Development in symplectic Floer theory

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Abstract. In the middle of the 1980s, Floer initiated a new theory, which is now called the Floer theory. Since then the theory has been developed in various ways. In this article we report some recent progress in Floer theory in symplectic geometry. For example, we give an outline of a proof of the flux conjecture, which states that the Hamiltonian diffeomorphism group is C^1-closed in the group of symplectomorphisms for closed symplectic manifolds. We also give a brief survey on the obstruction–deformation theory for Floer theory of Lagrangian submanifolds and explain some of its applications.

Mathematics Subject Classification (2000). Primary 53D40; Secondary 53D35.

Keywords. Symplectic manifold, Hamiltonian systems, Lagrangian submanifolds, Floer cohomology, A_∞-structure.

1. Introduction

In [9]–[14], Andreas Floer initiated "$\frac{\infty}{2}$-dimensional" (co)homology theory, which is now called Floer theory. He invented this theory to prove Arnold's conjecture for fixed points of Hamiltonian diffeomorphism and, under certain assumptions, its analogue for Lagrangian intersections. Roughly speaking, the conjecture states that there is a non-trivial topological lower bound for the number of fixed points of a Hamiltonian diffeomorphism. It is one of his conjectures which stimulated recent developments in symplectic geometry. This theory was soon adapted in Donaldson theory and he constructed the instanton homology theory. A lot of work has been done since and Floer theory has been developed in various directions. In this article, we will describe some recent development of Floer theory in symplectic geometry.

In these decades, symplectic geometry has been much developed. In particular, Gromov revealed many significant phenomena based on his theory of pseudo-holomorphic curves [18] and revolutionized the study in this area. Hamiltonian dynamics is one of main sources of symplectic geometry. The existence of periodic trajectories is a basic problem and there are many works on this subject up to now. In fact, the existence of periodic trajectories reflects so-called symplectic rigidity phenomena. Since trajectories of a Hamiltonian system are characterized by the least action principle, the variational method can be applied to the existence of periodic

*The author is partly supported by Grant-in-Aid for Scientific Research Nos. 14340019 and 17654009, Japan Society for the Promotion of Science.

Proceedings of the International Congress
of Mathematicians, Madrid, Spain, 2006

trajectories. Namely, closed trajectories are critical points of the action functional associated to the Hamiltonian system. Floer combined the variational framework with the theory of pseudo-holomorphic curves to construct an analogue of Morse theory for the action functional.

In the first part of this article, we discuss Floer theory for Hamiltonian systems and present some applications including the flux conjecture. In the second part, we discuss Floer theory for Lagrangian submanifolds. In general, Floer cohomology may not be defined for a pair of Lagrangian submanifolds. We briefly describe the obstruction to defining Floer cohomology as well as the filtered A_∞-algebra associated to a Lagrangian submanifold. We also present some applications, e.g., Lagrangian intersections, non-triviality of the Maslov class, etc. Although there will be some overlaps with Y. G. Oh's contribution to this proceedings, we will try to put different emphases on the theory in this lecture.

2. Floer theory for symplectomorphisms

2.1. Review on the construction. In this section, we briefly review the construction of Floer cohomology for symplectomorphisms, especially Hamiltonian diffeomorphisms, which was initiated in [14] and developed in e.g., [21], [36], [16], [29]. Let (M, ω) be a symplectic manifold. In this article, we assume that M is compact without boundary for simplicity. Denote by X_h the Hamiltonian vector field of h defined by

$$i(X_h)\omega = dh.$$

For $H = \{h_t\}_{t\in\mathbb{R}}$, we integrate the time-dependent vector field X_{h_t} to obtain the one-parameter family $\{\varphi_t^H\}$ of diffeomorphisms. We call such $\{\varphi_t^H\}$ a time-dependent Hamiltonian flow. A diffeomorphism φ of M is called a Hamiltonian diffeomorphism, when φ is the time-one map of $\{\varphi_t^H\}$ for some H. We may assume that $h_{t+1} = h_t$. Denote by $\mathrm{Ham}(M, \omega)$ resp. $\mathrm{Symp}(M, \omega)$ the group of Hamiltonian diffeomorphisms resp. the group of symplectomorphisms which are diffeomorphisms preserving ω. Clearly, $\mathrm{Ham}(M, \omega) \subset \mathrm{Symp}(M, \omega)$. Hamiltonian diffeomorphisms are fundamental in symplectic geometry and enjoy some distinguished properties, e.g., existence of fixed points (see Arnold's conjecture below), simplicity [1], existence of a biinvariant distance on (the universal covering group of) $\mathrm{Ham}(M, \omega)$, called Hofer's distance, etc. Now we recall the following:

Conjecture 2.1 (Arnold's conjecture). For $\varphi \in \mathrm{Ham}(M, \omega)$ there are as many fixed points of φ as the smallest number of critical points of smooth functions on M, namely,

$$\#\,\mathrm{Fix}(\varphi) \geq \min\{\#\mathrm{Crit}(f) \mid f \in C^\infty(M)\}.$$

If all the fixed points of φ are non-degenerate, i.e. 1 is not an eigenvalue of $d\varphi$ at any fixed point, then

$$\# \operatorname{Fix}(\varphi) \geq \min\{\#\operatorname{Crit}(f) \mid f \text{ is a Morse function on } M\}.$$

This conjecture has been verified for closed oriented surfaces, the torus, complex projective spaces, etc. A weaker version of the conjecture is formulated by replacing the lower bounds with the cup-length and the sum of Betti numbers, respectively. We call it homological Arnold conjecture.

Let φ be a symplectomorphism of (M, ω) such that all fixed points are non-degenerate. Following [6], we introduce the twisted loop space

$$\mathcal{P}_\varphi = \{\sigma : [0,1] \to M \mid \varphi(\sigma(1)) = \sigma(0)\},$$

and define a closed 1-form, in a formal sense, on $\mathcal{P}_\varphi$ by

$$\alpha_\varphi(\xi) = \int_0^1 \omega(\xi, \dot{\sigma})\, dt \quad \text{for } \xi \in T_\sigma \mathcal{P}_\varphi.$$

Clearly, fixed points of φ are in one-to-one correspondence with zeros of α_φ. We take the smallest covering space $\pi : \widetilde{\mathcal{P}}_\varphi \to \mathcal{P}_\varphi$ such that (1) $\pi^*\alpha_\varphi$ is exact, i.e., there exists a primitive function $\mathcal{A}_\varphi$ for α_φ, and (2) the integer valued Maslov index μ is well defined on $\operatorname{Crit}(\mathcal{A}_\varphi) = \pi^{-1}(\operatorname{Zero}(\alpha_\varphi))$. From now on we call such a covering space the Floer covering space. Pick an almost complex structure $J = \{J_t\}$ compatible with ω such that $\varphi_* J_1 = J_0$. Then the gradient of $\mathcal{A}_\varphi$ is formally written as

$$\operatorname{grad} \mathcal{A}_\varphi(\sigma) = -J\dot{\sigma},$$

and gradient flow lines are regarded as solutions of the following equation

$$\frac{\partial u}{\partial \tau} + J_t(u)\frac{\partial u}{\partial t} = 0$$

for

$$u = u(\tau, t) : \mathbb{R} \times [0,1] \to M \quad \text{such that } \varphi(u(\tau, 1)) = u(\tau, 0).$$

We set

$$CF^*(\varphi, J) = \Big\{ \sum_i a_i \tilde{\sigma}_i \mid a_i \in \mathbb{Q},\ \tilde{\sigma}_i \in \operatorname{Crit}(\mathcal{A}_\varphi) \text{ satisfy the following condition:}$$
$$\#\{i \mid a_i \neq 0,\ \mathcal{A}_\varphi(\tilde{\gamma}_i) < c\} \text{ is finite for any } c \in \mathbb{R} \Big\}.$$

The grading is given by the Maslov index μ on $\operatorname{Crit}(\mathcal{A}_\varphi)$. The coboundary operator $\delta = \delta^{\varphi, J}$ is defined by counting gradient flow lines connecting the critical points $\tilde{\sigma}^\pm$ of $\mathcal{A}_\varphi$ such that $\mu(\tilde{\sigma}^+) - \mu(\tilde{\sigma}^-) = 1$. Note that the covering transformation group $G_{M,\varphi}$ of $\pi : \widetilde{\mathcal{P}}_\varphi \to \mathcal{P}_\varphi$ naturally acts on the Floer complex. In fact, this action extends to the so-called Novikov ring associated to $\varphi \in \mathrm{Symp}(M, \omega)$, which is a certain completion of the group ring of $G_{M,\varphi}$. To make this construction rigorous, we need to study compactness properties, transversality, etc. for the moduli space of solutions of the J-holomorphic curve equation above. We can achieve these points as in [16], [29], see also [28], [41], [45] based on the notion of stable maps [23], [24]. The resulting cohomology is the Floer cohomology $HF^*(\varphi, J)$, which is a module over the Novikov ring associated to $\varphi \in \mathrm{Symp}(M, \omega)$. We also find that Floer cohomology is invariant under Hamiltonian deformations of φ. In the case that M is 2-dimensional, Seidel noticed that the Floer cohomology is invariant under a class of deformations which contains all Hamiltonian deformations [43].

When $\varphi \in \mathrm{Symp}_0(M, \omega)$, i.e., there is a path φ_t, $0 \leq t \leq 1$, such that $\varphi_0 = \mathrm{id}$ and $\varphi_1 = \varphi$, we can formulate the Floer theory on the loop space LM of M rather than the twisted loop space $\mathcal{P}_\varphi$. (From now on, we call φ_t with $\varphi_0 = \mathrm{id}$ a based path.) Namely, we identify them by

$$\sigma(t) \in \mathcal{P}_\varphi \mapsto \gamma(t) = (\varphi_t)^{-1}(\sigma(t)) \in LM.$$

In particular, when $\varphi \in \mathrm{Ham}(M, \omega)$ we choose a based path φ_t in $\mathrm{Ham}(M, \omega)$. Denote by H the time-dependent Hamiltonian function which generates φ_t. Then fixed points of φ are in one-to-one correspondence with 1-periodic orbits of the time-dependent flow φ_t, which are characterized as zeros of the following closed 1-form α_H on the loop space LM of M:

$$\alpha_H(\xi) = \int_0^1 \omega(\xi(t), \dot{\gamma}(t) - X_{H_t}(\gamma(t)))\, dt,$$

where $\gamma \in LM$ and $\xi \in T_\gamma LM$, i.e., a section of $\gamma^* TM$. Write $J_t' = (\varphi_t)_* J_t$. (Note that $J_0' = J_1'$.) Then gradient flow lines are solutions of the following equation:

$$\frac{\partial u}{\partial \tau} + J'(u) \left(\frac{\partial u}{\partial t} - X_{H_t}(u) \right) = 0,$$

for $u = u(\tau, t) \colon \mathbb{R} \times S^1 \to M$. Denote by $p \colon \tilde{L}M \to LM$ the Floer covering space of LM and by $\mathcal{A}_H \colon \tilde{L}M \to \mathbb{R}$ the action functional, i.e., $d\mathcal{A}_H = p^*\alpha_H$. Consider the graded module generated by the critical points of $\mathcal{A}_H$ with the grading given by μ, which is known as the Conley–Zehnder index in this setting. Then take its completion with respect to the filtration $\{\tilde{\gamma} \in \mathrm{Crit}(\mathcal{A}_H) \mid \mathcal{A}_H(\tilde{\gamma}) > c\}$ for $c \in \mathbb{R}$. We denote it by $CF^*(H)$. The coboundary operator $\delta = \delta^{H,J}$ is defined by counting the number of connecting orbits joining critical points. In this case Floer cohomology can be computed as follows:

$$HF^*(H, J) \cong H^{*+n}(M; \mathbb{Q}) \otimes \Lambda_\omega,$$

where Λ_ω is the Novikov ring of (M, ω).

As a corollary we have the following result ([16], [29]).

Theorem 2.2. *If $\varphi \in \mathrm{Ham}(M,\omega)$ has only non-degenerate fixed points then*

$$\#\,\mathrm{Fix}(\varphi) \geq \sum_k \mathrm{rank}\ H^k(M;\mathbb{Q}).$$

More precisely, we can find that the number of fixed points which correspond to contractible 1-periodic orbits of any based Hamiltonian path is at least the sum of Betti numbers in this theorem. As a consequence, we also find that there always exists a contractible 1-periodic orbit for any time-dependent periodic Hamiltonian system. For a based path $\{\psi_t\}$ in $\mathrm{Symp}_0(M,\omega)$ we can define the Floer cohomology, which we may call the Floer–Novikov cohomology, in a way similar to the case of Hamiltonian diffeomorphisms. Under the $\pm$-monotonicity assumption we have a similar computation for $\varphi \in \mathrm{Symp}_0(M,\omega)$ using Novikov cohomology of the flux of φ_t in place of the ordinary cohomology of M, see [27].

2.2. Application to the flux conjecture. The flux of a based path $\{\phi_t\}_{0\leq t\leq 1}$ in $\mathrm{Symp}_0(M,\omega)$ is defined to be

$$\widetilde{\mathrm{Flux}}(\phi_t) = \int_0^1 [i(X_t)\omega]\,dt \in H^1(M;\mathbb{R}),$$

where X_t is the family of symplectic vector fields generating ϕ_t. The flux depends only on the homotopy class of paths with fixed end points $\phi_0 = \mathrm{id}$ and ϕ_1, and induces a homomorphism from the universal covering group $\widetilde{\mathrm{Symp}}_0(M,\omega)$ of $\mathrm{Symp}_0(M,\omega)$ to $H^1(M;\mathbb{R})$. Denote by Γ_ω, which is called the flux group, the image of $\mathrm{Ker}\left(\widetilde{\mathrm{Symp}}_0(M;\omega) \to \mathrm{Symp}_0(M;\mathbb{R})\right) \cong \pi_1(\mathrm{Symp}_0(M;\mathbb{R}))$ under $\widetilde{\mathrm{Flux}}$. It is known that the path ϕ_t above can be homotoped to a path in $\mathrm{Ham}(M;\omega)$ keeping the end points fixed if and only if $\widetilde{\mathrm{Flux}}(\phi_t) = 0$. $\widetilde{\mathrm{Flux}}$ descends to a homomorphism $\mathrm{Flux}\colon \mathrm{Symp}_0(M;\omega) \to H^1(M;\mathbb{R})/\Gamma_\omega$. The group $\mathrm{Ham}(M;\mathbb{R})$ is also known to be the kernel of this homomorphism, see [1]. Hence, it is a basic question in order to understand $\mathrm{Ham}(M,\omega) \subset \mathrm{Symp}_0(M,\omega)$ how Γ_ω is embedded in $H^1(M;\mathbb{R})$.

Conjecture 2.3 (Flux conjecture). Γ_ω is discrete in $H^1(M;\mathbb{R})$.

This conjecture is equivalent to that $\mathrm{Ham}(M,\omega)$ is C^1-closed in $\mathrm{Symp}_0(M,\omega)$. There are various cases in which the flux conjecture is verified. For example, if $[\omega] \in H^1(M;\mathbb{Q})$ the conjecture clearly holds. A less trivial case is that (M,ω) is of Lefschetz type, i.e., $\wedge[\omega]^{n-1}\colon H^1(M;\mathbb{R}) \to H^{2n-1}(M;\mathbb{R})$ is an isomorphism. It was Lalonde, McDuff and Polterovich [25], [26] who noticed that the affirmative answer to the homological Arnold conjecture can be used to prove the flux conjecture. Among other things, they proved the following:

Theorem 2.4. *If $c_1(M)\colon \pi_2(M) \to \mathbb{Z}$ is trivial or its minimal positive value (the minimal Chern number) is at least $2n = \dim_{\mathbb{R}} M$, then the flux conjecture holds.*

Theorem 2.5. *The rank of the flux group Γ_ω is at most the first Betti number $b_1(M)$ of M. In particular, the flux conjecture holds if $b_1(M) = 1$.*

Remark 2.6. Note that Theorem 2.5 follows from the flux conjecture.

We give an outline of the proof of the flux conjecture. First of all, we collect some notation and fundamental properties of the Floer–Novikov cohomology. For any based path $\{\psi_t\}$ in $\mathrm{Symp}_0(M, \omega)$, we can deform it by a homotopy so that $i(X_t)\omega$ does not depend on t and is equal to $\theta = \widetilde{\mathrm{Flux}}(\psi_t)$. Here X_t is the family of symplectic vector fields generating ψ_t. Denote by $\pi : \overline{M} \to M$ the covering space of M associated to the homomorphism $I_\theta : \pi_1(M) \to \mathbb{R}$ obtained by integrating θ along loops. Then there exists $\widetilde{H} = \{\tilde{h}_t\}$, a smooth family of smooth functions on $\overline{M}$ such that $\pi^* i(X_t)\omega = d\tilde{h}_t$. Denote by $\tilde{L}^\theta M$ the Floer covering space of LM for $\{\psi_t\}$ which depends only on its flux θ, and by $G_{\omega,\theta}$ its covering transformation group. Then we can perform the Floer construction for $\mathcal{A}_{\widetilde{H}} : \tilde{L}^\theta M \to \mathbb{R}$ and obtain the cochain complex $(CFN^*(\widetilde{H}, J), \delta = \delta^{\widetilde{H},J})$. The group $G_{\omega,\theta}$ naturally acts on this complex. Moreover, the action extends to the Novikov completion $\Lambda_{\omega,\theta}$ of the group ring of $G_{\omega,\theta}$. Denote by $HFN^*(\{\psi_t\})$ the resulting cohomology, which is the Floer–Novikov cohomology of $\{\psi_t\}$ and which is a finitely generated module over $\Lambda_{\omega,\theta}$.

We collect its fundamental properties as follows.

Theorem 2.7. *For based paths $\{\psi_t^{(1)}\}$ and $\{\psi_t^{(2)}\}$ with $\widetilde{\mathrm{Flux}}(\{\psi_t^{(1)}\}) = \widetilde{\mathrm{Flux}}(\{\psi_t^{(2)}\})$ we have a natural isomorphism*

$$HFN^*(\{\psi_t^{(1)}\}) \cong HFN^*(\{\psi_t^{(2)}\}).$$

Theorem 2.8. *If $\widetilde{\mathrm{Flux}}(\{\psi_t\})$ is sufficiently small we have*

$$HFN^*(\{\psi_t\}) \cong HN^{*+n}(\theta) \otimes_{\overline{\Lambda}_\theta} \Lambda_{\omega,\theta}.$$

Here $HN^*(\theta)$ is the Novikov cohomology of θ and $\overline{\Lambda}_\theta$ is its coefficient ring.

Secondly, we note that the Floer construction can be performed with coefficients in a local system as in the ordinary cohomology theory, see e.g., [38], [39]. In particular, when the flux vanishes, i.e. $\{\psi_t\}$ is a Hamiltonian path, we obtain the Floer cohomology for based Hamiltonian paths with coefficients in a local system. Let $L \to M$ be a local system or a flat vector bundle. We denote by $HFN^*(\{\psi_t\}; L)$ the Floer–Novikov cohomology of $\{\psi_t\}$ with coefficients in L. Then Theorems 2.7 and 2.8 holds with coefficients in L. We state them for reference.

Theorem 2.9. *Let $L \to M$ be a flat vector bundle. For based paths $\{\psi_t^{(1)}\}$ and $\{\psi_t^{(2)}\}$ with $\widetilde{\mathrm{Flux}}(\{\psi_t^{(1)}\}) = \widetilde{\mathrm{Flux}}(\{\psi_t^{(2)}\})$ we have a natural isomorphism*

$$HFN^*(\{\psi_t^{(1)}\}; L) \cong HFN^*(\{\psi_t^{(2)}\}; L).$$

Theorem 2.10. *If* $\widetilde{\mathrm{Flux}}(\{\psi_t\})$ *is sufficiently small we have*

$$HFN^*(\{\psi_t\}; L) \cong HN^{*+n}(\theta; L) \otimes_{\bar{\Lambda}_\theta} \Lambda_{\omega,\theta}$$

for any flat vector bundle L. *Here* $HN^*(\theta; L)$ *is the Novikov cohomology of* θ *with coefficients in* L.

Based on the above preparation, we give an outline of the proof of the flux conjecture. Let $U \subset H^1(M; \mathbb{R})$ be a neighborhood of the origin consisting of $\theta \in U$, which is represented by a sufficiently C^1-small closed 1-form such that Theorems 2.8 and 2.10 holds for the flux $[\theta]$. We may assume that U is symmetric with respect to the origin. It is enough to show the following.

Claim. $\Gamma_\omega \cap U = \{0\}$.

If it is false, there is a based loop $\{\psi_t\}$ in $\mathrm{Symp}_0(M, \omega)$ such that $\theta = \widetilde{\mathrm{Flux}}(\{\psi_t\})$ belongs to $(\Gamma_\omega \cap U) \setminus \{0\}$. Denote by $\{\psi_t^{-\theta}\}$ the based symplectic isotopy generated by the vector field $X_{-\theta}$ which is the symplectic dual of $-\theta$. Then $\{\psi_t' = \psi_t^{-\theta} \circ \psi_t\}$ is a based symplectic isotopy, the flux of which vanishes. Hence, we can deform $\{\psi_t'\}$ up to homotopy keeping end points fixed to a Hamiltonian path $\{\phi_t\}$. Thus we obtain a based Hamiltonian path $\{\phi_t\}$ and a based symplectic path $\{\psi_t^{-\theta}\}$ with $\psi_1^{-\theta} = \phi_1$. Since $\psi_t^\theta = (\psi_t^{-\theta})^{-1}$, $\Phi_t = \phi_t \circ \psi_t^\theta$ is a based loop in $\mathrm{Symp}_0(M, \omega)$, which induces an isomorphism $\Phi\colon \gamma(t) \in LM \mapsto \Phi_t(\gamma(t)) \in LM$. It is clear that Φ restricts to one-to-one correspondence between 1-periodic orbits of $\{\phi_t^{-\theta}\}$ and 1-periodic orbits of $\{\phi_t\}$. Note that the former are constant loops at zeros of θ, since we assumed that θ is sufficiently C^1-small. On the other hand, Theorem 2.2 guarantees the existence of contractible 1-periodic orbits of $\{\phi_t\}$ as we noted there. Hence, Φ preserves the component of LM consisting of contractible loops. We have the following:

Lemma 2.11. $\Phi^*\alpha_{\{\phi_t\}} = \alpha_{\{\psi_t^{-\theta}\}}$.

As a consequence, we find that $\Phi\colon LM \to LM$ admits a lift $\tilde{\Phi}\colon \tilde{L}^{-\theta}M \to \tilde{L}^0M$. Note also that Φ_t preserves the homotopy class of almost complex structures compatible with ω, hence $c_1(M)(u) = c_1(M)[\Phi_\#(u)]$. Here $u\colon S^1 \times S^1 \to M$ and $\Phi_\#(u)(s,t) = \Phi_t(u(s,t))$. Therefore $\tilde{\Phi}$ induces an isomorphism between the Floer–Novikov cohomology of $\{\psi_t^{-\theta}\}$ and the Floer cohomology of $\{\phi_t\}$. (Φ also induces an isomorphism between the moduli spaces of gradient trajectories in the sense of Kuranishi structures, after choosing almost compatible structures appropriately.) We can also see that $\tilde{\Phi}$ induces an isomorphism between the Novikov rings $\Lambda_{\omega,-\theta}$ and $\Lambda_\omega = \Lambda_{\omega,0}$. Namely, we find

Proposition 2.12. *Let* $L \to M$ *be an arbitrary flat vector bundle. Then there exists* $c \in \mathbb{Z}$ *such that*

$$\tilde{\Phi}_*\colon HFN^*(\{\psi_t^{-\theta}\}; L) \cong HF^{*+c}(\{\phi_t\}; L).$$

Since $-\theta$ is sufficiently C^1-small, Theorem 2.10 implies that

$$HFN^*(\{\psi_t^{-\theta}\}; L) \cong HN^{*+n}(-\theta; L) \otimes_{\overline{\Lambda}_{-\theta}} \Lambda_{\omega,-\theta}.$$

On the other hand, we have

$$HF^*(\{\phi_t\}; L) \cong H^{*+n}(M; L) \otimes \Lambda_\omega.$$

Now we choose $L \to M$ as the flat real line bundle $L_{\varepsilon\theta}$ associated to $\ell \in \pi_1(M) \mapsto \exp(\varepsilon \int \ell^*\theta) \in \mathbb{R}^*$. Then the pull back $\pi^* L_{\varepsilon\theta}$ of $L_{\varepsilon\theta}$ by $\pi\colon \overline{M} \to M$, which is used to define the Novikov cohomology of $\pm\theta$, becomes trivial as a flat bundle. Hence $HN^*(-\theta; L_{\varepsilon\theta})$ is isomorphic to $HN^*(-\theta; \mathbb{R})$ after forgetting the module structure over the Novikov ring. On the other hand, for ordinary cohomology, we have the jumping phenomenon at $\varepsilon = 0$, i.e., since θ is not an exact 1-form, $H^0(M; L_{\varepsilon\theta}) = 0$ for $\varepsilon \neq 0$ while $H^0(M; \mathbb{R}) = \mathbb{R}$ for $\varepsilon = 0$. Based on this observation, we can derive a contradiction. Hence the flux conjecture is proved.

Theorem 2.13. *The flux conjecture holds for any closed symplectic manifolds.*

Remark 2.14. The action of Hamiltonian loops on Floer cohomologies was studied by Seidel [44]. Viterbo [47] developed the theory of generating functions and explored applications to symplectic invariants. Y. G. Oh is the first to apply the Floer theoretical framework to Hofer's geometry [34], [35], partly inspired by the work of Chekanov [3] to be mentioned later. Seidel's work also stimulated progress in the study of Hofer's geometry, e.g., Entov's work [7] and Schwarz [42]. Oh generalized Schwarz's result to closed symplectic manifolds which are not necessarily symplectically aspherical, cf. Oh's contribution to this proceedings. Based on this generalization, Entov and Polterovich constructed in [8] an $\mathbb{R}$-valued quasi-homomorphism from (the universal covering group of) $\mathrm{Ham}(M, \omega)$.

There are different kinds of development from those mentioned in this section. For example, Viterbo applied the Floer cohomology to a problem in real algebraic geometry and proved that hyperbolic manifolds cannot be realized as the real part of "sufficiently positively curved" complex projective manifolds; cf. [22].

3. Floer theory for Lagrangian submanifolds

3.1. Fundamental construction. Let L_0, L_1 be closed embedded Lagrangian submanifolds in a closed symplectic manifold (P, Ω). We assume that L_0 and L_1 intersect transversely. Consider the path space $\mathcal{P}(L_0, L_1) = \{\gamma\colon [0,1] \to P \mid \gamma(0) \in L_0,\ \gamma(1) \in L_1\}$ and define the action 1-form $\alpha = \alpha_{L_0,L_1}$ by

$$\alpha_{L_0,L_1}(\xi) = \int_0^1 \Omega(\xi(t), \dot\gamma(t))\, dt \quad \text{for } \xi = \{\xi(t)\} \in T_\gamma \mathcal{P}(L_0, L_1).$$

Then α_{L_0,L_1} is a "closed 1-form". In fact, a local primitive function around γ_0 is given by

$$\mathcal{A}^{\text{loc}}_{L_0,L_1}(\gamma) = \int_{[0,1]\times[0,1]} w^*\Omega,$$

where $w\colon [0,1]\times[0,1] \to P$ such that $w(s,i) \in L_i$ for $i = 0, 1$, $w(0,t) = \gamma_0(t)$ and $w(1,t) = \gamma(t)$. As long as the image of w is contained in a small neighborhood of γ_0, $\mathcal{A}^{\text{loc}}_{L_0,L_1}$ is well defined.

Before going further, we clarify the relation to the case of symplectomorphisms. Let ϕ be a symplectomorphism of (M,ω). Then its graph Γ_ϕ is a Lagrangian submanifold in $(M\times M, -\omega\oplus\omega)$. Denote by Δ the diagonal subset, which is the graph of the identity. Then we have the following identification:

$$G\colon L^\phi M \to \mathcal{P}(\Gamma_\phi,\Delta); \sigma(t) \mapsto \left(\sigma\left(1-\frac{t}{2}\right), \sigma\left(\frac{t}{2}\right)\right),$$

which satisfies $G^*\alpha_{\Gamma_\phi,\Delta} = \alpha_\phi$. In this way, the construction in this section is a generalization of the one in the previous section.

Pick a compatible almost complex structure J to equip $\mathcal{P}(L_0,L_1)$ with L^2-metric. Then the locally gradient flow line for α_{L_0,L_1} is described by $u\colon \mathbb{R}\times[0,1] \to P$ with $u(\tau,i) \in L_i$ for $i = 0, 1$, which satisfies

$$\frac{\partial u}{\partial \tau} + J(u)\frac{\partial u}{\partial t} = 0.$$

Existence of the limits $\lim_{\tau\to\pm\infty} u(\tau,t) \in L_0\cap L_1$ is equivalent to the condition that the energy $E(u)$ is finite. Note also that the zeros of α_{L_0,L_1} are exactly the constant paths at $L_0\cap L_1$.

In [9]–[13], Floer realized the idea of constructing an analogue of Morse complex for the action functional under the assumption that $\pi_2(P,L_i) = 0$ and that L_1 is a Hamiltonian deformation of L_0. In this situation the action functional admits a primitive function on $\mathcal{P}(L_0,L_1)$ and the grading of $L_0\cap L_1$, called the Maslov–Viterbo index $\mu = \mu_{L_0,L_1}$, is well defined with values in $\mathbb{Z}$. Define $CF^*(L_1,L_0)$ by the $\mathbb{Z}/2\mathbb{Z}$-module freely generated by $L_0\cap L_1$. Counting gradient flow lines connecting critical points of $\mathcal{A}_{L_0,L_1}$, we define the coboundary operator $\delta\colon CF^*(L_1,L_0) \to CF^{*+1}(L_1,L_0)$ by

$$\delta\langle p\rangle = \sum \#\mathcal{M}(p,q)\langle q\rangle,$$

where q runs over $L_0\cap L_1$ such that $\mu(q) = \mu(p)+1$, and $\mathcal{M}(p,q)$ is the moduli space of gradient flow lines, which we call connecting orbits, of $\mathcal{A}_{L_0,L_1}$ from p to q. Under the above assumption, for a generic choice of J, the moduli space $\mathcal{M}(p,q)$ is shown to be compact if $\mu(q)-\mu(p) = 1$. If $\mu(q)-\mu(p) = 2$, $\mathcal{M}(p,q)$ may not be compact, but its end is described as the union of $\mathcal{M}(p,r)\times\mathcal{M}(r,q)$ over $r \in L_0\cap L_1$

such that $\mu(r) - \mu(p) = 1$. Hence we find that $\delta \circ \delta = 0$ and obtain the Floer complex $(CF^*(L_1, L_0), \delta)$. We denote by $HF^*(L_1, L_0)$ the resulting cohomology. It is also shown that the Floer cohomology is invariant under Hamiltonian deformation of Lagrangian submanifolds. If L_1 is a sufficiently small Hamiltonian deformation of L_0, L_1 is regarded as the graph of a C^2-small Morse function on L_0 in T^*L_0. The Morse gradient trajectories appear as Floer connecting orbits. Although, in general, there may exist non-small Floer connecting orbits, the assumption that $\pi_2(P, L_i) = 0$ excludes such a possibility. Hence $HF^*(L_1, L_0)$ is isomorphic to $H^*(L_0; \mathbb{Z}/2\mathbb{Z})$ up to a shift in the grading. It is worth mentioning that Hofer [19], [20] developed an idea similar to Floer's and established the Lagrangian intersection property under the assumption that $\pi_2(P, L) = 0$.

Without the assumption that $\pi_2(P, L_i) = 0$, there arise some problems in the above argument. As we explain below, $\delta \circ \delta$ may not vanish[1], in general. It was Y. G. Oh [30], [31], [32] who extended Floer's construction to the case that L_i are monotone and their minimal Maslov number is at least 3. (He also computed Floer cohomology for some cases, e.g., $\mathbb{R}P^n \subset \mathbb{C}P^n$.) In general, the difficulties are caused by J-holomorphic discs with boundary on L_i as well as J-holomorphic spheres which arise as "bubbles" from sequences of connecting orbits with bounded energies. As in the case of symplectomorphisms, the bubbling-off of J-holomorphic spheres is expected to occur in real codimension 2 and does not cause any essential difficulty, which can be handled by Kuranishi structures. However, the bubbling-off of J-holomorphic discs occurs in real codimension 1 and we cannot avoid it, in general. If we restrict ourselves to some portion of $\mathcal{P}(L_0, L_1)$, on which the range of the action functional is sufficiently narrow, then there do not appear effects from J-holomorphic discs and J-holomorphic spheres. In fact, Chekanov [3] gave an alternative proof for the non-degeneracy of Hofer's distance on $\mathrm{Ham}(M, \omega)$ based on such an idea.

As we noticed, the bubbling-off of J-holomorphic discs is a codimension 1 phenomenon, hence we cannot, in general, avoid such a bubbling-off phenomenon from the moduli space $\mathcal{M}(p, q)$ even though $\mu(q) - \mu(p) \leq 2$. In order to understand how $\delta \circ \delta = 0$ fails to hold, we study all J-holomorphic discs systematically. From now on we follow our joint work with K. Fukaya, Y. G. Oh and H. Ohta [15]. Firstly, we arrange elements of $\pi_2(P, L)$[2], which are represented as the union of J-holomorphic discs $w\colon (D^2, \partial D^2) \to (P, L)$ and J-holomorphic spheres $v\colon S^2 \to P$ as $\beta_0 = 0, \beta_1, \beta_2, \ldots$ such that $\int_{\beta_i} \Omega \leq \int_{\beta_{i+1}} \Omega$ and $\int_{\beta_i} \Omega \to +\infty$ as $i \to +\infty$. This can be done with the so-called Gromov weak compactness. Denote by $\mu(w)$ the Maslov index of $(w^*TP, w|_{\partial D^2})^*TL) \to (D^2, \partial D^2)$. Denote by $\mathcal{M}_{k+1}(\beta)$ the moduli space of J-holomorphic discs[3] which represent class β, with $k+1$ marked points on ∂D^2. Then the moduli space $\mathcal{M}_{k+1}(\beta)$ is of dimension $n + \mu(\beta) + k - 2$, where

[1] I heard from A. Sergeev that Floer himself had (certainly) noticed this fact. This phenomenon is not only a bad news. We used this fact in [37].

[2] More precisely, we work with the image of $\pi_2(P, L)$ in $H^2(P, L; \mathbb{Z})$.

[3] More precisely, we use the stable maps from the prestable Riemann surface with 1 boundary component of genus 0.

$n = \dim L$. In general, the transversality, i.e., the surjectivity of the linearization of the J-holomorphic curve equation, may not hold. In order to overcome this trouble, we use the framework of Kuranishi structure. Since we use the multi-valued perturbation technique, we need a compatible system of orientations on various moduli spaces. However, the moduli space of J-holomorphic discs may not be orientable[4], in general. Therefore we assume the relative spin condition for Lagrangian submanifolds as follows. Pick a triangulation of L and extend it to a triangulation of P.

Definition 3.1 (Relative spin structure). Let L be a Lagrangian submanifold. If there is a cohomology class $w \in H^2(P; \mathbb{Z}/2\mathbb{Z})$ such that $w_2(L)$ is the restriction of w to L, we call L relatively spin. Under this condition, there is an orientable vector bundle V on the 3-skeleton $P^{(3)}$ of P with $w_2(V) = w$. A relative spin structure for L is the tuple of w, V and a spin structure on the restriction of $TL \oplus V$ to $L \cap P^{(2)}$. A relative spin structure on (L_0, L_1) is the above tuple which is chosen in common for L_i, $i = 0, 1$.

Then we have the following:

Theorem 3.2. (1) *A relative spin structure on L determines a canonical orientation on the moduli spaces $\mathcal{M}_{k+1}(\beta)$, which satisfies a certain compatibility condition under the gluing operation.*

(2) *A relative spin structure on (L_0, L_1) determines a canonical orientation on the moduli spaces $\mathcal{M}(p, q)$ of connecting orbits, which satisfies a certain compatibility condition under the gluing operation.*

From now on, we assume that a Lagrangian submanifold L or a pair (L_0, L_1) of Lagrangian submanifolds are equipped with a relative spin structure. We work with $\mathbb{Q}$-coefficients rather than $\mathbb{Z}/2\mathbb{Z}$-coefficients. Clearly, a spin structure on L gives a relative spin structure with a trivial bundle V.

We define obstruction classes for L to define Floer cohomology by inductive steps as follows[5]. Start with β_1, the first non-trivial case. Since the bubbling-off does not happen in $\mathcal{M}_1(\beta_1)$, the evaluation map $ev_0 \colon \mathcal{M}_1(\beta_1) \to L$ is a cycle with $\mathbb{Q}$-coefficients. This cycle represents the first obstruction class[6] $o_1 = o(\beta_1)$. Suppose that $o_i = o(\beta_i)$ is defined for $i = 1, \ldots, k$ and there exist $\mathbb{Q}$-chains $\mathcal{B}_i$ in L such that $o_i = (-1)^n \partial \mathcal{B}_i$ for $i = 1, \ldots, k$. (We call such a system of $\mathcal{B}_i$, $i = 1, 2, \ldots$ a bounding chain.) We define the next obstruction class $o_{k+1} = o(\beta_{k+1})$ as follows. The moduli space $\mathcal{M}_{k+1}(\beta)$ may have codimension 1 boundary, hence may not be a cycle. So we try to glue other (moduli) spaces along boundaries so that we finally obtain a cycle. Consider the moduli space $\mathcal{M}_{\ell+1}(\beta; \mathcal{B}_{i_1}, \ldots, \mathcal{B}_{i_\ell})$ consisting of J-holomorphic discs w representing the class β such that $\beta_{k+1} = \beta + \sum_{j=1}^{\ell} \beta_{i_j}$ and intersecting $\mathcal{B}_{i_1}, \ldots, \mathcal{B}_{i_\ell}$ along ∂D^2. The moduli space $\mathcal{M}_{\ell+1}(\beta; \mathcal{B}_{i_1}, \ldots, \mathcal{B}_{i_\ell})$ is

[4] Vin de Silva independently studied this problem in [5].

[5] The idea of this construction was inspired by Kontsevich around 1997.

[6] In the next subsection we adopt cohomological convention. Thus we take the Poincaré dual of o_k.

described as the fiber product of the spaces with Kuranishi structures:

$$\mathcal{M}_{\ell+1}(\beta)\ {}_{ev_1,\dots,ev_\ell} \times \prod_{j=1}^{\ell} \mathcal{B}_{i_j}.$$

We can assign to them an orientation so that the union $\widehat{\mathcal{M}}_1(\beta_{k+1})$ of $\mathcal{M}_1(\beta_{k+1})$ and all possible $\mathcal{M}_{\ell+1}(\beta; \mathcal{B}_{i_1}, \dots, \mathcal{B}_{i_\ell})$ becomes a $\mathbb{Q}$-virtual cycle. Note that we have the evaluation map $ev_0\colon \mathcal{M}_{\ell+1}(\beta; P_{i_1}, \dots, P_{i_\ell}) \to L$ at the remaining marked point after taking the fiber product. Then $ev_0\colon \widehat{\mathcal{M}}_1(\beta_{k+1}) \to L$ is a $\mathbb{Q}$-cycle of L, which represents the obstruction class $o_{k+1} = o(\beta_{k+1})$. Then we can find the following:

Theorem 3.3. *Suppose that a pair (L_0, L_1) of Lagrangian submanifolds is equipped with a relative spin structure. If all the obstruction classes for L_i, $i = 0, 1$ are defined and vanish, then we can revise the definition of Floer's coboundary operator to obtain the Floer complex $(CF^*(L_1, L_0), \delta)$. Moreover, the Floer cohomology $HF^*(L_1, L_0)$ is invariant under Hamiltonian deformation of L_i.*

Our construction depends on the choice of bounding chains for L_i, $i = 0, 1$. The invariance under Hamiltonian deformations also requires a subtle argument. Namely, we must describe the relation of bounding chains under Hamiltonian deformation. These points are clarified in terms of the filtered A_∞-algebras associated to L_i, which we discuss in the next subsection. We may weaken the assumption that the obstruction classes vanish. One of them is the deformation using $\mathbb{Q}$-cycles in P. It may also happen that the effects of J-holomorphic discs with boundary on L_i, $i = 0, 1$ cancel each other. When all non-vanishing obstruction classes for L_i are of top dimension, i.e., $\dim L$, then they are multiples of the fundamental class of L. We call the coefficient of the fundamental cycle as the potential function of L_i. If the potential function of L_i, $i = 0, 1$, coincide, they cancel each other in the construction of the Floer complex, hence the Floer cohomology. This is an extension of Oh's discovery that the Floer complex can be constructed for monotone Lagrangian submanifolds with minimal Maslov numbers are at least 2. Although we can define the Floer complex, hence the Floer cohomology under the assumption that all obstruction classes vanish, it is very difficult to compute it in general.

However, when L_1 is a Hamiltonian deformation of L_0, we can construct a certain spectral sequence with E_2-term being the ordinary cohomology with coefficients in the Novikov ring, which converges to the Floer cohomology, see Theorem 3.10 below.

3.2. The filtered A_∞-algebras associated to Lagrangian submanifolds. Based on [15], we describe the framework of the Floer theory for Lagrangian submanifolds. We generalize the idea of the construction of obstruction classes, which we mentioned in the previous subsection, and construct the filtered A_∞-algebras associated to Lagrangian submanifolds. We also include some applications at the end of this subsection.

We introduce the universal Novikov ring which we use from now on. Let R be a commutative ring with the unit. In this note, we mostly use the case that $R = \mathbb{Q}$. Let T and e be formal generators of degree 0 and 2, respectively. Set

$$\Lambda_{\text{nov}} = \Big\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} e^{\mu_i} \mid a_i \in R,\ \lambda_i \in \mathbb{R},\ \mu_i \in \mathbb{Z},\ \lim_{i\to\infty} \lambda_i = \infty \Big\}.$$

If R is a field, the degree 0-part of Λ_{nov} is also a field. We also set

$$\Lambda_{0,\text{nov}} = \Big\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} e^{\mu_i} \in \Lambda_{\text{nov}} \mid \lambda_i \geq 0 \Big\}.$$

These rings Λ_{nov} and $\Lambda_{0,\text{nov}}$ are complete with respect to the decreasing filtration by λ for T^{λ}. The Novikov rings, we mentioned before, are subrings of Λ_{nov}.

Now we shall present a rough idea of the construction of the A_∞-operations. Let $(f_i \colon P_i \to L)$, $i = 1, \dots, k$, be chains in L. We often abbreviate them as P_i. Take the fiber product

$$\mathcal{M}_{k+1}(\beta; P_1, \dots, P_k) = \mathcal{M}_{k+1}(\beta)_{ev_1,\dots,ev_k} \times_{f_1,\dots,f_k} \prod_{j=1}^{k} P_j.$$

We can give an orientation to these spaces with Kuranishi structure in such a way that the following construction works. Define a chain $(\mathcal{M}_{k+1}(\beta; P_1, \dots, P_k), ev_0)$ in L by taking the remaining marked point, i.e., $ev_0 \colon \mathcal{M}_{k+1}(\beta; P_1, \dots, P_k) \to L$. For $k \geq 2$, we set

$$\mathfrak{m}_{k,\beta}(P_1, \dots, P_k) = (\mathcal{M}_{k+1}(\beta; P_1, \dots, P_k), ev_0).$$

In the other cases, we set

$$\begin{aligned} \mathfrak{m}_{1,0}(P) &= (-1)^n \partial P, \\ \mathfrak{m}_{1,\beta}(P) &= (\mathcal{M}_2(\beta; P), ev_0), \quad \text{when } \beta \neq 0 \\ \mathfrak{m}_{0,\beta}(1) &= (\mathcal{M}_1(\beta), ev_0), \qquad \text{when } \beta \neq 0. \end{aligned}$$

In the last line, 1 is the unit of $R \subset \Lambda_{\text{nov}}$, which is regarded as an element in $B_0C(L; \Lambda_{0,\text{nov}})[1]$ below. We also set $\mathfrak{m}_{0,0}(1) = 0$. If we study the structure of compactifications of the moduli spaces $\mathcal{M}_{k+1}(\beta; P_1, \dots, P_k)$ in a heuristic way, we expect to obtain certain algebraic relations among these operations, the so-called A_∞-relations. However, when we perform this construction in a rigorous way, we encounter several problems, e.g., transversality of the moduli spaces, transversality for taking the fiber product, etc. So we have to clarify which class of chains of L we deal with and how to take the (multi-valued) perturbation for achieving transversality, etc. Here, we give some flavor of the argument. For details see [15].

First of all, we forget the effect of non-trivial J-holomorphic discs and consider only the contribution from $\beta = 0$ (classical case). Naively, $\mathfrak{m}_{2,0}(P_1, P_2)$ should be $P_1 \cap P_2$ up to sign. However, when $P_1 = P_2$, the transversality does not hold. Thus we are forced to perturb $\mathcal{M}_{3,0}(0; P_1, P_2)$ to define $\mathfrak{m}_{2,0}(P_1, P_2)$. (It is also necessary to take a suitable countable family of chains which spans a subcomplex of the chain complex of L. We also assume that its cohomology is isomorphic to the ordinary cohomology of L.) It causes a discrepancy between $\mathfrak{m}_{2,0}(\mathfrak{m}_{2,0}(P_1, P_2), P_3)$ and $\mathfrak{m}_{2,0}(P_1, \mathfrak{m}_{2,0}(P_2, P_3))$. Namely, $\mathfrak{m}_{2,0}$ does not satisfy the associativity. Nevertheless, the above discrepancy is described using $\mathfrak{m}_{3,0}(P_1, P_2, P_3)$, which is defined by the perturbation of $\mathcal{M}_{4,0}(0; P_1, P_2, P_3)$, as follows.

$$\begin{aligned}
&\mathfrak{m}_{2,0}(\mathfrak{m}_{2,0}(P_1, P_2), P_3) - (-1)^{\deg P_1}\mathfrak{m}_{2,0}(P_1, \mathfrak{m}_{2,0}(P_2, P_3)) \\
&\quad = -\{\mathfrak{m}_{1,0} \circ \mathfrak{m}_{3,0}(P_1, P_2, P_3) + \mathfrak{m}_{3,0}(\mathfrak{m}_{1,0}(P_1), P_2, P_3) \\
&\quad\quad - (-1)^{\deg P_1}\mathfrak{m}_{3,0}(P_1, \mathfrak{m}_{1,0}(P_2), P_3) \\
&\quad\quad + (-1)^{\deg P_1 + \deg P_2}(\mathfrak{m}_{3,0}(P_1, P_2, \mathfrak{m}_{1,0}(P_3))\}.
\end{aligned}$$

Here we define the degree of P by $\deg P = n - \dim P$ and work with the cohomological framework rather than the homological framework from now on. A series of similar formulae successively hold in higher order. We call these relations the A_∞-relations. We can show that this algebraic gadget, the A_∞-algebra, obtained by the chain level intersection theory is “equivalent” to the de Rham homotopy theory in the realm of A_∞-algebras.

Next we include the effect from non-trivial J-holomorphic discs. Then we first take a suitable countably generated subcomplex $C^*(L)$ of the (co)chain complex[7] and (multi-valued) perturbations of the moduli spaces $\mathcal{M}_{k+1}(\beta; P_1, \ldots, P_k)$ to define $\mathfrak{m}_{k,\beta}(P_1, \ldots, P_k)$. We assign the degree to $P \otimes T^\lambda e^\mu \in C^*(L; \Lambda_{\mathrm{nov}})$ by $\deg P + 2\mu$. We shift the degree as $C(L; \Lambda_{\mathrm{nov}})[1]^* = C^{*+1}(L; \Lambda_{\mathrm{nov}})$. Then we can easily see that

$$\mathfrak{m}_{k,\beta} \otimes T^{\int_\beta \Omega} e^{\mu(\beta)/2} \colon \bigotimes^k C(L; \Lambda_{\mathrm{nov}})[1]^* \to C(L; \Lambda_{\mathrm{nov}})[1]^*$$

shifts the degree by $+1$, in other words, they are operations of degree $+1$. Write

$$\mathfrak{m}_k = \sum_\beta \mathfrak{m}_{k,\beta} \otimes T^{\int_\beta} e^{\mu(\beta)/2}.$$

Write

$$BC[1]^* = \bigoplus_{k=0}^{\infty} B_k C[1]^*$$

and

$$B_k C[1]^* = B_k C(L; \Lambda_{0,\mathrm{nov}})[1]^* = \bigotimes^k C(L; \Lambda_{0,\mathrm{nov}})[1]^*,$$

[7] More precisely, we consider the quotient complex by identifying chains, which give the same current.

the bar construction of $C^* = C^*(L; \Lambda_{0,\mathrm{nov}})$. It is a free tensor coalgebra generated by the graded free module $C[1]^*$. We extend $\mathfrak{m}_k$ to $\widehat{\mathfrak{m}}_k \colon BC[1]^* \to BC[1]^*$ as a coderivation and define $\hat{d} = \sum \widehat{\mathfrak{m}}_k$. Then we find the following:

Theorem 3.4. $\hat{d} \circ \hat{d} = 0$.

The filtered A_∞-relations are the formulae which express the above equality in terms of $\mathfrak{m}_k$. We call $(BC(L; \Lambda_{0,\mathrm{nov}})[1], \hat{d})$ the filtered A_∞-algebra associated to the Lagrangian submanifold L. So far, this object depends on the choice of the compatible almost complex structure, the countably generated subcomplex $C(L)$, various (multi-valued) perturbations, etc. We can define the notion of (gapped filtered) A_∞-algebra morphisms, homotopy equivalences, homotopy units, etc., and find the following:

Theorem 3.5. (1) *The homotopy type of the filtered A_∞-algebra*

$$(BC(L; \Lambda_{0,\mathrm{nov}})[1], \hat{d})$$

depends only on the embedding of the Lagrangian submanifold $L \subset (P, \Omega)$. The fundamental cycle of L is a homotopy unit.

(2) *A symplectomorphism ψ of (P, Ω) induces a homotopy equivalence $\hat{\psi}$ between the filtered A_∞-algebras associated to L and $\psi(L)$.*

In fact, by the algebraic theory of the (filtered) A_∞-algebras, we can derive the A_∞-algebra structure, resp. the filtered A_∞-algebra structure on $H^*(L)$, resp. $H^*(L; \Lambda_{0,\mathrm{nov}})$. One of the advantages to work in the framework of (filtered) A_∞-algebras is that quasi-isomorphisms have homotopy inverses[8]. This is not true in the category of differential graded algebras.

In general, $\mathfrak{m}_0(1)$ may not vanish. From the A_∞-relation we have

$$\mathfrak{m}_1 \circ \mathfrak{m}_1(P) = -(\mathfrak{m}_2(\mathfrak{m}_0(1), P) + (-1)^{\deg P+1}\mathfrak{m}_2(P, \mathfrak{m}_0(1))),$$

which means that $\mathfrak{m}_1 \circ \mathfrak{m}_1$ does not necessarily vanish. This is the obstruction to define the Floer cohomology, which we discussed in the previous subsection.

Let $b \in C(L; \Lambda_{0,\mathrm{nov}})[1]^0$ with positive energy, i.e, b contains only terms with T^λ with $\lambda > 0$ and set

$$\mathfrak{m}_k^b(P_1, \dots, P_k) = \sum \mathfrak{m}_{k+\ell}(b, \dots, b, P_1, b, \dots, b, P_i, b, \dots, b, P_k, b, \dots, b),$$

where ℓ is the number of b's appearing above in all possible ways and the sum is taken over all possibilities. We define $\hat{d}^b$ using $\mathfrak{m}_k^b$ instead of $\mathfrak{m}_k$. Then $\hat{d}^b$ also satisfies the A_∞-relation $\hat{d}^b \circ \hat{d}^b = 0$. Write $e^b = 1 + b + b \otimes b + b \otimes b \otimes b + \cdots$. Then we find the following:

Theorem 3.6. *If there exists $b \in C(L; \Lambda_{0,\mathrm{nov}})[1]^0$ which satisfies $\hat{d}(e^b) = 0$, then we have $\mathfrak{m}_0^b(1) = 0$, hence $\mathfrak{m}_1^b \circ \mathfrak{m}_1^b = 0$.*

[8] We do not claim any priority in the unfiltered case.

We call the equation $\hat{d}(e^b) = 0$ the Maurer–Cartan equation for the filtered A_∞-algebra. If there is a solution b for the Maurer–Cartan equation, the complex $(C(L; \Lambda_{0,\mathrm{nov}}), \mathfrak{m}_1^b)$ and its extension $(C(L; \Lambda_{\mathrm{nov}}), \mathfrak{m}_1^b)$ are the Bott–Morse Floer complex in the case that $L = L_0 = L_1$. We denote the resulting cohomology groups by $HF^*((L, b); \Lambda_{0,\mathrm{nov}})$ and $HF^*((L, b); \Lambda_{\mathrm{nov}})$, respectively. For a bounding chain $\mathcal{B}_i$ in the previous subsection we set

$$b = \sum \mathcal{B}_i \otimes T^{\int_\beta \Omega} e^{\mu(\beta)/2}.$$

Then b is a solution of the Maurer–Cartan equation. There is a notion of the gauge equivalence relation among solutions of the Maurer–Cartan equation. We can see that the Floer cohomologies are isomorphic for gauge equivalent b and b'. Note that the filtered A_∞-morphism maps a solution of the Maurer–Cartan equation for the source to a solution of the Maurer–Cartan equation for the target. Hence, for a symplectomorphism ψ of (P, Ω), $\psi_*(b)$, the B_1-component of $\widehat{\psi}(e^b)$ is a solution of the Maurer–Cartan equation for $\psi(L)$ if b is a solution for L. With respect to them we have the following:

$$\widehat{\psi}\colon\ HF^*((L, b); \Lambda_{0,\mathrm{nov}}) \cong HF^*((\psi(L), \psi_*(b)); \Lambda_{0,\mathrm{nov}}).$$

Now we consider a pair (L_0, L_1) of Lagrangian submanifolds. By counting Floer connecting orbits intersecting k chains in L_1 and ℓ chains in L_0, we define the operation

$$\begin{aligned}\mathfrak{n}_{k,\ell}\colon\ B_k C(L_1; \Lambda_{0,\mathrm{nov}})[1] \otimes C(L_1, L_0; \Lambda_{0,\mathrm{nov}}) \otimes B_\ell C(L_0; \Lambda_{0,\mathrm{nov}})[1]\\ \longrightarrow C(L_1, L_0; \Lambda_{0,\mathrm{nov}}).\end{aligned}$$

Using the filtered A_∞-algebra structures on L_1 and L_0 as well as $\mathfrak{n}_{k,\ell}$, we obtain the coderivation $\hat{d}_{(L_1,L_0)}$ on

$$BC(L_1; \Lambda_{0,\mathrm{nov}})[1] \otimes C(L_1, L_0; \Lambda_{0,\mathrm{nov}}) \otimes BC(L_0; \Lambda_{0,\mathrm{nov}})[1].$$

We have $\hat{d}_{(L_1,L_0)} \circ \hat{d}_{(L_1,L_0)} = 0$. We call $(BC(L_1; \Lambda_{0,\mathrm{nov}})[1] \otimes C(L_1, L_0; \Lambda_{0,\mathrm{nov}}) \otimes BC(L_0; \Lambda_{0,\mathrm{nov}})[1], \hat{d}_{(L_1,L_0)})$ the filtered A_∞-bimodule associated to the pair (L_0, L_1). More precisely, we say that it is a left $C(L_1; \Lambda_{0,\mathrm{nov}})$, right $C(L_0; \Lambda_{0,\mathrm{nov}})$ filtered A_∞-bimodule. Similar to the case of the filtered A_∞-algebras, we obtain the following:

Theorem 3.7. (1) *For a pair (L_0, L_1) of Lagrangian submanifold equipped with a relative spin structure as a pair, the filtered A_∞-bimodule is uniquely defined up to homotopy equivalences.*

(2) *A pair of Hamiltonian diffeomorphisms ϕ_i, $i = 0, 1$, induces a homotopy equivalence between the filtered A_∞-bimodules with coefficients in Λ_{nov} of (L_0, L_1) and $(\phi_0(L_0), \phi_1(L_1))$.*

If there exists solutions b_i of the Maurer–Cartan equations for L_i, we can revise the Floer coboundary operator as follows:

$$\delta^{b_1,b_0}(p) = \sum \mathfrak{n}_{k,\ell}(b_1, \dots b_1, p, b_0, \dots, b_0).$$

Then we have the following:

Theorem 3.8. *Let b_i be solutions of the Maurer–Cartan equations for L_i, $i = 0, 1$. Then $\delta^{b_1,b_0} \circ \delta^{b_1,b_0} = 0$ holds.*

We denote the resulting cohomology by $HF^*((L_1, b_1), (L_0, b_0); \Lambda_{0,\mathrm{nov}})$ and its coefficient extension to Λ_{nov} by $HF^*((L_1, b_1), (L_0, b_0); \Lambda_{\mathrm{nov}})$ Then we have the following:

Corollary 3.9. *Let b_i be solutions of the Maurer–Cartan equation for L_i and ϕ_i Hamiltonian diffeomorphisms of $(P.\Omega)$. Then (ϕ_1, ϕ_0) induces an isomorphism*

$$HF^*((L_1.b_1), (L_0, b_0); \Lambda_{\mathrm{nov}}) \cong HF^*((\phi_1(L_1), \phi_{1*}(b_1)), (\phi_0(L_0), \phi_{0*}(b_0)); \Lambda_{\mathrm{nov}}).$$

In a similar way to the case of filtered A_∞-algebras, if b_i is gauge equivalent to b_i', $i = 0, 1$, the corresponding Floer cohomologies are isomorphic. Suppose that $\mathfrak{m}_0(1) = c[L]$ for some $c \in \Lambda_{0,\mathrm{nov}}$. We set $c = \mathfrak{PO}(L)$, the potential function. If $\mathfrak{PO}(L_0) = \mathfrak{PO}(L_1)$ we can modify the above construction to obtain the Floer complex. For example, if L_0 is a Lagrangian submanifold such that $\mathfrak{m}_0(1) = c[L_0]$ and L_1 is a Hamiltonian deformation of L_0, then we can obtain the Floer cohomology for (L_0, L_1).

It is not easy to compute the Floer cohomology $HF^*((L_1, b_1), (L_0, b_0))$, even when $L_1 = \phi(L_0)$ and $b_1 = \phi_*(b_0)$ for some Hamiltonian diffeomorphism ϕ. In such a case we find that it is isomorphic to the Bott–Morse Floer cohomology $HF^*((L_0, b_0); \Lambda_{\mathrm{nov}})$. Using the energy filtration, we have a spectral sequence as follows.

Theorem 3.10. *There is a spectral sequence with E_2-term being $H^*(L; \Lambda_{0,\mathrm{nov}})$ and converging to $HF^*((L_0, b_0); \Lambda_{0,\mathrm{nov}})$.*

We can also use a cycle in the ambient space P to deform the filtered A_∞-algebra associated to L. Pick a cycle $\boldsymbol{b}$ in P. Consider the moduli space of stable maps with one boundary component. In addition to the $k + 1$ boundary marked points put ℓ interior marked points. Take the fiber product

$$\mathcal{M}_{k+1,\ell}(\beta; P_1, \dots, P_k) = \mathcal{M}_{k+1,\ell}(\beta) \times_{\prod^k L \times \prod^\ell P} \Big(\prod_{i=1}^k P_i\Big) \times \Big(\prod^\ell \boldsymbol{b}\Big).$$

Summing up these moduli spaces for all ℓ, we obtain the deformed operation $\mathfrak{m}_k^{\boldsymbol{b}}$. The corresponding $\hat{d}^{\boldsymbol{b}}$ gives a deformation of the filtered A_∞-algebra structure. We can also discuss the Maurer–Cartan equation for the deformed structure, gauge equivalences, etc. Thanks to this larger class of deformations, we have the following:

Theorem 3.11. *Let L be a relatively spin Lagrangian submanifold. If the embedding $L \subset P$ induces an injection $H^*(L;\mathbb{Q}) \to H^*(P;\mathbb{Q})$, there is a $\Lambda^+_{0,\mathrm{nov}}$-cycle*[9] $\boldsymbol{b}$ *of P such that the deformed Maurer–Cartan solution $\hat{d}^{\boldsymbol{b}}(e^b) = 0$ has a solution.*

The following theorem is a direct consequence.

Theorem 3.12. *Let L be a relatively spin Lagrangian submanifold. Suppose that the embedding $L \subset P$ induces an injection on homology with rational coefficients. Then, for any Hamiltonian diffeomorphism ϕ of (P,Ω), we have*

$$\#L \cap \phi(L) \geq \sum_p \mathrm{rank}\ H^p(L;\mathbb{Q}).$$

Note that the graph of a Hamiltonian diffeomorphism satisfies the above assumption, hence Theorem 3.12 is a generalization of Theorem 2.2. Although the complete computation is difficult, there are cases where we have the non-vanishing result.

Theorem 3.13. *Let L be a relatively spin Lagrangian submanifold. Suppose that there is a $\Lambda^+_{0,\mathrm{nov}}$-cycle $\boldsymbol{b}$ in P and $b \in C(L;\Lambda_{0,\mathrm{nov}})[1]^0$ such that $\hat{d}^{\boldsymbol{b}}(e^b) = 0$. Suppose also that the Maslov index of any J-holomorphic disc with boundary on L is non-positive. Then, after adding correction terms which are of positive energy, the cycle $[pt]$ and the cyle $[L]$ become linearly independent, non-trivial cohomology classes in $HF^*((L,\mathfrak{b});\Lambda_{\mathrm{nov}})$.*

Here we denote by $\mathfrak{b}$ the pair $(\boldsymbol{b}, b)$. When the Maslov class μ vanishes for L, all obstruction classes belong to $H^2(L;\mathbb{Q})$. Hence we obtain the following:

Theorem 3.14. *Let L be a relatively spin Lagrangian submanifold with vanishing Maslov class such that $H^2(L;\mathbb{Q}) = 0$. Then, for any Hamiltonian diffeomorphism ϕ, $L \cap \phi(L) \neq \emptyset$. Moreover, there is $p \in L \cap \phi(L)$ with Viterbo–Maslov index 0.*

Thomas and Yau [46] used this theorem to establish the uniqueness of special Lagrangian homology spheres. From an opposite viewpoint, if L is a relatively spin Lagrangian submanifold with vanishing second rational cohomology and admits a Hamiltonian diffeomorphism ϕ such that $L \cap \phi(L) = 0$, then the Maslov class μ_L does not vanish. For instance, we have the following:

Theorem 3.15. *Let L be a Lagrangian submanifold in the symplectic vector space $(\mathbb{R}^{2n}, \omega_{can})$. If $H^2(L;\mathbb{Q}) = 0$ then $\mu_L \neq 0$. Moreover, its minimal Maslov number is at most $n+1$.*

Some results in a similar spirit were also obtained by Biran and Cieliebak [2]. Y. G. Oh obtained a more precise upper bound for the minimal Maslov number for Lagrangian tori up to a certain dimension [33]. Once we know that there exists a Hamiltonian diffeomorphism ϕ of (P,Ω) such that $L \cap \phi(L) = \emptyset$, either some

[9] $\Lambda^+_{0,\mathrm{nov}} = \left\{ \sum a_i T^{\lambda_i} e^{\mu_i} \in \Lambda_{\mathrm{nov}} | \lambda_i > 0 \right\}$.

obstruction class does not vanish, or some differential in the spectral sequence in Theorem 3.10 is non-trivial. In each case we obtain the existence of non-trivial J-holomorphic discs with boundary on L. Thus, for example, we can find that any Lagrangian submanifolds in symplectic vector spaces are not exact. Namely, the restriction of the Liouville form $\lambda = \sum p_i dq^i$ to L is not an exact 1-form on L (Gromov).

Finally we discuss an analogue of the flux conjecture for Lagrangian submanifolds. Denote by $\mathrm{Lag}(L)$ the space of all Lagrangian submanifolds which are Lagrangian isotopic to L with C^1-topology. Consider the quotient $\mathrm{Lag}(L)/\mathrm{Ham}(P,\Omega)$ by the obvious action of $\mathrm{Ham}(P,\Omega)$. The question is whether $\mathrm{Lag}(L)/\mathrm{Ham}(P,\Omega)$ is Hausdorff or not. This is false in general. In fact, Chekanov's example in [4] provides a counterexample. In his example the Maslov class is non-zero. As an application of our theory [15] we find the following result which is an analogue to Theorem 2.4 (the case that the Chern number is 0).

Theorem 3.16. *Let L be a relatively spin Lagrangian submanifold L with vanishing Maslov class. Suppose that the (deformed) Maurer–Cartan equation for L has a solution. If $L' = \phi(L)$, for some $\phi \in \mathrm{Ham}(P,\Omega)$, is sufficiently C^1-close to L, then L' is regarded as the graph of an exact 1-form on L via Weinstein's tubular neighborhood theorem.*

We expect that $\mathrm{Lag}(L)/\mathrm{Ham}(P,\Omega)$ is Hausdorff under the above assumption.

Finally, we make a remark that if L is a so-called semi-positive Lagrangian submanifold, we can work with $\mathbb{Z}/2\mathbb{Z}$-coefficients rather than $\mathbb{Q}$-coefficients. We do not need the relative spin condition in this case. There is also an approach to the Floer cohomology with $\mathbb{Z}$-coefficients [17]. There are also applications in relation to "mirror symmetry" which we do not discuss here.

Acknowledgement. I would like to thank my collaborators, K. Fukaya, Y. G. Oh, H. Ohta in [15] and H. V. Le in [27].

References

[1] Banyaga, A., Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique. *Comment. Math. Helv.* **53** (1978), 174–227.

[2] Biran, P., and Cieliebak, K., Lagrangian embeddings into subcritical Stein manifolds. *Israel J. Math.* **127** (2002), 221–244.

[3] Chekanov, Y., Lagrangian intersections, symplectic energy and areas of holomorphic curves. *Duke Math. J.* **95** (1998), 213–226.

[4] Chekanov, Y., Lagrangian torus in a symplectic vector space and global symplectomorphisms. *Math. Z.* **223** (1996), 547–559.

[5] de Silva, V., Products in the symplectic Floer homology of Lagrangian intersections. Ph.D. thesis, Oxford University, 1998.

[6] Dostoglou, S., and Salamon, D., Self-dual instantons and holomorphic curves. *Ann. of Math.* (2) **139** (1994), 581–640.

[7] Entov, M., K-area, Hofer metric and geometry of conjugacy classes in Lie groups. *Invent. Math.* **146** (2001), 93–141.

[8] Entov, M., and Polterovich, L., Calabi quasimorphisms and quantum homology. *Internat. Math. Res. Notices* **30** (2003), 1635–1676.

[9] Floer, A., Morse theory for lagrangian intersections. *J. Differential Geom.* **28** (1988), 513–547.

[10] Floer, A., The unregularized gradient flow of the symplectic action. *Comm. Pure Appl. Math.* **41** (1988), 775–813.

[11] Floer, A., A relative Morse index for the symplectic action. *Comm. Pure Appl. Math.* **41** (1988), 393–407.

[12] Floer, A., Witten's complex and infinite dimensional Morse theory. *J. Differential Geom.* **30** (1989), 207–221.

[13] Floer, A., Cup length estimate on lagrangian intersections. *Comm. Pure Appl. Math.* **42** (1989), 335–357.

[14] A. Floer, Holomorphic spheres and symplectic fixed points. *Comm. Math. Phys.* **120** (1989), 575–611.

[15] Fukaya, K., Oh, Y. G., Ohta, H., and Ono, K., Lagrangian intersection Floer theory – obstruction and anomaly. Preprint 2000 and revised version, in preparation.

[16] Fukaya, K., and Ono, K., Arnold conjecture and Gromov-Witten invariant. *Topology* **38** (1999), 933–1048.

[17] Fukaya, K., and Ono, K., Floer homology and Gromov-Witten invariant over integer for general symplectic manifolds. In *Taniguchi Conference on Mathematics Nara '98* (ed. by Masaki Maruyama and Toshikazu Sunada), Adv. Stud. Pure Math. 31, Mathematical Society of Japan, Tokyo 2001, 75–91.

[18] Gromov, M., Pseudoholomorphic curves in symplectic manifolds. *Invent. Math.* **82** (1985), 307–347.

[19] Hofer, H., Lagrangian embeddings and critical point theory. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **2** (1985), 407–462.

[20] Hofer, H., Lusternik-Schnirelmann theory for Lagrangian intersections. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **5** (1988), 465–499.

[21] Hofer, H., and Salamon, D., Floer homology and Novikov rings. In *The Floer memorial volume* (ed. by H. Hofer, C. Taubes, A. Weinstein and E. Zehnder), Progr. Math. 133, Birkhäuser, Basel 1995, 483–524.

[22] Kharlamov, V., Variétés de Fano réels (d'après C. Viterbo). Séminaire Bourbaki, 1999/2000, *Astérisque* **276** (2002), 189–206.

[23] Kontsevich, M., Enumeration of rational curves by torus action. In *Moduli space of curves* (ed. by H. Dijkgraaf, C. Faber, G. v. d. Geer), Progr. Math. 129, Birkhäuser, Basel 1995, 335–368.

[24] Kontsevich, M., and Manin, Y., Gromov-Witten classes, quantum cohomology and enumerative geometry. *Comm. Math. Phys.* **164** (1994), 525–562.

[25] Lalonde, F., , McDuff, D., and Polterovich, L., On the flux conjectures. In *Geometry, topology and dynamics*, CRM Proc. Lecture Notes 15, Amer. Math. Soc., Providence, RI, 1998, 69–85.

[26] Lalonde, F., , McDuff, D., and Polterovich, L., Topological rigidity of Hamiltonian loops and quantum homology. *Invent. Math.* **135** (1999), 369–385.

[27] Lê, H. V., and Ono, K., Symplectic fixed points, the Calabi invariant and Novikov homology. *Topology* **34** (1995), 155–176.

[28] Li, J., and Tian, G., Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds. In *Topics in symplectic* 4*-manifolds*, First Int. Press Lect. Ser. 1, International Press, Cambridge, MA, 1998, 47–83.

[29] Liu, G., and Tian, G., Floer homology and Arnold conjecture. *J. Differential Geom.* **49** (1998), 1–74.

[30] Oh, Y. G., Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks, I. *Comm. Pure Appl. Math.* **46** (1993), 949–994 Addendum ibid. 48 (1995), 1299–1302.

[31] Oh, Y. G., Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks, II. *Comm. Pure Appl. Math.* **46** (1993), 995–1012

[32] Oh, Y. G., Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks, III. In *The Floer memorial volume* (ed. by H. Hofer, C. Taubes, A. Weinstein and E. Zehnder), Progr. Math. 133, Birkhäuser, Basel 1995, 555–573.

[33] Oh, Y. G., Floer cohomology, spectral sequences and the Maslov class of Lagrangian embeddings. *Internat. Math. Res. Notices* **7** (1996), 305–346.

[34] Oh, Y. G., Symplectic topology as the geometry of the action functional, I. *J. Differential Geom.* **46** (1997), 499–577.

[35] Oh, Y. G., Symplectic topology as the geometry of the action functional, II. *Commun. Anal. Geom.* **7** (1999), 1–55.

[36] Ono, K., On the Arnold conjecture for weakly monotone symplectic manifolds. *Invent. Math.* **119** (1995), 519–537.

[37] Ono, K., Lagrangian intersection under legendrian deformations. *Duke Math. J.* **85** (1996), 209–225.

[38] Ono, K., Floer-Novikov cohomology and the flux conjecture. Preprint.

[39] Ono, K., Floer-Novikov cohomology and symplectic fixed points. To appear in *Proceedings of Workshop on Symplectic Topology, Stare Jablonki, 2004*.

[40] Polterovich, L., Monotone Lagrangian submanifolds of linear spaces and the Maslov class in cotangent bundles. *Math. Z.* **207** (1991), 217–222.

[41] Ruan, Y., Virtual neighborhoods and pseudoholomorphic curves. *Turkish J. Math.* **23** (1999), 161–231.

[42] Schwarz, M., On the action spectrum for closed symplectically aspherical manifolds. *Pacific J. Math.* **193** (2000), 419–461.

[43] Seidel, P., Symplectic Floer homology and the mapping class group. *Pacific J. Math.* **206** (2002), 219–229.

[44] Seidel, P., π_1 of symplectic automorphism groups and invertibles in quantum cohomology rings. *Geom. Funct. Anal.* **7** (1997), 1046–1095.

[45] Siebert, B., Gromov-Witten invariants for general symplectic manifolds. Preprint, 1996.

[46] Thomas, R., and Yau, S. T., Special Lagrangians, stable bundles and mean curvature flow. *Comm. Geom. Anal.* **10** (2002), 1075–1113.

[47] Viterbo, C., Symplectic topology as the geometry of generating functions. *Math. Ann.* **292** (1992), 685–710.

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