is the *Brouwer degree* of f and only depends on the homotopy class of f.

Now choose a small enough open n-ball $\mathcal{B}$ around x such that $f^{-1}(\mathcal{B})$ is a disjoint union of an open n-balls $\mathcal{B}_y$ around each y's. The quotient space $S^n/(S^n - \bigcup \mathcal{B}_y)$ is

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homeomorphic to the wedge of spheres $\vee_y S^n$ and the quotient map $S^n \to S^n/(S^n - \mathcal{B})$ is a homotopy equivalence. The induced commutative square

$$\begin{tikzcd}
S^n \arrow[rr, "f"] \arrow[d] & & S^n \arrow[d, "\wr"] \\
\vee_y S^n = S^n/(S^n - \bigcup \mathcal{B}_y) \arrow[rr, "\vee_y f_y"'] & & S^n/(S^n - \mathcal{B})
\end{tikzcd} \tag{1.1}$$

expresses the homotopy class of f as the sum of the homotopy classes of the f_y's, each of which being the one point compactification of the differential map df_y. This proves that the degree homomorphism $\pi_n(S^n) \to \mathbb{Z}$ is injective, thus an isomorphism.

We illustrate the algebraic situation by a simple close example. Let k be a field, let $f \in k(T)$ be a rational fraction and denote still by $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ the k-morphism from the projective line to itself corresponding to f. Assume, for simplicity, that f admits a regular value x in the following strong sense (which is not the generic one): x is a rational point in $\mathbb{A}^1 \subset \mathbb{P}^1$ such that f is étale over x, such that the finite étale k-scheme $f^{-1}(x)$ consists of finitely many rational points $y \in \mathbb{A}^1$ (none being ∞), and that the differentials $\frac{df}{dt}(y)$ are each units α_y. Observe that $\mathbb{P}^1 - \{x\}$ is isomorphic to the affine line $\mathbb{A}^1$ and thus the quotient morphism $\mathbb{P}^1 \to \mathbb{P}^1/\mathbb{A}^1 := T$ a "weak $\mathbb{A}^1$-equivalence". The commutative diagram (in some category of spaces over k, see below)

$$\begin{tikzcd}
\mathbb{P}^1 \arrow[rr, "f"] \arrow[d] & & \mathbb{P}^1 \arrow[d, "\wr"] \\
\vee_y T = \mathbb{P}^1/(\mathbb{P}^1 - f^{-1}(x)) \arrow[rr, "\vee \hat{\alpha}_y"'] & & T
\end{tikzcd}$$

analogous to (1.1), also expresses f, up to $\mathbb{A}^1$-weak homotopy, as the sum of the classes of the morphisms $\hat{\alpha}_y\colon T \to T$ induced by the multiplication by α_y. The idea is that in algebraic geometry the analogue of the "sign" of a unit $u \in k^\times$, or the $\mathbb{A}^1$-homotopy class of $\hat{u}$, is its class in $k^\times/(k^\times)^2$. The set $k^\times/(k^\times)^2$ should also be considered as the set of *orientations* of the affine line over k. We observe that $\hat{u}$ is $\mathbb{A}^1$-equivalent to the "1-point compactification" of the multiplication by $u\colon \mathbb{P}^1 \to \mathbb{P}^1$, $[x, y] \mapsto [ux, y]$. If $u = v^2$, the latter is $[x, y] \mapsto [ux, y] = [vx, v^{-1}y]$ which is given by the action of the matrix $\left(\begin{smallmatrix} v & 0 \\ 0 & v^{-1} \end{smallmatrix}\right)$ of $\mathrm{SL}_2(k)$ and thus, being a product of elementary matrices, is $\mathbb{A}^1$-homotopic to the identity.

Using the same procedure as in topology, we have "expressed" the $\mathbb{A}^1$-homotopy class of f as a sum of units modulo the squares and the Brouwer degree of a morphism $\mathbb{P}^1 \to \mathbb{P}^1$ in the $\mathbb{A}^1$-homotopy category $\mathsf{H}(k)$ over k should have this flavor. Denote by $GW(k)$ the Grothendieck–Witt ring of non-degenerate symmetric bilinear forms over k, that is to say the group completion of the monoid – for the direct sum – of isomorphism classes of such forms over k, see [27]. It is a quotient of the free abelian group on units $k^\times$. We will find that the algebraic Brouwer degree over k takes it values in $GW(k)$ by constructing for $n \geq 2$ an isomorphism

$$\mathrm{Hom}_{\mathsf{H}(k)}((\mathbb{P}^1)^{\wedge n}, (\mathbb{P}^1)^{\wedge n}) \cong \mathrm{Hom}_{\mathsf{H}_\bullet(k)}((\mathbb{P}^1)^{\wedge n}, (\mathbb{P}^1)^{\wedge n}) \cong GW(k)$$

where $\mathsf{H}_\bullet(k)$ is the pointed $\mathbb{A}^1$-homotopy category over k and $\wedge$ denotes the smash-product [54], [38]. For $n = 1$ the epimorphism $\mathrm{Hom}_{\mathsf{H}_\bullet(k)}(\mathbb{P}^1, \mathbb{P}^1) \to GW(k)$ has a kernel isomorphic to the subgroup of squares $(k^\times)^2$.

The ring $GW(k)$ is actually the cartesian product of $\mathbb{Z}$ and $W(k)$ (the Witt ring of isomorphism classes of anisotropic forms) over $\mathbb{Z}/2$, fitting into the cartesian square

$$\begin{tikzcd}
GW(k) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\
W(k) \arrow[r] & \mathbb{Z}/2 .
\end{tikzcd}$$

The possibility of defining the Brouwer degree with values[1] in $GW(k)$ and the above cartesian square emphasizes one of our constant intuition in this paper and should be kept in mind: from the degree point of view, the (top horizontal) rank homomorphism corresponds to “taking care of the topology of the complex points” and the projection $GW(k) \to W(k)$ corresponds to “taking care of the topology of the real points”. Indeed, given a real embedding $k \to \mathbb{R}$, with associated signature $W(k) \to \mathbb{Z}$, the signature of the degree of f is the degree of the associated map $f(\mathbb{R})\colon \mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$. This idea of taking care of these two topological intuitions at the same time is essential in the present work.

We do not pretend to be exhaustive in such a short paper; we have mostly emphasized the progress in unstable $\mathbb{A}^1$-homotopy theory and we will almost not address stable $\mathbb{A}^1$-homotopy theory.

Notations. We fix a base field k of any characteristic; Sm_k will denote the category of smooth quasi-projective k-schemes. Given a presheaf of sets on Sm_k, that is to say a functor $F\colon (\mathsf{Sm}_k)^{\mathrm{op}} \to \mathsf{Sets}$, and an essentially smooth k-algebra A, which means that A is the filtering union of its sub-k-algebras A_α which are smooth and finite type over k, we set $F(A) := \mathrm{colimit}_\alpha F(\mathrm{Spec}(A_\alpha))$. For instance, for each point $x \in X \in \mathsf{Sm}_k$ the local ring $\mathcal{O}_{X,x}$ of X at x as well as its henselization $\mathcal{O}^h_{X,x}$ are essentially smooth k-algebras.

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I thank J. Lannes for his influence and interest on my first proof of the Milnor conjecture on quadratic forms in [29], relying on Voevodsky’s results and on the use of the Adams spectral sequence based on mod. 2 motivic cohomology. Since then I considerably simplified the topological argument in [33].

[1] Barge and Lannes have defined and studied a related degree from the set of naive $\mathbb{A}^1$-homotopy classes of k-morphisms $\mathbb{P}^1 \to \mathbb{P}^1$ to $GW(k)$, unpublished.

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2. A quick recollection on $\mathbb{A}^1$-homotopy

A convenient category of spaces. We will always consider that Sm_k is endowed with the Nisnevich topology [40], [38]. We simply recall the following characterization for a presheaf of sets on Sm_k to be a sheaf in this topology.

Proposition 2.1 ([38]). *A functor $F\colon (\mathsf{Sm}_k)^{\mathrm{op}} \to \mathsf{Sets}$ is a sheaf in the Nisnevich topology if and only if for any cartesian square in Sm_k of the form*

$$\begin{array}{ccc} W & \subset & V \\ \downarrow & & \downarrow \\ U & \subset & X \end{array} \tag{2.1}$$

where U is an open subscheme in X, the morphism $f\colon V \to X$ is étale and the induced morphism $(f^{-1}(X-U))_{\mathrm{red}} \to (X-U)_{\mathrm{red}}$ is an isomorphism, the map

$$F(X) \to F(U) \times_{F(W)} F(V)$$

is a bijection.

Squares like (2.1) are call *distinguished squares*. We denote by $\Delta^{\mathrm{op}}\mathsf{Shv}_k$ the category of simplicial sheaves of sets over Sm_k (in the Nisnevich topology); these objects will be just called “spaces” (this is slightly different from [54] where “space” only means a sheaf of sets, with no simplicial structure). This category contains the category Sm_k as the full subcategory of representable sheaves.

$\mathbb{A}^1$-weak equivalence and $\mathbb{A}^1$-homotopy category. Recall that a *simplicial weak equivalence* is a morphism of spaces $f\colon \mathcal{X} \to \mathcal{Y}$ such that each of its stalks

$$\mathcal{X}(\mathcal{O}^h_{X,x}) \to \mathcal{Y}(\mathcal{O}^h_{X,x})$$

at $x \in X \in \mathsf{Sm}_k$ is a weak equivalence of simplicial sets. Inverting these morphisms in $\Delta^{\mathrm{op}}\mathsf{Shv}_k$ yields the classical *simplicial homotopy category of sheaves* [10], [21]. The notion of $\mathbb{A}^1$-*weak equivalence* is generated in some natural way by that of simplicial weak equivalences and the projections $\mathcal{X} \times \mathbb{A}^1 \to \mathcal{X}$ for any space $\mathcal{X}$. Inverting the class of $\mathbb{A}^1$-weak equivalence yields now the $\mathbb{A}^1$-homotopy category $\mathsf{H}(k)$ [54], [38]. We denote by $\mathsf{H}_\bullet(k)$ the $\mathbb{A}^1$-homotopy category of pointed spaces.

The smash product with the simplicial circle S^1 induces the simplicial suspension functor $\Sigma\colon \mathsf{H}_\bullet(k) \to \mathsf{H}_\bullet(k)$, $\mathcal{X} \mapsto \Sigma(\mathcal{X})$. For a pointed morphism $f\colon \mathcal{X} \to \mathcal{Y}$ we may define the $\mathbb{A}^1$-homotopy fiber $\Gamma(f)$ together with an $\mathbb{A}^1$-fibration sequence $\Gamma(f) \to \mathcal{X} \to \mathcal{Y}$, which moreover induces for any other pointed space $\mathcal{Z}$ a long homotopy exact sequence (of pointed sets, groups, abelian groups as usual)

$$\begin{aligned}\cdots \to \mathrm{Hom}_{\mathsf{H}_\bullet(k)}(\Sigma(\mathcal{Z}), \mathcal{X}) \to \mathrm{Hom}_{\mathsf{H}_\bullet(k)}(\Sigma(\mathcal{Z}), \mathcal{Y}) \to \mathrm{Hom}_{\mathsf{H}_\bullet(k)}(\mathcal{Z}, \Gamma(f))\\ \to \mathrm{Hom}_{\mathsf{H}_\bullet(k)}(\mathcal{Z}, \mathcal{X}) \to \mathrm{Hom}_{\mathsf{H}_\bullet(k)}(\mathcal{Z}, \mathcal{Y}).\end{aligned}$$

In a dual way a distinguished square like (2.1) above is $\mathbb{A}^1$-homotopy cocartesian and induces corresponding Mayer–Vietoris type long exact sequences by mapping its vertices to $\mathcal{Z}$.

The geometric ideas on the Brouwer degree recalled in the introduction lead in general (for $d > n$) to the interpretation, due to Pontryagin, of the stable homotopy groups of spheres in terms of *parallelized cobordism groups*, and even more generally to the *Thom–Pontryagin construction* used by Thom to compute most of the cobordism rings. Recall that given a closed embedding $i\colon Z \hookrightarrow X$ between differentiable manifolds (with Z compact for simplicity) and a *tubular neighborhood* $Z \subset U \subset X$ of Z in X there is a pointed continuous map (indeed homeomorphism)

$$X/(X - U) \to Th(\nu_i) \tag{2.2}$$

to the Thom space (the one point compactification of the total space $E(\nu_i) \cong U$ of the normal bundle ν_i) which is independent up to pointed homotopy of the choices of the tubular neighborhood.

The choice of the topology on Sm_k (see [38]) was very much inspired by the this Thom–Pontryagin construction and the definition of the $\mathbb{A}^1$-homotopy category of smooth schemes over a base in [54], [38] allows to construct, for any closed immersion $i\colon Z \to X$ between smooth k-schemes, a pointed $\mathbb{A}^1$-weak equivalence $X/(X-Z) \to Th(\nu_i)$ [38], although no tubular neighborhood is available in general in algebraic geometry. In that case we get an $\mathbb{A}^1$-cofibration sequence

$$(X - Z) \to X \to Th(\nu_i).$$

Let $\mathcal{X}$ be a space. We let $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ denote the associated sheaf (of sets) on Sm_k to the presheaf $U \mapsto \mathrm{Hom}_{\mathsf{H}(k)}(U, \mathcal{X})$. If moreover $\mathcal{X}$ is pointed, and $n \geq 1$ we denote by $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ the sheaf on Sm_k associated to the presheaf $U \mapsto \mathrm{Hom}_{\mathsf{H}_\bullet(k)}(\Sigma^n(U_+), \mathcal{X})$ (where U_+ means U together with a base point added outside), a sheaf of groups for $n = 1$, of abelian groups for $n \geq 2$.

It is also very useful for the intuition to recall from [38] the existence of the topological realization functors. When $\rho\colon k \to \mathbb{C}$ (resp. $k \to \mathbb{R}$) is a complex (resp. real) embedding there is a canonical functor $\mathsf{H}(k) \to \mathsf{H}$ to the usual homotopy category of C.W.-complexes, induced by sending $X \in \mathsf{Sm}_k$ to the set of complex points $X(\mathbb{C})$ (resp. real points $X(\mathbb{R})$) with its classical topology.

3. $\mathbb{A}^1$-homotopy and $\mathbb{A}^1$-homology: the basic theorems

We recall that everywhere in this paper the topology to be understood is the Nisnevich topology.

Strictly $\mathbb{A}^1$-invariant sheaves

Definition 3.1. 1) A presheaf of sets M on Sm_k is said to be $\mathbb{A}^1$-*invariant* if for any $X \in \mathsf{Sm}_k$, the map $M(X) \to M(X \times \mathbb{A}^1)$ induced by the projection $X \times \mathbb{A}^1 \to X$, is a bijection.

2) A sheaf of groups M is said to be *strongly* $\mathbb{A}^1$-*invariant* if for any $X \in \mathsf{Sm}_k$ and any $i \in \{0, 1\}$ the map $H^i(X; M) \to H^i(X \times \mathbb{A}^1; M)$ induced by the projection $X \times \mathbb{A}^1$, is a bijection.

3) A sheaf of abelian groups M is said to be *strictly* $\mathbb{A}^1$-*invariant* if for any $X \in \mathsf{Sm}_k$ and any $i \in \mathbb{N}$ the map $H^i(X; M) \to H^i(X \times \mathbb{A}^1; M)$ induced by the projection $X \times \mathbb{A}^1$ is a bijection.

These notions, except 2), appear in Voevodsky's study of cohomological properties of presheaves with transfers [55] and were extensively studied in [34] over a general base, though very few is known except when the base is a field. Hopefully, given a sheaf of abelian groups the *a priori* different properties 2) and 3) coincide.

Theorem 3.2 ([36]). *A sheaf of abelian groups which is strongly $\mathbb{A}^1$-invariant is strictly $\mathbb{A}^1$-invariant.*

This result can be used to simplify some of the proofs of [55]. Let us denote by Ab_k the abelian category of sheaves of abelian groups on Sm_k. Another easy application is that the full sub-category $\mathsf{Ab}_k^{\mathbb{A}^1} \subset \mathsf{Ab}_k$, consisting of strictly $\mathbb{A}^1$-invariant sheaves, is an abelian category for which the inclusion functor is exact. From Theorem 3.3 below, these strictly $\mathbb{A}^1$-invariant sheaves and their cohomology play in $\mathbb{A}^1$-algebraic topology the role played in classical algebraic topology by the abelian groups and the singular cohomology with coefficients in those.

The constant sheaf $\mathbb{Z}$, the sheaf represented by an abelian variety over k are examples of strictly $\mathbb{A}^1$-invariant sheaves, in fact the higher cohomology groups, $H^i_{Nis}(X; -)$, $i > 0$, for these sheaves automatically vanish. Another well known example is the multiplicative group $\mathbb{G}_m = \mathbb{A}^1 - \{0\}$. More elaborated examples were produced by Voevodsky over a perfect field: for each $\mathbb{A}^1$-homotopy invariant presheaf with transfers F its associated sheaf F_{Nis} a strictly $\mathbb{A}^1$-invariant sheaf [55]. In particular if F itself is an $\mathbb{A}^1$-homotopy invariant sheaf with transfers, it is strictly $\mathbb{A}^1$-invariant. By [12] these sheaves are very closely related to Rost's cycle modules [46], which also produce strictly $\mathbb{A}^1$-invariant sheaves, like the unramified Milnor K-theory sheaves introduced in [19]. There are other types of strictly $\mathbb{A}^1$-invariant sheaves given for instance by the unramified Witt groups $\underline{\mathbf{W}}$ as constructed in [42], or [36], as well as their subsheaves of unramified power of the fundamental ideal $\underline{\mathbf{I}}^n$ used in [33].

$\mathbb{A}^1$-homotopy sheaves

Theorem 3.3 ([36]). *Let $\mathcal{X}$ be a pointed space. Then the sheaf $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ is strongly $\mathbb{A}^1$-invariant, and the sheaves $\pi_n^{\mathbb{A}^1}$, for $n \geq 2$, are strictly $\mathbb{A}^1$-invariant.*

Curiously enough, we are unable to prove that the sheaf $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is $\mathbb{A}^1$-invariant, though it is true in all the cases we can compute.

Remark 3.4. One of the main tool used in the proof of the Theorem 3.3 is the presentation Lemma of Gabber [14] as formalized in [11]. Then a "non-abelian" variant of [11] and ideas from [46] lead to the result. In fact one can give a quite concrete description of a sheaf of groups which is strongly $\mathbb{A}^1$-invariant [36].

A pointed space $\mathcal{X}$ such that the sheaves $\pi_i^{\mathbb{A}^1}(\mathcal{X})$ vanish for $i \leq n$ will be called n-$\mathbb{A}^1$-connected. In case $n = 0$ we simply say $\mathbb{A}^1$-connected.

Corollary 3.5 (Unstable $\mathbb{A}^1$-connectivity theorem). *Let $\mathcal{X}$ be a pointed space and n be an integer ≥ 0 such that $\mathcal{X}$ is* simplicially *n-connected. Then it is n-$\mathbb{A}^1$-connected.*

This result was only known in the case $n = 0$ in [38], over a general base. As a consequence, the simplicial suspension of an $(n-1)$-$\mathbb{A}^1$-connected pointed space is n-$\mathbb{A}^1$-connected.

The main example of a simplicially n-connected space is the $(n+1)$-th simplicial suspension of a pointed space.

For n and i two natural numbers we set $S^n(i) = (S^1)^{\wedge(n)} \wedge (\mathbb{G}_m)^{\wedge i}$ where $\wedge$ denotes the smash-product. Observe that these are actually mapped to spheres (up to homotopy) through any topological realization functors (real or complex). Note also the following isomorphisms in $\mathsf{H}_\bullet(k)$: $\mathbb{A}^n - \{0\} \cong S^{(n-1)}(n)$ and $(\mathbb{P}^1)^{\wedge n} \cong S^1 \wedge (\mathbb{A}^n - \{0\}) \cong S^n(n)$.

From the previous Corollary $S^n(i)$ is $(n-1)$-$\mathbb{A}^1$-connected. Actually we will see below that it is exactly $(n-1)$-$\mathbb{A}^1$-connected, as $\pi_n^{\mathbb{A}^1}(S^n(i))$ is always non trivial. The $\mathbb{A}^1$-connectivity corresponds to the connectivity of the space of real points.

$\mathbb{A}^1$-fundamental group and universal $\mathbb{A}^1$-covering. An $\mathbb{A}^1$-trivial cofibration $\mathcal{A} \to \mathcal{B}$ is a monomorphism between spaces which is also an $\mathbb{A}^1$-weak equivalence. The following definition is the obvious analogue of the definition of a covering in topology:

Definition 3.6. An $\mathbb{A}^1$-covering $\mathcal{Y} \to \mathcal{X}$ is a morphism of spaces which has the unique right lifting property with respect to $\mathbb{A}^1$-trivial cofibrations. This means that given any commutative square of spaces

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{B} & \longrightarrow & \mathcal{X} \end{array}$$

in which $\mathcal{A} \to \mathcal{B}$ is an $\mathbb{A}^1$-trivial cofibration, there exists one and exactly one morphism $\mathcal{B} \to \mathcal{Y}$ which makes the whole diagram commutative.

Example 3.7. 1) Any finite étale covering $Y \to X$ between smooth k-varieties, in characteristic 0, is an $\mathbb{A}^1$-covering. Any Galois étale covering $Y \to X$ with Galois group of order prime to the characteristic of k is an $\mathbb{A}^1$-covering.

2) Any $\mathbb{G}_m$-torsor $\mathcal{Y} \to \mathcal{X}$ is an $\mathbb{A}^1$-covering. Remember to think about the real points! A $\mathbb{G}_m$-torsor gives (up to homotopy) a $\mathbb{Z}/2$-covering.

Theorem 3.8. *Any pointed $\mathbb{A}^1$-connected space $\mathcal{X}$ admits a universal pointed $\mathbb{A}^1$-covering $\tilde{\mathcal{X}} \to \mathcal{X}$ in the category of pointed coverings of $\mathcal{X}$. The fiber of this universal $\mathbb{A}^1$-covering at the base point is isomorphic to $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ and $\tilde{\mathcal{X}} \to \mathcal{X}$ is* (*up to canonical isomorphism*) *the unique pointed $\mathbb{A}^1$-covering with $\tilde{\mathcal{X}}$ being* 1-$\mathbb{A}^1$-*connected.*

Remark 3.9. A pointed $\mathbb{A}^1$-connected smooth k-scheme (X, x) admits no non-trivial étale pointed covering. Thus the $\pi_1^{\mathbb{A}^1}$ is in some sense orthogonal to the étale one and gives a more combinatorial information, as shown by the example of the $\mathbb{P}^n$'s below. On the other hand the pointed étale coverings always come from the $\pi_0^{\mathbb{A}^1}$: for instance an abelian variety X is discreet, in the sense that $\pi_0^{\mathbb{A}^1}(X) = X$, and have huge étale π_1. We did not try to further study the $\mathbb{A}^1$-fundamental groupoid which cares about both aspects, the combinatorial and the étale.

Lemma 3.10. *Let $n \geq 2$. The canonical $\mathbb{G}_m$-torsor*

$$(\mathbb{A}^{n+1} - \{0\}) \to \mathbb{P}^n$$

is the universal covering of $\mathbb{P}^n$. As a consequence the morphism $\pi_1^{\mathbb{A}^1}(\mathbb{P}^n) \to \mathbb{G}_m$ is an isomorphism.

Indeed, $\mathbb{A}^{n+1} - \{0\}$ is 1-$\mathbb{A}^1$-connected. For $n = 1$ the problem is that $\mathbb{A}^2 - \{0\}$ is no longer 1-$\mathbb{A}^1$-connected. See the next section for more information.

$\mathbb{A}^1$-derived category, $\mathbb{A}^1$-homology and Hurewicz Theorem. Let us denote by $\mathbb{Z}(\mathcal{X})$ the free abelian sheaf generated by a space $\mathcal{X}$ and by $C_*(\mathcal{X})$ its the associated chain complex; if moreover $\mathcal{X}$ is pointed, let us denote by $\mathbb{Z}_\bullet(\mathcal{X}) = \mathbb{Z}(\mathcal{X})/\mathbb{Z}$ and $\tilde{C}_*(\mathcal{X}) = C_*(\mathcal{X})/\mathbb{Z}$ the reduced versions obtained by collapsing the base point to 0.

We may perform in the derived category of chain complexes in Ab_k exactly the same process as for spaces and define the class of $\mathbb{A}^1$-weak equivalences, rather $\mathbb{A}^1$-quasi isomorphisms; these are generated by quasi-isomorphisms and collapsing $\mathbb{Z}_\bullet(\mathbb{A}^1)$ to 0. Formally inverting these morphisms yields the $\mathbb{A}^1$-*derived category* $\mathsf{D}_{\mathbb{A}^1}(k)$ of k [34]. The functor $\mathcal{X} \mapsto C_*(\mathcal{X})$ obviously induces a functor $\mathsf{H}(k) \to \mathsf{D}_{\mathbb{A}^1}(k)$ which admits a right adjoint given by the usual Eilenberg–MacLane functor $K : \mathsf{D}_{\mathbb{A}^1}(k) \to \mathsf{H}(k)$.

As for spaces, one may define $\mathbb{A}^1$-homology sheaves of a chain complex C_*. An abelian version of Theorem 3.3 implies that for any complex C_* these $\mathbb{A}^1$-homology sheaves are strictly $\mathbb{A}^1$-invariant [36], [34].

Definition 3.11. For a space $\mathcal{X}$ and for each integer $n \in \mathbb{Z}$, we let $\mathbb{H}_n^{\mathbb{A}^1}(X)$ denote the n-th $\mathbb{A}^1$-homology sheaf of $C_*(\mathcal{X})$ and call $\mathbb{H}_*^{\mathbb{A}^1}(\mathcal{X})$ the $\mathbb{A}^1$-homology of $\mathcal{X}$ (with integral coefficients). In case $\mathcal{X}$ is pointed, we let $\tilde{\mathbb{H}}_*^{\mathbb{A}^1}(\mathcal{X})$ denote the reduced version obtained by collapsing the base point to 0.

Observe that these $\mathbb{A}^1$-homology sheaves are strictly $\mathbb{A}^1$-invariant and that $\mathbb{H}_i^{\mathbb{A}^1}(\mathcal{X}) = 0$ for $i < 0$ by the abelian analogue of Corollary 3.5. As a consequence for a space $\mathcal{X}$ the sheaf $\mathbb{H}_0^{\mathbb{A}^1}(\mathcal{X})$ is the *free strictly* $\mathbb{A}^1$*-invariant sheaf* generated by $\mathcal{X}$. These sheaves play a fundamental role in $\mathbb{A}^1$-algebraic topology. For instance we have suspension isomorphisms $\tilde{\mathbb{H}}_*^{\mathbb{A}^1}(S^n(i)) \cong \tilde{\mathbb{H}}_{*-n}^{\mathbb{A}^1}((\mathbb{G}_m)^{\wedge i})$ for our spheres $S^n(i)$. In particular the first *a priori* non trivial sheaf is $\tilde{\mathbb{H}}_n^{\mathbb{A}^1}(S^n(i)) \cong \tilde{\mathbb{H}}_0^{\mathbb{A}^1}((\mathbb{G}_m)^{\wedge i})$. We will compute these sheaves in the next section in terms of Milnor–Witt K-theory.

The computation of the higher $\mathbb{A}^1$-homology sheaves is at the moment highly non trivial and mysterious[2].

Remark 3.12. There exists a natural morphism of sheaves $\mathbb{H}_n^{\mathbb{A}^1}(X; \mathbb{Z}) \to \mathbb{H}_n^S(X)$ where the right hand side denotes Suslin–Voevodsky singular homology sheaves [52], [55]. In general, this is not an isomorphism. More generally let $\mathsf{DM}(k)$ be Voevodsky's triangulated category of motives [56]. Then there exists a canonical functor of "adding transfers"

$$\mathsf{D}_{\mathbb{A}^1}(k) \to \mathsf{DM}(k).$$

It is not an equivalence. One explanation is given by the (pointed) algebraic Hopf map:

$$\eta\colon \mathbb{A}^2 - \{0\} \to \mathbb{P}^1.$$

The associated morphism on $\mathbb{H}_1^{\mathbb{A}^1}$ defines a morphism[3]:

$$\eta\colon \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathbb{G}_m) \otimes_{\mathbb{A}^1} \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathbb{G}_m) \cong \mathbb{H}_1^{\mathbb{A}^1}(\mathbb{A}^2 - \{0\}) \to \mathbb{H}_1^{\mathbb{A}^1}(\mathbb{P}^1) \cong \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathbb{G}_m).$$

The latter is never nilpotent (use the same argument as in the proof of Theorem 4.7). On the other hand, the computation of the motive of $\mathbb{P}^2$, which is the cone of η, shows that $\mathbb{P}^1 \to \mathbb{P}^2$ admits a retraction in $\mathsf{DM}(k)$ and thus that the image of η in $\mathsf{DM}(k)$ is the zero morphism.

Theorem 3.13 (Hurewicz Theorem, [36]). *Let $\mathcal{X}$ be a pointed $\mathbb{A}^1$-connected space. Then the Hurewicz morphism*

$$\pi_1^{\mathbb{A}^1}(\mathcal{X}) \to \mathbb{H}_1^{\mathbb{A}^1}(\mathcal{X})$$

[2] We do not know any example which does not use the Bloch–Kato conjecture.

[3] Here for sheaves M and N, we denote by $M \otimes_{\mathbb{A}^1} N$ the $\mathbb{H}_0^{\mathbb{A}^1}$ of the sheaf $M \otimes N$, and call it the $\mathbb{A}^1$-tensor product.

is the universal morphism from $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ *to a strictly* $\mathbb{A}^1$*-invariant sheaf*[4]. *If moreover* $\mathcal{X}$ *is* $(n-1)$*-connected for some* $n \geq 2$ *then the Hurewicz morphism*

$$\pi_i^{\mathbb{A}^1}(\mathcal{X}) \to \mathbb{H}_i^{\mathbb{A}^1}(\mathcal{X})$$

is an isomorphism for $i \leq n$ *and an epimorphism for* $i = (n+1)$.

We now may partly realize our program of proving the analogue of Theorem 1.1. Given a sphere $S^n(i)$ with $n \geq 2$, we have $\pi_m^{\mathbb{A}^1}(S^n(i)) = 0$ for $m < n$ and

$$\pi_n^{\mathbb{A}^1}(S^n(i)) \cong \tilde{\mathbb{H}}_0^{\mathbb{A}^1}((\mathbb{G}_m)^{\wedge n}) \cong \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathbb{G}_m)^{\otimes_{\mathbb{A}^1}(n)}.$$

In the next section we will describe those sheaves.

Remark 3.14. Of course, the Hurewicz Theorem has a lot of classical consequences. We do not mention them here, see [36].

4. $\mathbb{A}^1$-homotopy and $\mathbb{A}^1$-homology: computations involving Milnor–Witt K-theory

Milnor–Witt K-theory of fields. The following definition was obtained in collaboration with Mike Hopkins.

Definition 4.1. Let F be a commutative field. The Milnor–Witt K-theory $K_*^{MW}(F)$ of F is the graded associative ring generated by the symbols $[u]$, for each unit $u \in F^\times$, of degree $+1$, and η of degree -1 subject to the following relations:

(1) (Steinberg relation) For each $a \in F^\times - \{1\}$, one has $[a].[1-a] = 0$.

(2) For each pair $(a, b) \in (F^\times)^2$ one has $[ab] = [a] + [b] + \eta.[a].[b]$.

(3) For each $a \in F^\times$, one has $[a].\eta = \eta.[a]$.

(4) One has $\eta^2.[-1] + 2\eta = 0$.

This Milnor–Witt K-theory groups were introduced by the author in a different complicated way. The previous one, is very simple and natural (but maybe the 4-th relation which will be explained below): all the relations easily come from natural $\mathbb{A}^1$-homotopies, see Theorem 4.8.

The quotient $K_*^{MW}(F)/\eta$ of the Milnor–Witt K-theory of F by η is the *Milnor K-theory* $K_*^M(F)$ of F as defined in [26]; indeed after η is killed, the symbol $[a]$ becomes additive and there is only the Steinberg relation.

For any unit $a \in F^\times$, set $\langle a \rangle = \eta[a] + 1 \in K_0^{MW}(F)$. One can show that $[1] = 0$, $\langle 1 \rangle = 1$ and $\langle ab \rangle = \langle a \rangle \langle b \rangle$. Set $\varepsilon := -\langle -1 \rangle$ and $h = 1 + \langle -1 \rangle$. Observe that $h = \eta.[-1] + 2$ and the fourth relation can be written $\eta\varepsilon = \eta$ or equivalently $\eta \,.\, h = 0$.

[4] it is not yet known whether this is the abelianization nor an epimorphism

This η will be interpreted below in term of the algebraic Hopf map (see also Remark 3.12 above). Observe that the relation $\eta^2.[-1] + 2\eta = 0$ is compatible with the complex points (where $[-1] = 0$ and stably $2.\eta = 0$) and the real points (where $[-1] = -1$, $\eta = 2$ and $-2^2 + 2 \times 2 = 0$).

It is natural to call the quotient ring $K_*^{MW}(F)/h$ the *Witt K-theory* of F and to denote it by $K_*^W(F)$. The mod 2-Milnor K-theory $k_*(F) := K_*^M(F)/2$ is thus also the mod η Witt K-theory $K_*^W(F)/\eta = K_*^{MW}(F)/(h, \eta)$.

It is not hard to check that $K_0^{MW}(F)$ admits the following presentation as an abelian group: a generator $\langle u\rangle$ for each unit of $F^\times$ and the relations of the form: $\langle u(v^2)\rangle = \langle u\rangle$, $\langle u\rangle + \langle v\rangle = \langle u+v\rangle + \langle (u+v)uv\rangle$ if $(u+v) \neq 0$ and $\langle u\rangle + \langle -u\rangle = 1 + \langle -1\rangle$. Moreover one checks that the morphism $\eta^n : K_0^{MW}(F) \to K_{-n}^{MW}(F)$ induces an isomorphism $K_0^W(F) \cong K_{-n}^{MW}(F)$ for $n > 0$. Thus in particular $K_*^{MW}(F)[\eta^{-1}] \to K_0^W(F)[\eta, \eta^{-1}]$ is an isomorphism.

Remark 4.2. In the above presentation of $K_0^{MW}(F)$ one recognizes the presentation of the Grothendieck–Witt ring $GW(F)$, see [47] in the case of characteristic $\neq 2$ and [27] in the general case. The element h becomes the hyperbolic plane. The quotient group (actually a ring) $K_0^W(F) = GW(F)/h$ is exactly the Witt ring $W(F)$ of F.

Let us define the *fundamental ideal* $I(F)$ of $K_0^W(F)$ to be the kernel of the mod 2 rank homomorphism $K_0^W(F) \to \mathbb{Z}/2$. Set $I^*(F) = \bigoplus_{n\in\mathbb{Z}} I^n(F)$ (with the convention $I^n(F) = K_0^W(F)$ for $n \leq 0$). We observe that the obvious correspondence $[u] \mapsto \langle u\rangle - 1 \in I(F)$ induces an (epi)morphism

$$S_F\colon K_*^W(F) \to I^*(F)$$

where η acts through the inclusions $I^n(F) \subset I^{n-1}(F)$. Killing η in this morphism yields the Milnor morphism [26]:

$$s_F\colon k_*(F) \to i^*(F) \tag{4.1}$$

where $i^*(F)$ denotes $\oplus I^n(F)/I^{(n+1)}(F)$.

Theorem 4.3 ([32]). *For any field F of characteristic $\neq 2$ the homomorphism*

$$S_F\colon K_*^W(F) \to I^*(F)$$

is an isomorphism.

This statement cannot be trivial as it implies the Milnor conjecture on quadratic forms that morphism (4.1) is an isomorphism. This statement is a reformulation of [1] and thus uses the proof of the Milnor conjecture on mod 2 Galois cohomology by Voevodsky [57], [41].

As a consequence we obtained in [32] that the commutative square of graded rings,

$$\begin{tikzcd} K_*^{MW}(F) \arrow[r] \arrow[d] & K_*^M(F) \arrow[d] \\ K_*^W(F) \arrow[r] & k_*(F) \end{tikzcd} \tag{4.2}$$

is cartesian (for a field of characteristic $\neq 2$).

Remark 4.4. 1) Using Kato's proof [20] of the analogue of the Milnor conjecture in characteristic 2, we can also show the previous result holds in characteristic 2.

2) The fiber products of the form $I^n(F) \times_{i^n(F)} K_n^M(F)$ where considered in [5] in characteristic not 2.

For $n \geq 1$ we simply set

$$\mathbb{Z}_{\mathbb{A}^1}(n) := \tilde{\mathbb{H}}_0^{\mathbb{A}^1}((\mathbb{G}_m)^{\wedge n})$$

for the free (reduced) strictly $\mathbb{A}^1$-invariant sheaf on $(\mathbb{G}_m)^{\wedge n}$. The Hopf morphism $\eta\colon \mathbb{A}^2 - \{0\} \to \mathbb{P}^1$ induces on $\tilde{\mathbb{H}}_1^{\mathbb{A}^1}$ a morphism of the form $\eta\colon \mathbb{Z}_{\mathbb{A}^1}(2) \to \mathbb{Z}_{\mathbb{A}^1}(1)$. Observe that $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}((\mathbb{G}_m)^{\wedge 0}) = \mathbb{H}_0^{\mathbb{A}^1}(\mathrm{Spec}(k)) = \mathbb{Z}$ but that we did not set $\mathbb{Z}_{\mathbb{A}^1}(0) = \mathbb{Z}$. We will in fact extend this family of sheaves $\mathbb{Z}_{\mathbb{A}^1}(n)_{n\geq 1}$ to integers $n \leq 0$ using a construction of Voevodsky.

Given a presheaf of pointed sets M one defines the *pointed $\mathbb{G}_m$-loop space M_{-1}* on M so that for $X \in \mathsf{Sm}_k$, $M_{-1}(X)$ is the "Kernel" of the restriction through the unit section $M(X \times \mathbb{G}_m) \to M(X)$. If M is a sheaf of abelian groups, so is M_{-1}. We may iterate this construction to get M_n for $n < 0$; we set, for $n \leq 0$

$$\mathbb{Z}_{\mathbb{A}^1}(n) = \mathbb{Z}_{\mathbb{A}^1}(1)_{n-1}.$$

The canonical morphism $\mathbb{Z} \to \mathbb{Z}_{\mathbb{A}^1}(0)$ is far from being an isomorphism. The tensor product (and internal Hom) defines natural pairings $\mathbb{Z}_{\mathbb{A}^1}(n) \otimes \mathbb{Z}_{\mathbb{A}^1}(m) \to \mathbb{Z}_{\mathbb{A}^1}(n+m)$ for any integers $(n,m) \in \mathbb{Z}^2$. The element η becomes now an element $\eta \in \mathbb{Z}_{\mathbb{A}^1}(-1)(k)$. Any unit $u \in F^\times$ in a separable field extension $F|k$, viewed as an element in $\mathbb{G}_m(F)$ defines an element $[u] \in \mathbb{Z}_{\mathbb{A}^1}(1)(F)$.

The following result own very much to the definition of the Milnor–Witt K-theory found with Hopkins:

Theorem 4.5 ([28]). *For any separable field extension $F|k$, the symbols $[u] \in \mathbb{Z}_{\mathbb{A}^1}(1)(F)$, for any $u \in F^\times$, and $\eta \in \mathbb{Z}_{\mathbb{A}^1}(-1)(F)$, satisfy the* 4 *relations of Definition* 4.1 *in the graded ring $\mathbb{Z}_{\mathbb{A}^1}(*)(F)$. We thus obtain a canonical homomorphism of graded rings*

$$\Theta_*(F)\colon K_*^{MW}(F) \to \mathbb{Z}_{\mathbb{A}^1}(*)(F).$$

The Steinberg relation (1) is a consequence of the following nice result of P. Hu and I. Kriz [17]. Consider the canonical closed immersion $\mathbb{A}^1 - \{0, 1\} \hookrightarrow \mathbb{G}_m \times \mathbb{G}_m$, $x \mapsto (x, 1-x)$. Then its (unreduced) suspension $\tilde{\Sigma}^1(\mathbb{A}^1 - \{0, 1\}) \to \Sigma^1(\mathbb{G}_m \times \mathbb{G}_m)$ composed with $\Sigma^1(\mathbb{G}_m \times \mathbb{G}_m) \to \Sigma^1(\mathbb{G}_m \wedge \mathbb{G}_m)$ is trivial in $\mathsf{H}_\bullet(k)$. Applying $\mathbb{H}_1^{\mathbb{A}^1}$ yields the Steinberg relation.

The last 3 relations are consequences of the following fact: let $\mu\colon \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ denote the product morphism of the group scheme $\mathbb{G}_m$, then the induced morphism on $\mathbb{H}_1^{\mathbb{A}^1}$, $\mathbb{Z}_{\mathbb{A}^1}(1) \oplus \mathbb{Z}_{\mathbb{A}^1}(1) \oplus \mathbb{Z}_{\mathbb{A}^1}(2) \to \mathbb{Z}_{\mathbb{A}^1}(1)$ is of the form $\mathrm{Id}_{\mathbb{Z}_{\mathbb{A}^1}(1)} \oplus \mathrm{Id}_{\mathbb{Z}_{\mathbb{A}^1}(1)} \oplus \eta$. The relation (2) follows clearly form this fact. The relations (3) and (4) follow from the commutativity of μ. $\square$

Unramified Milnor–Witt K-theory and the main computation. We next define for each $n \in \mathbb{Z}$ an explicit sheaf $\underline{\mathbf{K}}_n^{MW}$ called the sheaf of *unramified Milnor–Witt K-theory* in weight n. To do this, let us give some recollection. For the Milnor K-theory [26], for any discrete valuation v on a field F, with valuation ring $\mathcal{O}_v \subset F$, residue field $\kappa(v)$, one can define a unique homomorphism (of graded groups)

$$\partial_v\colon K_*^M(F) \to K_{*-1}^M(\kappa(v))$$

called "residue" homomorphism, such that

$$\partial_v(\{\pi\}\{u_2\}\dots\{u_n\}) = \{\overline{u_2}\}\dots\{\overline{u_n}\}$$

for any uniformizing element π (of v) and units $u_i \in \mathcal{O}_v^\times$, and where $\bar{u}$ denotes the image of $u \in \mathcal{O}_v \cap F^\times$ in $\kappa(v)$.

In the same way, given a uniformizing element π, one can define a residue morphism

$$\partial_v^\pi\colon K_*^{MW}(F) \to K_{*-1}^{MW}(\kappa(v))$$

satisfying the formula:

$$\partial_v^\pi([\pi].[u_2]\dots[u_n]) = [\overline{u_2}]\dots[\overline{u_n}].$$

However, one important feature is that in the case of Milnor K-theory, these residues do not depend on the choice of π, only on the valuation, but in the case of Milnor–Witt K-theory, they do depend on the choice of π: for $u \in \mathcal{O}^\times$, as one has $\partial_v^\pi([u.\pi]) = \partial_v^\pi([\pi]) + \eta.[\bar{u}] = 1 + \eta.[\bar{u}]$.

To make this residue homomorphism "canonical" (see [5], [6], [48] for instance), one defines for a field κ and a one dimensional κ-vector space L, twisted Milnor–Witt K-theory groups: $K_*^{MW}(\kappa; L) = K_*^{MW}(\kappa) \otimes_{\mathbb{Z}[\kappa^\times]} \mathbb{Z}[L - \{0\}]$, where the group ring $\mathbb{Z}[\kappa^\times]$ acts through $u \mapsto \langle u\rangle$ on $K_*^{MW}(\kappa)$ and through multiplication on $\mathbb{Z}[L - \{0\}]$. The canonical residue homomorphism is of the following form

$$\partial_v\colon K_*^{MW}(F) \to K_{*-1}^{MW}(\kappa(v); m_v/(m_v)^2)$$

with $\partial_v([\pi].[u_2]\dots[u_n]) = [\overline{u_2}]\dots[\overline{u_n}]\otimes\overline{\pi}$, where $m_v/(m_v)^2$ is the cotangent space at v (a one dimensional $\kappa(v)$-vector space).

Using these residue homomorphisms, one may define for any smooth k-scheme $X \in \mathsf{Sm}_k$, irreducible say, with function field K, and any $n \in \mathbb{Z}$, the group $\underline{\mathbf{K}}^{MW}(X)$ of unramified Milnor–Witt K-theory in weight n as the kernel of the (locally finite) sum of the residues at points x of codimension 1, viewed as discrete valuations on K:

$$K_n^{MW}(K) \xrightarrow{\Sigma_x \partial_x} \bigoplus_{x \in X^{(1)}} K_{n-1}^{MW}(\kappa(x); m_x/(m_x)^2)$$

and extends this to a sheaf $X \mapsto \underline{\mathbf{K}}_n^{MW}(X)$.

Example 4.6. 1) In [18] Kato considered first the sheaves of unramified Milnor K-theory $\underline{\mathbf{K}}_n^M$ defined exactly in the same way on the Zariski site of X. It was turned into a strictly $\mathbb{A}^1$-invariant sheaf (on Sm_k) by Rost in [46].

2) One may also define unramified Witt K-theory $\underline{\mathbf{K}}_n^W$, unramified mod 2 Milnor K-theory $\underline{\mathbf{k}}_n$ in the same way, etc.

These types of cohomology theories easily give the non nilpotence of η:

Theorem 4.7. *Let $n \geq 1$ and $i \geq 1$ be natural numbers. The n-th suspension in* $\mathsf{H}_\bullet(k)$

$$\Sigma^n(\eta^i)\colon S^{n+1}(i+1) \to S^{n+1}(1)$$

of the i-th iteration of the Hopf map $\eta\colon S^1(2) \to S^1(1)$, is never trivial. Thus the algebraic Hopf map is not stably nilpotent.

This is trivial if one has a real embedding as $\eta(\mathbb{R})$ is the degree 2 map. In general, one uses the cohomology theory $H^*(-; \underline{\mathbf{K}}_*^{MW}[\eta^{-1}])$, in which η induces an isomorphism. To conclude remember that $K_*^{MW}(k)[\eta^{-1}] = K_0^W(k)[\eta, \eta^{-1}]$ and that $K_0^W(k)$ is never 0 (for k algebraically closed it is $\mathbb{Z}/2$).

We can now state our main computational result. Any strictly $\mathbb{A}^1$-invariant sheaf M has residue homomorphisms (see [34] for instance) and one proves that the homomorphism of Theorem 4.8

$$\Theta_*(F)\colon K_*^{MW}(F) \to \mathbb{Z}_{\mathbb{A}^1}(*)(F)$$

is compatible with residues. Thus (by [33, A.1] it induces a morphism of sheaves

$$\Theta_*\colon \underline{\mathbf{K}}_*^{MW} \to \mathbb{Z}_{\mathbb{A}^1}(*). \tag{4.3}$$

Theorem 4.8 ([28]). *The above morphism* (4.3) *is an isomorphism.*

We observe that the product $\mathbb{G}_m \wedge \underline{\mathbf{K}}_n^{MW} \to \underline{\mathbf{K}}_1^{MW} \wedge \underline{\mathbf{K}}_n^{MW} \to \underline{\mathbf{K}}_{n+1}^{MW}$ induces an isomorphism $\underline{\mathbf{K}}_n^{MW} \cong (\underline{\mathbf{K}}_{n+1}^{MW})_{-1}$. We deduce the existence for each $n > 0$, each $i > 0$, of a canonical $\mathsf{H}_\bullet(k)$-morphism

$$S^n(i) \to K(\underline{\mathbf{K}}_i^{MW}, n). \tag{4.4}$$

Some consequences and applications. The previous result and the Hurewicz Theorem imply:

Theorem 4.9. *For any $n \geq 2$, any $i > 0$:*

1) *The morphism* (4.4) *induces an isomorphism*

$$\pi_n^{\mathbb{A}^1}(S^n(i)) \cong \underline{\mathbf{K}}_i^{MW}.$$

2) *For any $m \in \mathbb{N}$, any $j \in \mathbb{N}$, the previous isomorphism induces canonical isomorphisms*

$$\mathrm{Hom}_{\mathsf{H}(k)}(S^m(j), S^n(i)) \cong \begin{cases} 0 & \text{if } m < n, \\ K_{i-j}^{MW}(k) & \text{if } m = n. \end{cases}$$

In case $i = 0$, $\pi_n^{\mathbb{A}^1}(S^n) = \mathbb{Z}$ and $\mathrm{Hom}_{\mathsf{H}(k)}(S^m(j), S^n) = \begin{cases} 0 \text{ if } m < n & \text{or } j \neq 0, \\ \mathbb{Z} \text{ if } m = n & \text{and } j = 0. \end{cases}$
In general, for $n = 1$ the question is much harder, and in fact unknown. We only know $\pi_1^{\mathbb{A}^1}(S^1(i))$ in the cases $i = 0, 1, 2$. For $i = 0$, $\pi_1^{\mathbb{A}^1}(S^1(i))(S^1) = \mathbb{Z}$.

For $i = 2$, as $\mathrm{SL}_2 \to \mathbb{A}^2 - \{0\} \cong S^1(2)$ is an $\mathbb{A}^1$-weak equivalence, the sphere $S^1(2)$ is an h-space and (by Hurewicz Theorem and Theorem 3.2) $\pi_1^{\mathbb{A}^1}(S^1(2)) = \mathbb{H}_1^{\mathbb{A}^1}(S^1(2)) = \underline{\mathbf{K}}_2^{MW}$. In fact the universal $\mathbb{A}^1$-covering given by Theorem 3.8 admits a group structure and we thus get an extension of sheaves of groups (in fact in the Zariski topology as well)

$$0 \to \underline{\mathbf{K}}_2^{MW} \to \widetilde{\mathrm{SL}}_2 \to \mathrm{SL}_2 \to 1.$$

This is a central extension which also arises in the following way. Let $B(\mathrm{SL}_2)$ denote the simplicial classifying space of SL_2. Then the canonical cohomology class $\Sigma(\mathrm{SL}_2) \cong S^2(2) \to K(\underline{\mathbf{K}}_2^{MW}, 2)$ can be uniquely extended to a $\mathsf{H}_\bullet(k)$-morphism:

$$B(\mathrm{SL}_2) \to K(\underline{\mathbf{K}}_2^{MW}, 2)$$

because the quotient $B(\mathrm{SL}_2)/\Sigma(\mathrm{SL}_2)$ is 3-$\mathbb{A}^1$-connected. It is well-known that such an element in $H^2(B(\mathrm{SL}_2); \underline{\mathbf{K}}_2^{MW})$ corresponds to a central extension of sheaves as above. It is the universal $\mathbb{A}^1$-covering for SL_2.

Remark 4.10. 1) In view of [13] it should be interesting to determine the possible $\pi_1^{\mathbb{A}^1}$ of linear algebraic groups.

2) A. Suslin has computed in [49] the group $H_2(\mathrm{SL}_2(k))$ for most field k and found exactly $\underline{\mathbf{K}}_2^{MW}(k) = I^2(k) \times_{i^2(k)} K_2^M(k)$. This computation has clearly influenced our work.

To understand $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ we use the $\mathbb{A}^1$-fibration sequence

$$\mathbb{A}^2 - \{0\} \to \mathbb{P}^1 \to \mathbb{P}^\infty \tag{4.5}$$

which, using the long exact sequence of $\mathbb{A}^1$-homotopy sheaves, gives a short exact sequence of the form:

$$1 \to \underline{\mathbf{K}}_2^{MW} \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to \mathbb{G}_m \to 1$$

because $\underline{\mathbf{K}}_2^{MW} = \pi_1^{\mathbb{A}^1}(\mathbb{A}^2 - \{0\})$ and because $\mathbb{P}^\infty \cong B(\mathbb{G}_m)$ has only non-trivial $\pi_1^{\mathbb{A}^1}$ equal to $\mathbb{G}_m$. This extension of (sheaves of) groups can be completely explicited [36]. In particular $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ is non abelian!

The Brouwer degree. Now we can deduce as particular case of Theorem 4.9 what we announced in the introduction.

Corollary 4.11. *For any $n \geq 2$, any $i > 0$, the degree morphism induced by the morphism* (4.4)

$$\mathrm{Hom}_{\mathsf{H}(k)}(S^n(i), S^n(i)) \to K_0^{MW}(k)$$

is an isomorphism. As a consequence, the endomorphism ring of the $\mathbb{P}^1$-sphere spectrum $\mathbb{S}^0$, which by definition is

$$\pi_0^{\mathbb{A}^1}(\mathbb{S}^0) = \mathrm{colim}_{n\to\infty} \mathrm{Hom}_{\mathsf{H}(k)}(S^n(n), S^n(n)),$$

is isomorphic to the Grothendieck–Witt ring $GW(k) = K_0^{MW}(k)$ of k (see [31], [30] *for the case of a perfect field of characteristic $\neq 2$).*

When $n = 1$, $i = 1$, $S^1(1) \cong \mathbb{P}^1$, using the $\mathbb{A}^1$-fibration sequence (4.5) one may entirely describe $\mathrm{Hom}_{\mathsf{H}(k)}(\mathbb{P}^1, \mathbb{P}^1)$ [36]. One may check the morphism $\mathbb{P}^1 \to K(\underline{\mathbf{K}}_1^{MW}, 1)$ induces a degree morphism $\mathrm{Hom}_{\mathsf{H}_\bullet(k)}(\mathbb{P}^1, \mathbb{P}^1) \to K_0^{MW}(k)$, which coincides with the one sketched in the introduction, for an actual morphism $\mathbb{P}^1 \to \mathbb{P}^1$ which has a regular value. However it is not an isomorphism in general: its kernel is isomorphic to the subgroup of squares $(k^\times)^2$ in $k^\times$.

Remark 4.12. 1) *Transfers.* It is well know that, given a finite separable field extension $k \subset L$ together with a primitive element $x \in L$ (which generates $L|k$), one can define a transfer morphism in $\mathsf{H}_\bullet(k)$ of the form

$$tr_x \colon \mathbb{P}^1 \to \mathbb{P}^1 \wedge (\mathrm{Spec}(L)_+).$$

This follows from the Purity Theorem of [38] (or the Thom–Pontryagin construction) applied to the closed immersion $\mathrm{Spec}(L) \to \mathbb{P}^1$ determined by x. Using our computations and methods, we have been able to show that the induced morphism on $\mathbb{H}_1^{\mathbb{A}^1}$ does not depend on the choice of x. As a consequence we obtain that for any strictly $\mathbb{A}^1$-invariant sheaf M the strictly $\mathbb{A}^1$-invariant sheaf M_{-1} has canonical transfers morphisms for finite separable extensions between separable extensions of k. This can be used to simplify the construction of transfers in Milnor K-theory [18], [7].

Beware however that this notion of transfers for finite extension is slightly more general than Voevodsky's notion. The sheaf M_{-1} is automatically a sheaf of modules over $\underline{\mathbf{K}}_0^{MW}$. Given a finite separable extension $k \subset L$ as above, the composition $M_{-1}(k) \to M_{-1}(L) \xrightarrow{\mathrm{tr}} M_{-1}(k)$ is precisely the multiplication by the class of L in $K_0^{MW}(k)$ (which is its Euler characteristic by the remark below). In characteristic $\neq 2$, this is (up to an invertible element) the trace form of $L|k$ in the Grothendieck–Witt group. In the case of Voevodsky's structure this composition is just the multiplication by $[L:k] \in \mathbb{N}$.

2) Using the previous computations as well as the classical ideas on Atiyah duality [2] and [16] in $\mathbb{A}^1$-algebraic topology[5] one may define for any morphism f (in fact in $\mathsf{H}(k)$) from a smooth projective k-variety X to itself a Lefschetz number $\lambda(f) \in K_0^{MW}(k)$ which satisfies all the usual properties (like the Lefschetz fixed point formula). In particular the Euler characteristic of X lies in $K_0^{MW}(k)$.

3) In view of the cartesian diagram (4.2) and our philosophy, the part coming from the Milnor K-theory is the one compatible with the intuition coming from the topology of complex points (or motives), and the part coming from the Witt K-theory is the one compatible with the intuition on the topology of real points. For any $X \in \mathsf{Sm}_k$ the graded ring $\bigoplus_n H^n(X; \underline{\mathbf{K}}_n^{MW})$ maps surjectively to the Chow ring $CH^*(X) = \bigoplus_n H^n(X; \underline{\mathbf{K}}_n^{M})$ and to the graded ring $\bigoplus_n H^n(X; \underline{\mathbf{K}}_n^{W})$ (however it does not inject into the product in general: one has a Mayer–Vietoris type long exact sequence). Given a real embedding there exists a morphism of rings $\bigoplus_n H^n(X; \underline{\mathbf{K}}_n^{W}) \to H^*(X(\mathbb{R}); \mathbb{Z})$. Note that it is known that the Chow ring only maps to $H^*(X(\mathbb{R}); \mathbb{Z}/2)$.

5. Some results on classifying spaces in $\mathbb{A}^1$-homotopy theory

Serre's splitting principle and $\mathbb{H}_0^{\mathbb{A}^1}$ of some classifying spaces. The Serre's splitting principle was stated in [15] only in terms of étale cohomology groups §24 or in terms of Witt groups §29, but we may easily generalize it to our situation.

Let us briefly recall from [53] and also [38] the notion of *geometric classifying space* $B_{gm}(G)$ for a linear algebraic group G. Choose a closed immersion of k-groups $\rho\colon G \subset \mathbb{G}L_n$. For each $r > 0$, denote by $U_r \subset \mathbb{A}^{rn}$ the open subset where G acts freely (in the étale topology) in the direct sum of r copies of the representation ρ. $B_{gm}(G)$ is then the union over r of the quotient k-varieties U_r/G, which are smooth k-varieties. We proved in [38] that for G a finite group of order prime to $\mathrm{char}(k)$ and X a smooth k-variety:

$$\mathrm{Hom}_{\mathsf{H}(k)}(X, B_{gm}(G)) \cong H^1_{\text{ét}}(X; G).$$

For n an integer, denote by $m = [\frac{n}{2}]$ and by $(\mathbb{Z}/2)^m \subset \Sigma_n$ the natural embedding. The following result is a variation on the Splitting principle [15, §24] (using the fact

[5]These ideas are also present in much more elaborated form in Voevodsky formalism of cross-functors [59], see also [3].

that strictly $\mathbb{A}^1$-invariant sheaves have also residues [34] as well as [15, Appendix C, A letter from B. Totaro to J.-P. Serre]):

Theorem 5.1 (Serre's splitting principle). *For any strictly $\mathbb{A}^1$-invariant sheaf M the restriction map*

$$H^0(B_{gm}(\Sigma_n); M) \to H^0(B_{gm}((\mathbb{Z}/2)^m); M)$$

is injective.

Corollary 5.2. *The homomorphism*

$$\mathbb{H}_0^{\mathbb{A}^1}(B_{gm}((\mathbb{Z}/2)^m) \to \mathbb{H}_0^{\mathbb{A}^1}(B_{gm}(\Sigma_n))$$

is an epimorphism.

We observe that $B_{gm}((\mathbb{Z}/2)^m)$ is $\mathbb{A}^1$-equivalent to a point in characteristic 2, see [38]. In that case we get $\mathbb{H}_0^{\mathbb{A}^1}(B_{gm}(\Sigma_n)) = \mathbb{Z}$.

In characteristic $\neq 2$, one has an exact sequence $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathbb{G}_m) \to \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(\mathbb{G}_m) \to \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(B_{gm}(\mathbb{Z}/2)) \to 0$ where the left morphism is induced by the squaring map (this comes from the fact that $B_{gm}(\mathbb{Z}/2)$ is the union of the quotients $(\mathbb{A}^n - \{0\})/(\mathbb{Z}/2)$). Thus $\tilde{\mathbb{H}}_0^{\mathbb{A}^1}(B_{gm}(\mathbb{Z}/2)) = \underline{\mathbf{K}}_1^{MW}/h = \underline{\mathbf{K}}_1^{W}$ and $\mathbb{H}_0^{\mathbb{A}^1}(B_{gm}(\mathbb{Z}/2)) = \mathbb{Z} \oplus \underline{\mathbf{K}}_1^{W}$.

Now the $\mathbb{A}^1$-tensor product $\underline{\mathbf{K}}_n^{W} \otimes_{\mathbb{A}^1} \underline{\mathbf{K}}_m^{W}$ is $\underline{\mathbf{K}}_{n+m}^{W}$. Using this we may compute $\mathbb{H}_0^{\mathbb{A}^1}(B_{gm}((\mathbb{Z}/2)^m))$ by the Künneth formula and as the morphism of Theorem 5.1 is invariant under the action of Σ_m we get in characteristic $\neq 2$ an epimorphism of sheaves

$$\bigoplus_{i\in\{0,\dots,m\}} \underline{\mathbf{K}}_i^{W} \twoheadrightarrow \mathbb{H}_0^{\mathbb{A}^1}(B_{gm}(\Sigma_n)). \tag{5.1}$$

Theorem 5.3. *In characteristic $\neq 2$ the epimorphism* (5.1) *is an isomorphism.*

The method is to construct refined Stiefel–Whitney classes $W_i\colon K_0^{MW}(F) \to K_i^{W}(F)$ lifting the usual ones w_i in $k_i(F)$ using the same method as in [26, §3]. The composition $\mathbb{H}_0^{\mathbb{A}^1}(B_{gm}(\Sigma_n)) \to \mathbb{H}_0^{\mathbb{A}^1}(B_{gm}(O_n)) \xrightarrow{\oplus W_i} \bigoplus_{i\in\{0,\dots,m\}} \underline{\mathbf{K}}_i^{W}$ is the required left inverse.

Remark 5.4. 1) This result implies the Baratt–Priddy–Quillen Theorem in dimension 0 (at least in characteristic $\neq 2$), stating that the morphism induced by the stable transfers

$$\amalg_{n\in\mathbb{N}} B_{gm}(\Sigma_n) \to Q_{\mathbb{P}^1}\mathbb{S}^0$$

where $Q_{\mathbb{P}^1}\mathbb{S}^0$ means the colimit of the iterated $\mathbb{P}^1$-loop spaces[6], is an $\mathbb{A}^1$-stable group completion[7], see [37].

[6] $\mathrm{colim}_n R\underline{\mathrm{Hom}}_\bullet((\mathbb{P}^1)^{\wedge n}, (\mathbb{P}^1)^{\wedge n})$

[7] Voevodsky proved that it is not the usual group completion.

2) The same computation holds for Suslin singular homology [52] of $B_{gm}(\Sigma_n)$: one gets in characteristic $\neq 2$: $\mathbb{H}_0^S(B_{gm}(\Sigma_n)) = \bigoplus_{i\in\{0,\dots,m\}} \underline{\mathbf{k}}_i$.

3) Using the refined Stiefel–Whitney classes W_i considered previously and [15] we can also compute in characteristic $\neq 2$: $\mathbb{H}_0^{\mathbb{A}^1}(B_{gm}(O_n)) = \oplus_{i\in\{0,\dots,n\}}\underline{\mathbf{K}}_i^W$ and $\mathbb{H}_0^S(B_{gm}(O_n)) = \oplus_{i\in\{0,\dots,n\}}\underline{\mathbf{k}}_i$. We observe as a consequence that the natural map (of sets)

$$H^1_{\text{ét}}(k; O_n) \to \mathbb{H}_0^{\mathbb{A}^1}(B_{gm}(O_n))(k)$$

is injective (but is not if one consider the Suslin $\mathbb{H}_0^S$ instead !). It is a natural question to ask for which algebraic k-groups the analogous map is injective. It is wrong for finite groups in general (but the abelian ones). It could be however true for a general class of algebraic groups G, in connection with a conjecture of Serre addressing the injectivity of the extension map $H^1_{\text{ét}}(k; G) \to H^1_{\text{ét}}(L_1; G) \times H^1_{\text{ét}}(L_2; G)$ when the finite field extensions L_1 and L_2 have coprime degrees over k.

$\mathbb{A}^1$-homotopy classification of algebraic vector bundles. Lindel has proven in [25] that for any n and for any smooth affine k-scheme X the projection $X \times \mathbb{A}^1 \to X$ induces a bijection

$$H^1_{Zar}(X; \mathbb{G}L_n) \to H^1_{Zar}(X \times \mathbb{A}^1; \mathbb{G}L_n)$$

(after the fundamental cases obtained by Quillen [45] and Suslin [50] on the Serre problem). As a consequence if one denotes by $\mathbb{G}r_n$ the "infinite Grassmanian of n-plans" the natural map $\mathrm{Hom}_k(X; \mathbb{G}r_n) \to H^1_{Zar}(X; \mathbb{G}L_n)$ which to a morphism assigns the pull-back of the universal rank n bundle, induces a map $\pi(X; \mathbb{G}r_r) \to H^1_{Zar}(X; \mathbb{G}L_n)$ (where the source means the set of morphisms modulo naive $\mathbb{A}^1$-homotopies); it is moreover easy to show this map is a bijection.

Theorem 5.5 ([35]). *For any integer $n \geq 3$ and any affine smooth k-scheme X the obvious map*

$$H^1_{Zar}(X; \mathbb{G}L_n) \cong \pi(X; \mathbb{G}r_r) \to \mathrm{Hom}_{\mathsf{H}(k)}(X, \mathbb{G}r_r)$$

is a bijection.

For $n = 1$ this is well-known [38]. The proof of this result relies on the works of Quillen, Suslin, Lindel cited above and also on the works of Suslin [51] and Vorst [60] on the generalized Serre problem for the general linear group. In these latter works n has to be assumed $\neq 2$. We conjecture however that the statement of the previous theorem should remain true also for $n = 2$.

One then observes that one has an $\mathbb{A}^1$-fibration sequence of pointed spaces:

$$\mathbb{A}^n - \{0\} \to \mathbb{G}r_{n-1} \to \mathbb{G}r_n \tag{5.2}$$

because the simplicial classifying space $B(\mathbb{G}L_m)$ is $\mathbb{A}^1$-equivalent to $\mathbb{G}r_m$, for any m, and because $\mathbb{G}L_n/\mathbb{G}L_{n-1} \to \mathbb{A}^n - \{0\}$ is an $\mathbb{A}^1$-weak equivalence. From Theorem 4.9

we know that the space $\mathbb{A}^n - \{0\}$ is $(n-2)$-connected and that there exists a canonical isomorphism of sheaves: $\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}) \cong \underline{\mathbf{K}}_n^{MW}$.

Euler class and Stably free vector bundles. For a given smooth affine k-scheme X and an integer $n \geq 4$ we may now study the map:

$$H^1_{Zar}(X; \mathbb{G}L_{n-1}) \to H^1_{Zar}(X; \mathbb{G}L_n)$$

of adding the trivial line bundle following the classical method of obstruction theory in homotopy theory:

Theorem 5.6 (Theory of Euler class, [35]). *Assume $n \geq 4$. Let X be a smooth affine k-scheme, together with an oriented algebraic vector bundle ξ of rank n (this means a vector bundle of rank n and a trivialization of $\Lambda^n(\xi)$). Define its Euler class*

$$e(\xi) \in H^n(X; \underline{\mathbf{K}}_n^{MW}) = H^n(X; \pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}))$$

to be the obstruction class obtained from Theorem 5.5 *and the $\mathbb{A}^1$-fibration sequence* (5.2). *If dimension $X \leq n$ we have the following equivalence:*

$$\xi \text{ split off a trivial line bundle} \Leftrightarrow e(\xi) = 0 \in H^n(X; \underline{\mathbf{K}}_n^{MW}).$$

Remark 5.7. 1) In case $\mathrm{char}(k) \neq 2$, the group $H^n(X; \underline{\mathbf{K}}_n^{MW})$ coincides with the oriented Chow group $\widetilde{CH}^n(X)$ as defined in [5] and our Euler class coincides also with the one defined in *loc. cit.* There is an epimorphism from the Euler class group of Nori [8] to ours but we do not know whether this is an isomorphism. We observe that in [8] an analogous result is proven, and our result implies the result in [8]. If $\mathrm{char}(k) \neq 2$, in [5] the case of rank $n = 2$ was settled by some other method.

2) If ξ is an algebraic vector bundle of rank n over X, let $\lambda_\xi = \Lambda^n(\xi) \in \mathrm{Pic}(X)$ denotes its first Chern class. The obstruction class $e(\xi)$ obtained by the $\mathbb{A}^1$-fibration sequence (5.2) lives now in the corresponding cohomology group $H^n(X; \underline{\mathbf{K}}_n^{MW}(\lambda_\xi))$ obtained by twisting the sheaf $\underline{\mathbf{K}}_n^{MW}$ by λ_ξ.

3) The obvious morphism

$$H^n(X; \underline{\mathbf{K}}_n^{MW}) \to H^n(X; \underline{\mathbf{K}}_n^{M}) = CH^n(X)$$

maps the Euler class to the top Chern class $c_n(\xi)$. When k is algebraically closed and $\dim(X) \leq n$, this homomorphism is an isomorphism. This case of the Theory is due to Murthy [39].

4) Given a real embedding of the base field $k \to \mathbb{R}$, the canonical morphism from Remark 4.12 3): $H^n(X; \underline{\mathbf{K}}_n^{MW}) \to H^n(X(\mathbb{R}); \mathbb{Z})$ maps the Euler class $e(\xi)$ to the Euler class of the real vector bundle $\xi(\mathbb{R})$.

The long exact sequence in homotopy for the $\mathbb{A}^1$-fibration sequence (5.2) (applied to $(n+1)$) also gives the following theorem (compare [9]):

Theorem 5.8 (Stably free vector bundles, [35]). *Assume $n \geq 3$. Let X be a smooth affine k-scheme. The canonical map*

$$\mathrm{Hom}_{\mathsf{H}(k)}(X, \mathbb{A}^{n+1} - \{0\}) \,/\, \mathrm{Hom}_{\mathsf{H}(k)}(X, \mathbb{G}L_{n+1}) \to \mathrm{Hom}_{\mathsf{H}(k)}(X, \mathbb{G}r_n)$$

is injective and its image $\Psi_n(X) \subset H^1_{Zar}(X; \mathbb{G}L_n) = \mathrm{Hom}_{\mathsf{H}(k)}(X, \mathbb{G}r_n)$ consists exactly of the set of isomorphism classes of algebraic vector bundles of rank n over X such that $\xi \oplus \theta^1$ is trivial.

Moreover if the dimension of X is $\leq n$, the natural map

$$\mathrm{Hom}_{\mathsf{H}(k)}(X, \mathbb{A}^{n+1} - \{0\}) \to H^n(X; \underline{\mathbf{K}}^{MW}_{n+1})$$

is a bijection and the natural action of $\mathrm{Hom}_{\mathsf{H}(k)}(X, \mathbb{G}L_{n+1})$ factors trough the determinant as an action of $\mathcal{O}(X)^\times$. In that case, we get a bijection

$$H^n(X; \underline{\mathbf{K}}^{MW}_{n+1})/\mathcal{O}(X)^\times \cong \Psi_n(X).$$

Remark 5.9. Using Popescu's approximation result [43] it is possible, with some care, to extend the results of this paragraph to affine regular schemes defined over a field k.

6. Miscellaneous

Proofs of the Milnor conjecture on quadratic forms. Using Voevodsky's result [57] we have produced two proofs of the Milnor conjecture on quadratic forms asserting that for a field F of characteristic $\neq 2$ the Milnor epimorphism $s_F : k_*(F) \to i^*(F)$ is an isomorphism.

The first one is only sketched in [29], however it is very striking in the context of $\mathbb{A}^1$-algebraic topology. We consider the Adams spectral sequence based on mod 2 motivic cohomology "converging" to $\pi_*^{\mathbb{A}^1}(\mathbb{S}^0)$. Using an unpublished work of Voevodsky on the computation of the mod 2 motivic Steenrod algebra we showed that $E_2^{s,u} = \mathrm{Ext}^s_{\mathcal{A}_k}(H^*(k; \mathbb{Z}/2(*)), H^*(k; \mathbb{Z}/2(*))[s+u])$ and could compute enough. First $E_2^{s,u}$ vanishes for $u < 0$ which is compatible with the $\mathbb{A}^1$-connectivity result 3.5, which implies $\pi_u^{\mathbb{A}^1}(\mathbb{S}^0) = 0$ for $u < 0$. More striking is the computation of the column $E_2^{s,0}$ converging to $\pi_0^{\mathbb{A}^1}(\mathbb{S}^0) = GW(k)$ (in characteristic $\neq 2$). We found that $E_2^{0,0} = \mathbb{Z}/2$ and that for $s > 0$

$$E^{s,0} = \mathbb{Z}/2 \oplus k_s(k).$$

This is exactly the predicted form of the associated graded ring for $GW(k)$ by the Milnor conjecture. The terms $\mathbb{Z}/2$ are detected (in the bar complex) by the tensor powers of the Bockstein $\beta^{\otimes s}$ and the mod 2 Milnor K-theory terms are detected by the tensor powers of the $\mathbb{S}q^2$-operation[8] of Voevodsky $(\mathbb{S}q^2)^{\otimes s}$. The proof of the Milnor

[8] This relationship is explained again by "taking" the real points: the operation $\mathbb{S}q^2$ "induces" the Bockstein operation on mod 2 singular cohomology of real points

conjecture then amounts to showing that the Adams spectral sequence degenerates from the $E_2^{*,*}$-term on the column $u = 0$.

The degenerescence was obtained by a careful study of the column $E_2^{*,1}$ from which the potential differentials start to reach $E_2^{*,0}$, using the Milnor conjecture on mod 2 Galois cohomology of fields of characteristic 2 established by Voevodsky in [57]. The idea was to observe that the groups $E_2^{*,1}$ are enough "divisible" by some suitable mod 2-Milnor K-theory groups. We realized recently in [33] that this argument could be made much simpler and that everything amounts to proving some "$\mathbb{P}^1$-cellularity" of the sheaves $\underline{\mathbf{k}}_n$ in the $\mathbb{A}^1$-derived category, which again is given by the main result of [57].

Global properties of the stable $\mathbb{A}^1$-homotopy category. We have unfortunately no room available to discuss much recent developments in the global properties of the stable $\mathbb{A}^1$-homotopy category. Let us just mention briefly: our work (in preparation) on the rational stable homotopy category and its close relationship with Voevodsky's category of rational mixed motives. The slice filtration and motivic Atiyah–Hirzebruch's type spectral sequence approach due to Voevodsky (see [58] for instance); we must also mention Levine's recent work in this direction, for instance [22]. There is also a work in preparation by Hopkins and the author starting from the Thom spectrum $M\mathbb{G}L$, where is proven that the "homotopical quotient" $M\mathbb{G}L/(x_1, \dots, x_n, \dots)$ obtained by killing the generators of the Lazard ring is, in characteristic 0, the motivic cohomology spectrum of Voevodsky. This gives an Atiyah–Hirzebruch spectral sequence for $M\mathbb{G}L$ (and also K-theory) and gives an other (purely homotopical) proof of the general degree formula of [24], [23].

We must mention Voevodsky's formalism of cross functors [59] and Ayoub's work [3] in which is established the analogue of the theory of vanishing cycles in the context of Voevodsky's triangulated category of motives.

From $\mathbb{A}^1$-homotopy to algebraic geometry? We conclude this paper by an observation. All the tools and notions concerning the classical approach to surgery in classical differential topology seem now available in $\mathbb{A}^1$-algebraic topology: degree, homology, fundamental groups, cobordism groups [24], [23], Poincaré complexes, classification of vector bundles, etc. We also have natural candidates of surgery groups using Balmer's Witt groups [4] of some triangulated category of $\pi_1^{\mathbb{A}^1}$-modules. Why not then dreaming about a surgery approach also for smooth projective k-varieties? Of course there is no obvious analogues for surgery. There is also a major new difficulty: we have observed that even the simplest varieties like the projective spaces are never simply connected. This fact obstructs any hope of "h-cobordism" theorem[9], but now we also understand the reason: the $\mathbb{A}^1$-fundamental group of a pointed projective smooth k-scheme is almost never trivial. A major advance would then be to find the analogue of the "s-cobordism" theorem, the generalization of the h-cobordism theorem in the presence of π_1.

[9] Marc Levine indeed produced a counter-example

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