

The cohomology of automorphism groups of free groups

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Abstract. There are intriguing analogies between automorphism groups of finitely generated free groups and mapping class groups of surfaces on the one hand, and arithmetic groups such as $GL(n, \mathbb{Z})$ on the other. We explore aspects of these analogies, focusing on cohomological properties. Each cohomological feature is studied with the aid of topological and geometric constructions closely related to the groups. These constructions often reveal unexpected connections with other areas of mathematics.

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1. Introduction

In the 1920s and 30s Jakob Nielsen, J. H. C. Whitehead and Wilhelm Magnus invented many beautiful combinatorial and topological techniques in their efforts to understand groups of automorphisms of finitely-generated free groups, a tradition which was supplemented by new ideas of J. Stallings in the 1970s and early 1980s. Over the last 20 years mathematicians have been combining these ideas with others motivated by both the theory of arithmetic groups and that of surface mapping class groups. The result has been a surge of activity which has greatly expanded our understanding of these groups and of their relation to many areas of mathematics, from number theory to homotopy theory, Lie algebras to bio-mathematics, mathematical physics to low-dimensional topology and geometric group theory.

In this article I will focus on progress which has been made in determining cohomological properties of automorphism groups of free groups, and try to indicate how this work is connected to some of the areas mentioned above. The concept of assigning cohomology groups to an abstract group was originally motivated by work of Hurewicz in topology. Hurewicz proved that the homotopy type of a space with no higher homotopy groups (an *aspherical space*) is determined by the fundamental group of the space, so the homology groups of the space are in fact invariants of the group. Low-dimensional homology groups were then found to have interpretations in terms of algebraic invariants such as group extensions and derivations which had long been studied by algebraists, and a purely algebraic definition of group cohomology

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was also introduced. These were the beginning steps of a rich and fruitful interaction between topology and algebra via cohomological methods.

Borel and Serre studied the cohomology of arithmetic and S -arithmetic groups by considering their actions on homogeneous spaces and buildings. Thurston studied surface mapping class groups by considering their action on the Teichmüller space of a surface, and this same action was used later by Harer to determine cohomological properties of mapping class groups. Outer automorphism groups of free groups are neither arithmetic groups nor surface mapping class groups, but they have proved to share many algebraic features with both classes of groups, including many cohomological properties. The analogy is continually strengthened as more and more techniques from the arithmetic groups and mapping class groups settings are adapted to the study of automorphism groups of free groups. The adaptation is rarely straightforward, and often serves more as a philosophy than a blueprint. The connection is more than strictly empirical and philosophical, however, due to the natural maps

$$\text{Out}(F_n) \rightarrow \text{GL}(n, \mathbb{Z})$$

and

$$\Gamma_{g,s} \rightarrow \text{Out}(F_{2g+s-1}).$$

The first map is induced by the abelianization map $F_n \rightarrow \mathbb{Z}^n$; it is always surjective and is an isomorphism for $n = 2$. In the second map, the group $\Gamma_{g,s}$ is the mapping class group of a surface of genus g with $s > 0$ punctures, which may be permuted. The map is defined using the observation that a homeomorphism of a surface induces a map on the (free) fundamental group of the surface; it is always injective and is an isomorphism for $g = s = 1$.

2. Outer space and homological finiteness results

In order to adapt techniques Borel and Serre used for arithmetic groups, and those Thurston and Harer used for mapping class groups to the context of automorphism groups of free groups, the first thing one needs is a replacement for the homogeneous or Teichmüller space. A suitable space \mathcal{O}_n , now called *Outer space*, was introduced by Culler and Vogtmann in 1986 [12]. The most succinct definition of Outer space is that it is the space of homothety classes of minimal free simplicial actions of $\text{Out}(F_n)$ on \mathbb{R} -trees. (Here an \mathbb{R} -tree is a metric space with a unique arc, isometric to an interval of \mathbb{R} , joining any two points and actions are by isometries; an action is *simplicial* if every orbit is discrete and *minimal* if there is no proper invariant subtree.) The topology on the space can be taken to be the equivariant Gromov–Hausdorff topology, or it can be topologized as a space of projective length functions on F_n . Pre-composing an action with an element of $\text{Out}(F_n)$ gives a new action, and this defines the action of $\text{Out}(F_n)$ on the space. This description is efficient, but it is often easier to visualize and to work with Outer space when it is presented instead in terms of *marked graphs*.

2.1. Marked graphs. The quotient of a free action of F_n on a simplicial \mathbb{R} -tree is a finite graph with fundamental group F_n and a metric determined by the lengths of the edges. The action is minimal if and only if the quotient graph has no univalent or bivalent vertices. If we have a specific identification of F_n with the fundamental group of the graph then the tree, with its F_n -action, can be recovered as the universal cover of the graph. Thus another way to describe a point in Outer space is to fix a graph R_n and identification $F_n = \pi_1(R_n)$; a point is then an equivalence class of pairs (g, G) , where

- G is a finite connected metric graph with no univalent or bivalent vertices,
- $g: R_n \rightarrow G$ is a homotopy equivalence.

We normalize the metric so that the sum of the lengths of the edges in G is equal to one; then two pairs (g, G) and (g', G') are *equivalent* if there is an isometry $h: G \rightarrow G'$ with $h \circ g \simeq g'$. An equivalence class of pairs is called a *marked graph*.

Varying the lengths of edges in a marked graph with k edges and total length one allows one to sweep out an open $(k-1)$ -simplex of points in Outer space. Collapsing an edge which is not a loop determines a new open simplex which is a face of the original simplex. We topologize Outer space as the union of these open simplices, modulo these face identifications. A picture of Outer space for $n = 2$ is given in Figure 1.

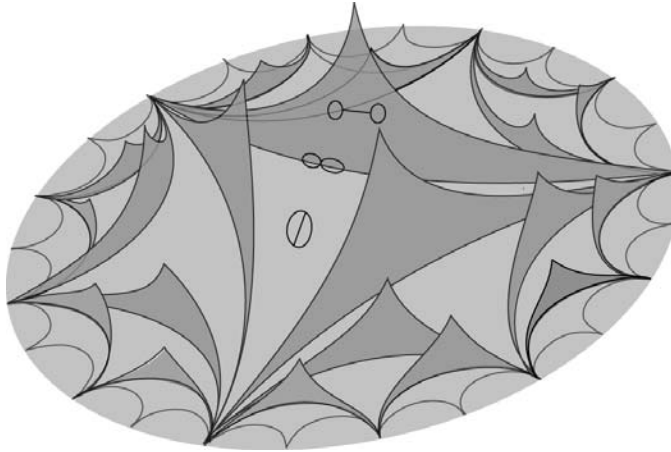


Figure 1. Outer space in rank 2.

2.2. Virtual cohomological dimension. The first cohomological results about automorphism groups of free groups were finiteness results, which followed directly by considering the topology and combinatorial structure of Outer space. Unlike homogeneous spaces and Teichmüller spaces, Outer space and its quotient are not manifolds,

as can be seen already for $n = 2$. However, observed through the prism of cohomology they resemble manifolds in several important ways, including satisfying various finiteness properties and a type of duality between homology and cohomology. As a start, we have

Theorem 2.1 ([12]). *Outer space is contractible of dimension $3n - 4$, and $\text{Out}(F_n)$ acts with finite stabilizers.*

$\text{Out}(F_n)$ has torsion-free subgroups of finite index, which by the above theorem must act freely on \mathcal{O}_n . The quotient of \mathcal{O}_n by such a subgroup Γ is thus an aspherical space with fundamental group Γ , and the following corollary follows immediately:

Corollary 2.2. *The cohomological dimension of any torsion-free subgroup of finite index in $\text{Out}(F_n)$ is at most $3n - 4$.*

Serre proved that in fact the cohomological dimension of a torsion-free subgroup of finite index in any group is independent of the choice of subgroup; this is called the *virtual cohomological dimension*, or VCD, of the group. As in the case of $\text{GL}(n, \mathbb{Z})$ and mapping class groups, the quotient of \mathcal{O}_n by $\text{Out}(F_n)$ is not compact, and the dimension of the homogeneous space does not give the best upper bound on the virtual cohomological dimension. A solution to this problem for $\text{Out}(F_n)$ is given by considering an equivariant deformation retract of \mathcal{O}_n , called the *spine of Outer space*. This spine can be described as the geometric realization of the partially ordered set of open simplices of \mathcal{O}_n , so has one vertex for every homeomorphism type of marked graph with fundamental group F_n and one k -simplex for every sequence of k forest collapses.

Theorem 2.3 ([12]). *The spine of Outer space is an equivariant deformation retract of Outer space. It has dimension $2n - 3$ and the quotient is compact.*

This theorem allows one to compute the virtual cohomological dimension of $\text{Out}(F_n)$ precisely:

Corollary 2.4. *The virtual cohomological dimension of $\text{Out}(F_n)$ is equal to $2n - 3$.*

Proof. For $i > 1$, let λ_i be the automorphism of $F_n = F\langle x_1, \dots, x_n \rangle$ which multiplies x_i by x_1 on the left and fixes all other x_j . Similarly, define ρ_i to be the automorphism which multiplies x_i by x_1 on the right. The subgroup of $\text{Out}(F_n)$ generated by the λ_i and ρ_i is a free abelian subgroup of $\text{Out}(F_n)$ of dimension $2n - 3$, giving a lower bound on the virtual cohomological dimension. The upper bound is given by the dimension of the spine. \square

2.3. Finite generation of homology. The fact that the quotient of the spine by $\text{Out}(F_n)$ is compact implies immediately that any torsion-free finite-index subgroup is the fundamental group of an acyclic space with only finitely many cells in each dimension, and in particular its homology is finitely generated in all dimensions. This implies the same result for the entire group $\text{Out}(F_n)$:

Corollary 2.5. *The homology of $\text{Out}(F_n)$ is finitely-generated in all dimensions.*

The focus of this article is cohomology, but we would like to point out that Outer space and its spine can also be used to prove properties of $\text{Out}(F_n)$ which are not strictly cohomological. As an example, we note that any finite subgroup of $\text{Out}(F_n)$ can be realized as automorphisms of a finite graph with fundamental group F_n (see, e.g. [11]). This is equivalent to saying that any finite subgroup of $\text{Out}(F_n)$ fixes some vertex of the spine of Outer space, so compactness of the quotient immediately gives the following information about the subgroup structure of $\text{Out}(F_n)$.

Corollary 2.6. *$\text{Out}(F_n)$ has only finitely many conjugacy classes of finite subgroups.*

Going even farther afield, we remark that the very concrete description of the spine in terms of graphs and forest collapses allows one to determine the local structure quite precisely. In particular, a neighborhood of a *rose*, i.e. a graph with one vertex and n edges, is easily identified with the space of trees with $2n$ labeled leaves, and can be used as a model for the space of phylogenetic trees in biology. This neighborhood can be given a metric of non-positive curvature, which has a computational advantage for applications to biology (see [3]).

3. The bordification and duality

There is another approach to resolving difficulties arising from the fact that the action of $\text{Out}(F_n)$ on Outer space is not cocompact. Instead of finding a spine inside Outer space, one can extend Outer space and the action of $\text{Out}(F_n)$ by adding cells “at infinity” to produce a larger space whose quotient is compact. This was the approach taken by Borel and Serre in their work on arithmetic groups. They defined a *bordification* of the homogeneous space and used Poincaré–Lefschetz duality for this “manifold with corners” to prove that arithmetic groups satisfy a form of duality between homology and cohomology. Specifically, a group Γ is said to be a *duality group* if there is a module D , integer d and isomorphisms $H^i(\Gamma; M) \rightarrow H_{d-i}(\Gamma; M \otimes D)$ for any integer i and Γ -module M . If Γ is a duality group, then the integer d is equal to the virtual cohomological dimension, and this is how Borel and Serre determined the (virtual) cohomological dimension of arithmetic groups.

Although Outer space is not a manifold, Bestvina and Feighn showed that torsion-free finite index subgroups of $\text{Out}(F_n)$ are duality groups, i.e. $\text{Out}(F_n)$ is a *virtual duality group*. They accomplished this by defining a bordification $\widehat{\mathcal{O}}_n$ of Outer space and studying its topology at infinity. This bordification has the structure of a locally finite cell complex on which $\text{Out}(F_n)$ acts with finite stabilizers.

3.1. Cells in the bordification. There are only a finite number of orbit classes of open simplices in \mathcal{O}_n , leading one to expect that the quotient should be compact. The reason it is not is that we are leaving out some of the faces of simplices; we go to a

face by collapsing edges in the graph, but we are not allowed to reduce the rank of the graph. One might think of simply adding the missing faces to achieve cocompactness, but this destroys essential features of the action: in particular, the result is not locally finite, and simplex stabilizers are infinite. Bestvina and Feighn found a way around this by keeping track of exactly how subgraphs degenerate as you approach a missing face. Thus there is a cell “at infinity” for each marked metric graph G and sequence of nested subgraphs $G = G_0 \supset G_1 \supset \cdots \supset G_k$. Each subgraph G_i comes with its own metric of volume 1, and G_{i+1} is spanned by the edges of length zero in G_i . The idea is that the sequence consists of subgraphs which are collapsing to zero faster and faster, and the metrics keep track of the direction one is going as one approaches infinity.

3.2. Virtual duality. Bieri and Eckman showed that a group is a virtual duality group if and only if the cohomology of the group with coefficients in its integral group ring vanishes in all but one dimension, where it is free. The cohomology of $\text{Out}(F_n)$ with coefficients in its integral group ring is isomorphic to the cohomology with compact supports of the bordification $\widehat{\mathcal{O}}_n$, and this in turn can be shown to satisfy Bieri and Eckmann’s criteria by showing that $\widehat{\mathcal{O}}_n$ is sufficiently connected at infinity. This is what Bestvina and Feighn prove, using Morse theory techniques:

Theorem 3.1 ([2]). *The bordification $\widehat{\mathcal{O}}_n$ of Outer space is $(2n - 3)$ -connected at infinity.*

Corollary 3.2 ([2]). *$\text{Out}(F_n)$ is a virtual duality group.*

4. The Degree Theorem and rational homology stability

We now turn attention to the group $\text{Aut}(F_n)$. An advantage that $\text{Aut}(F_n)$ has over $\text{Out}(F_n)$ is that it comes equipped with natural inclusions $\text{Aut}(F_n) \rightarrow \text{Aut}(F_{n+1})$. The analogous inclusions in the general linear case induce maps $H_k(\text{GL}(n, \mathbb{Z})) \rightarrow H_k(\text{GL}(n+1, \mathbb{Z}))$ which were shown by Charney to be isomorphisms for n sufficiently large with respect to k [6]. The fact that the homology of $\text{GL}(n, \mathbb{Z})$ stabilizes in this way serves to considerably simplify the problem of computing the homology. For example, one may determine $H_k(\text{GL}(n, \mathbb{Z}))$ for any large n by performing the computations with relatively small values of n , where the size of the computation is more manageable. A more subtle and more powerful advantage is that one may work instead with the stable groups $\text{GL}_\infty(\mathbb{Z}) = \lim_{n \rightarrow \infty} \text{GL}(n, \mathbb{Z})$ which carry additional multiplicative structure and are amenable to homotopy theoretic methods such as the plus construction.

There is a construction completely analogous to the construction of Outer space using basepointed graphs, where the basepoint may be at a vertex or in the interior of an edge. This space \mathcal{A}_n is also contractible [16], has a cocompact spine, and acts as a homogeneous space for $\text{Aut}(F_n)$, which acts with finite stabilizers. (A French

colleague suggested that the space \mathcal{A}_n should be called “Autre espace,” but the name “Auter space” seems to have taken hold instead.) An advantage to this space is that the basepoint determines a natural Morse function on a marked metric graph, and we can use parameterized Morse theory methods to study the space. The basepoint also allows us to define a filtration of \mathcal{A}_n by highly connected subspaces which, as we will see, are very useful in homology calculations and in particular for proving homology stability theorems.

4.1. The Degree Theorem. We will filter \mathcal{A}_n by the *degree* of a marked graph, where the degree is defined as the number of vertices away from the basepoint counted with multiplicity (multiplicity is the valence of the vertex minus 2). The degree of a graph with fundamental group F_n is then equal to $2n$ minus the valence of the basepoint. Thus a rose has degree zero, a graph with one trivalent vertex away from the basepoint has degree one, and any graph with basepoint in the interior of an edge has degree $2n - 2$. The fact that $\text{Aut}(F_n)$ is generated by Nielsen automorphisms, which can be modeled by a homotopy equivalence which wraps one leaf of a rose around another, implies that the degree 1 subspace of \mathcal{A}_n is connected. An analogous statement is true for higher degrees:

Theorem 4.1 (Degree Theorem, [18]). *The subspace $\mathcal{A}_{n,k}$ of Auter space consisting of marked graphs of degree at most k is $(k - 1)$ -connected.*

Thus $\mathcal{A}_{n,k}$ acts as a kind of k -skeleton for Auter space. The action of $\text{Aut}(F_n)$ on \mathcal{A}_n changes the marking on a graph but not the homeomorphism type, so that it preserves the degree, i.e. restricts to an action on $\mathcal{A}_{n,k}$. Since $\mathcal{A}_{n,k}$ is $(k - 1)$ -connected and $\text{Aut}(F_n)$ acts with finite stabilizers, the homology of $\text{Aut}(F_n)$ with trivial rational coefficients can be identified with the rational homology of the quotient in dimensions less than k .

Because a degree k graph has only k vertices away from the basepoint (counted with multiplicity), if we ignore loops at the basepoint there are only a finite number of possibilities for such a graph. Thus the map from $\mathcal{A}_{n,k}$ to $\mathcal{A}_{n+1,k}$ given by attaching an extra loop at the basepoint is a homeomorphism for n large. As an immediate consequence, we see that the map $H_{k-1}(\text{Aut}(F_n); \mathbb{Q}) \rightarrow H_{k-1}(\text{Aut}(F_{n+1}); \mathbb{Q})$ induced by inclusion is an isomorphism on homology n for n large.

For computational purposes, it is obviously advantageous to know exactly how large n has to be, i.e. to have the best possible bound on the stability range. An easy Euler characteristic argument shows that the map from $\mathcal{A}_{n,k}$ to $\mathcal{A}_{n+1,k}$ is a homeomorphism for $n \geq 2k$; for example for any $n \geq 4$, the degree 2 subcomplex of the spine has a contractible quotient consisting of 7 triangles glued together as in Figure 2. As a consequence we can say that $\text{Aut}(F_n)$ satisfies *homology stability with slope 2*. The slope can in fact be improved to $3/2$ fairly easily by showing that the quotients in this range, though not actually homeomorphic, are nevertheless homotopy equivalent [18]; further improvement is possible using some calculations in rank 3. The best known result at this time is the following.

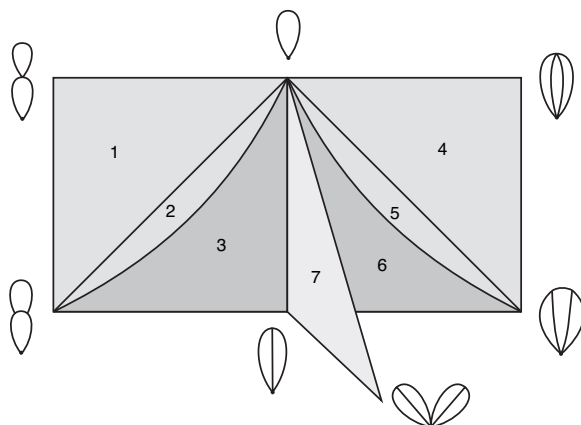


Figure 2. The degree 2 quotient.

Theorem 4.2 ([19]). *The map $H_{k-1}(\text{Aut}(F_n); \mathbb{Q}) \rightarrow H_{k-1}(\text{Aut}(F_{n+1}); \mathbb{Q})$ induced by inclusion is an isomorphism on homology n for $n \geq 5(k+1)/4$.*

Homology stability is a property also shared by surface mapping class [15], with appropriate inclusions. The exact slope, for trivial rational coefficients, is known for both $\text{GL}(n, \mathbb{Z})$ and for mapping class groups of bounded surfaces, but the question remains open for automorphism groups of free groups.

5. Sphere complexes and integral homology stability

5.1. Presentations of $\text{Aut}(F_n)$ and stability for H_1 . The first integral homology of $\text{Aut}(F_n)$ is isomorphic to $\mathbb{Z}/2$ for all $n \geq 3$, as can be seen by abelianizing a presentation. There are several different presentations of $\text{Aut}(F_n)$ available, including presentations due to J. Nielsen, B. Neumann, and J. McCool. A new way of obtaining a presentation is supplied by the Degree Theorem, which tells us that the degree 2 subcomplex of Auter space is simply-connected. The method of K. Brown [5] for calculating a presentation from the action of a group on a simply-connected CW-complex can then be used to find a presentation. This was carried out in [1]. The generators, for any $n \geq 4$, are the stabilizers of the seven graphs pictured in Figure 2, with appropriate numbers of loops added at the basepoint. The relations are the relations in these stabilizers, together with relations given by inclusions of edge stabilizers into vertex stabilizers and by inclusions composed with conjugations.

5.2. Quillen's method. More delicate techniques are needed to show that the k -th homology of $\text{Aut}(F_n)$ with trivial integral coefficients stabilizes for n large. The general idea is borrowed from Quillen's work on homology stability for general linear

groups of fields. The simplest possible setup for using this method to prove homology stability for a sequence of groups $\{G_n\}$ is provided by having contractible simplicial complexes X_n and actions of G_n on X_n which are transitive on p -simplices for all p . The stabilizer of a p -simplex should be isomorphic to G_{n-p-1} and the quotient of X_n by the action should be highly connected. Given these conditions, one looks at the equivariant homology spectral sequence for this action, which has

$$E_{p,q}^1 = \begin{cases} \bigoplus_{\sigma_p} H_q(\text{stab}(\sigma_p)) & p \geq 0, \\ H_q(G_n) & p = -1, \end{cases}$$

where the direct sum is over all orbits of p -simplices. This spectral sequence converges to 0, and the map $H_k(G_{n-1}) \rightarrow H_k(G_n)$ induced by inclusion appears as the map $d^1: E_{0,k}^1 \rightarrow E_{-1,k}^1$. By induction, we may assume that we understand what happens below the k -th row of the spectral sequence, specifically that the E^2 page vanishes below that row. Since the entire spectral sequence converges to zero, this implies that the d^1 map in question must be onto. Further argument is needed to show that d^1 is injective; this can either be done by increasing the dimension n and applying induction again or by carefully analyzing the next d^1 map, $d^1: E_{k,1}^1 \rightarrow E_{k,0}^1$ and showing that this is the zero map.

In practice, conditions are usually not quite this nice: the space X_n may not be contractible, the stabilizers may not be precisely G_{n-p-1} , the action may not be transitive on simplices, etc. These difficulties can sometimes be overcome at the cost of introducing further spectral sequence arguments and/or settling for a weaker stability range.

For $G_n = \text{Aut}(F_n)$, homology stability was first proved by Hatcher and Vogtmann in [18] using a complex of free factorizations of the free group F_n . They reproved this theorem in [20] using a different complex, which we describe below. The second paper (together with the erratum [21]) also contains a proof that the map from $\text{Aut}(F_n)$ to $\text{Out}(F_n)$ induces isomorphisms on k -th homology for n large.

The complexes used in [20] involve isotopy classes of 2-spheres embedded in a connected sum of n copies of $S^2 \times S^1$, with a small ball removed. This 3-manifold M_n has fundamental group F_n , and Laudenbach proved that the natural map from the mapping class group of M_n to $\text{Aut}(F_n)$ is surjective, with kernel a 2-torsion subgroup generated by Dehn twists along 2-spheres. This kernel acts trivially on the set of isotopy classes of 2-spheres in M_n , so we obtain an action of $\text{Aut}(F_n)$ on the complex formed by taking a k -simplex for every set of $k + 1$ isotopy classes which can be represented by disjoint spheres. The idea of using the complex of isotopy classes of 2-spheres in M_n originated in [16], where Hatcher established many of the basic tools needed for working with such complexes.

A vertex in the complex used in [20] is a non-separating sphere together with an extra, “enveloping” sphere which cuts the sphere and the boundary of M_n off from the rest of the manifold. An alternate description of such a vertex is obtained by using embedded arcs to *tether* each side of the sphere to the boundary of M_n . A set of

$k + 1$ isotopy classes of tethered 2-spheres forms a k -simplex if representatives can be found so that all spheres and tethers are disjoint. The stabilizer of a vertex is then isomorphic to $\text{Aut}(F_{n-1})$, and the following theorem allows us to apply Quillen's method:

Theorem 5.1 ([20]). *The complex of isotopy classes of tethered 2-spheres in M_n is $(n - 3)/2$ -connected.*

As a result, we obtain the following homology stability theorem:

Theorem 5.2 ([20]). *The map $i_*: H_k(\text{Aut}(F_n)) \rightarrow H_k(\text{Aut}(F_{n+1}))$ induced by the natural inclusion is an isomorphism for $n \geq 2k + 2$.*

We are also interested in showing that the map $p_*: H_k(\text{Aut}(F_n)) \rightarrow H_k(\text{Out}(F_n))$ induced by the natural projection is an isomorphism for n large. In the course of proving this, we are forced to consider the mapping class group of a connected sum of n copies of $S^1 \times S^2$ with $s \geq 0$ balls removed, modulo Dehn twists on 2-spheres. For $s = 0$ this is $\text{Out}(F_n)$. We prove the homology in dimension k is independent of n and s for n and s sufficiently large. In particular, we obtain

Theorem 5.3 ([20], [21]). *The map $p_*: H_k(\text{Aut}(F_n)) \rightarrow H_k(\text{Out}(F_n))$ is an isomorphism for $n \geq 2k + 4$.*

The idea of using “tethers” to tie geometric objects to a basepoint turns out to be useful in other contexts. The extra structure obtained from the tethers has the effect that the conditions for applying Quillen's method are close to ideal, so that the spectral sequence arguments needed to prove homology stability are relatively simple. In particular, tethers have led to simplified proofs of homology stability for mapping class groups of surfaces and braid groups, as well as to new proofs that several related series of groups have homology stability [17].

5.3. Galatius' theorem. Since we know that the homology of $\text{Aut}(F_n)$ stabilizes, the next problem is then to compute the stable homology. Computations in dimensions less than 7 were done in [19], and produced no stable rational homology classes. Igusa showed that the map from the stable rational homology of $\text{Aut}(F_n)$ to that of $\text{GL}(n, \mathbb{Z})$ is the zero map [22]. This evidence led to the conjecture that the stable rational homology is trivial. On the other hand, Hatcher showed that the stable homology contains the stable homology of the symmetric group Σ_n as a direct factor, so there are lots of torsion classes [16]. The entire situation has recently been resolved by S. Galatius [13] using methods adapted from Madsen and Weiss' work on the stable homology of mapping class groups.

The commutator subgroup of $\text{Aut}(F_\infty)$ is a perfect normal subgroup, so that Quillen's plus construction can be applied to the classifying space $B \text{Aut}(F_\infty)$. The resulting space $B \text{Aut}(F_\infty)^+$ is an infinite loop space whose homology is equal to the stable homology of $\text{Aut}(F_n)$. The natural inclusions of the symmetric groups Σ_n

into $\text{Aut}(F_n)$ induce an infinite loop space map $B\Sigma_\infty^+ \rightarrow B\text{Aut}_\infty^+$, and a theorem of Barratt–Priddy and Quillen says that $B\Sigma_\infty^+$ is homotopy equivalent to $\Omega^\infty S^\infty$. The space $\Omega^\infty S^\infty$ is the most fundamental example of an infinite loop space; its homotopy groups are the stable homotopy groups of spheres. In [13] Galatius proves that $B\text{Aut}(F_\infty)^+$ is also homotopy equivalent to $\Omega^\infty S^\infty$, showing in particular that the symmetric group and the automorphism group of a free group have the same stable homology. The proof relies on the contractibility of Outer space and the homology stability results of [20] and [21]. Galatius proceeds by defining maps of $B\text{Out}(F_n)$ to a certain “graph spectrum” E , whose n -th space is the space of all graphs in \mathbb{R}^n . He proves that after passing to infinite loop spaces this map becomes a homotopy equivalence, and then that $\Omega^\infty E$ is in fact homotopy equivalent to $\Omega^\infty S^\infty$. The homology of $\Omega^\infty S^\infty$ is torsion, so that the stable rational homology of $\text{Aut}(F_n)$ is trivial as conjectured.

6. Graph complexes and unstable homology

Though the stable homology of $\text{Aut}(F_n)$ and $\text{Out}(F_n)$ has been completely determined, at this writing the unstable homology is still largely mysterious. In this section we consider the unstable *rational* homology.

6.1. Low-dimensional calculations. The simplices in the spines of Outer space and Outer space naturally group themselves into cubes, giving these spines the structure of cube complexes. Specifically, an m -dimensional cube corresponds to a marked graph (g, G) together with a subforest Φ of G with m edges, since the set of simplices which can be obtained by collapsing the edges in the subforest in any one of the 2^{m-1} possible orders fit together to form a cube.

The quotient of a cube by a linear map is a rational homology cell, so that the cube complex structure on the spine descends to a “cell structure” on the quotient, which can be used to compute the rational homology of $\text{Out}(F_n)$ and $\text{Aut}(F_n)$. For $\text{Aut}(F_n)$, the Degree Theorem can be used to reduce the number of cubes one must consider when computing the k th homology, and further reductions are possible by examining the structure of the quotient. In the end it is possible for $k = 2, 3$ and even 4 to do the computations by hand. For $k > 4$, however, the aid of a computer becomes essential. Computations for both $\text{Aut}(F_n)$ and $\text{Out}(F_n)$ for $k \leq 7$ were carried out by Hatcher and Vogtmann, Jensen and Gerlits. They showed that that $H_k(\text{Aut}(F_n); \mathbb{Q}) = H_k(\text{Out}(F_n); \mathbb{Q}) = 0$ for $k \leq 7$ except that $H_4(\text{Aut}(F_4); \mathbb{Q}) = H_4(\text{Out}(F_4); \mathbb{Q}) \cong \mathbb{Q}$ [19]. This agrees with Galatius’ theorem in the stable range, and gives a tantalizing glimpse into the unstable homology. We now understand this very interesting non-trivial unstable homology class in a much more general context, as we will see below.

6.2. Graph homology of a cyclic operad. In [23], [24] Kontsevich found a remarkable correspondence between the cohomology of certain infinite-dimensional

symplectic Lie algebras and the homology of outer automorphism groups of free groups. This Lie algebra cohomology can be computed using the subcomplex of the Chevalley–Eilenberg complex spanned by symplectic invariants. The connection with $\text{Out}(F_n)$ is made via Weyl’s invariant theory, which allows one to interpret the complex of symplectic invariants as a chain complex indexed by finite graphs, where the vertices of these graphs are decorated by elements of the Lie operad. The homology of this graph complex can then be interpreted in terms of Outer space; this is carried out explicitly in [8].

The same formalism using the associative operad in place of the Lie operad gives a chain complex of “ribbon graphs”, which computes the homology of surface mapping class groups. The commutative operad gives rise to a type of graph homology which includes information about diffeomorphism groups of odd-dimensional homology spheres. Kontsevich’s construction in fact makes sense using any cyclic operad to decorate the vertices of graphs (see [8]), and it would be interesting to study the functorial properties of the resulting homology theories.

The connection with Lie algebra homology reveals new structure on the level of chain complexes. Specifically, the graph homology chain complex for any cyclic operad supports a Lie bracket and cobracket, which were studied in [10]; the Lie bracket can be shown to correspond to the classical Schouten bracket on the Lie algebra. The bracket and cobracket do not in general form a compatible Lie bialgebra structure, but do on the subcomplex spanned by connected graphs with no separating edges. For the associative and Lie operads, this subcomplex is quasi-isomorphic to the whole complex, so in particular has the same homology. For the commutative operad, this is not true, but the Lie bracket and cobracket do induce a bracket and cobracket on an appropriate quotient complex which is quasi-isomorphic to the whole complex [8]. These brackets come from a second boundary operator on the graph complex, and measure the deviation of this boundary operator from being a derivation (resp. coderivation). This second boundary operator anti-commutes with the standard boundary operator, so induces a map on graph homology. The Lie bracket and cobracket vanish on the level of homology, making graph homology together with this induced map into a differential graded algebra.

6.3. Morita cycles. Kontsevich’s work also led to new discoveries by S. Morita, who had been studying some of the same Lie algebras in his work on surface mapping class groups. In particular, Morita found an infinite sequence of cocycles for these Lie algebras based on his “trace” map and showed that the first of these cocycles is non-trivial on cohomology [28]. Via Kontsevich’s theorem, this cocycle produces a nontrivial homology class in $H_4(\text{Out}(F_4); \mathbb{Q})$. Since we know the rational homology $H_4(\text{Out}(F_4); \mathbb{Q})$ is one-dimensional, this class in fact completely computes the homology in this dimension.

Conant and Vogtmann translated Morita’s cocycles into cocycles on the complex of Lie graphs. They then showed that the combinatorial information contained in an oriented graph with vertices decorated by basic elements of the Lie operad is captured

more simply by a trivalent graph together with a subforest whose edges are ordered. This allows one to reinterpret the Morita cocycles directly in terms of such forested graphs, and to give a quick proof that the first of Morita’s cocycles is non-trivial. With a little more work it can be used to show that second one is also non-trivial, giving a rational homology class in $H_8(\text{Out}(F_6); \mathbb{Q})$ [9]. Morita reports that R. Ohashi has recently shown that in fact $H_8(\text{Out}(F_6); \mathbb{Q}) \cong \mathbb{Q}$, so that this class gives all of the homology [27].

There is a Morita cocycle corresponding to every graph consisting of two vertices joined by an odd number of edges. Both Morita and Conant-Vogtmann found generalizations which give a class in $H_{2k+r-2}(\text{Out}(F_{k+r}))$ for every odd-valent graph of rank r with k vertices. One expects that all of these should be non-trivial classes, and it is possible that they determine all of the unstable rational homology of $\text{Out}(F_n)$. We would therefore like to understand these cocycles as well as possible. Since they determine rational homology classes in the homology of $\text{Out}(F_n)$, we should be able to find representatives in the quotient of the spine of Outer space and in the quotient of the bordification of Outer space, both of which are rationally acyclic spaces for $\text{Out}(F_n)$. This is indeed possible, as is shown in [7]. In the case of the bordification, the cycles are found “at infinity.” Recall that a cell at infinity is given by a marked filtered graph (see section 3). The action of $\text{Out}(F_n)$ is transitive on markings, so that they disappear in the quotient. The first Morita cycle is the quotient of the single cell shown in Figure 3.

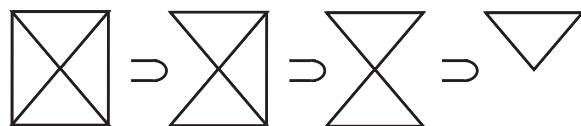


Figure 3. Generator of $H_4(\text{Out}(F_4); \mathbb{Q})$ in the bordification.

For the description of this class in terms of the spine, recall that the spine of Outer space has the structure of a cube complex, where a cube is given by a marked graph together with a subforest. Again, the action of $\text{Out}(F_n)$ is transitive on the markings, and the first Morita cycle is the union of the quotient of the three cubes shown in Figure 4.

Conant and Vogtmann use these descriptions in [7] to show that all of the Morita classes are unstable in the strongest possible sense: they vanish when the rank of the free group increases by one.

6.4. Rational Euler characteristic. A homological invariant of infinite groups G which is often easier to compute than the complete cohomology is the rational Euler characteristic $\chi(G)$. This is defined as the usual alternating sum of the Betti numbers for any torsion-free subgroup of finite index, divided by the index of the subgroup, and was shown by Serre to be independent of the choice of the subgroup.



Figure 4. Generator of $H_4(\text{Out}(F_4); \mathbb{Q})$ in the spine.

For $\text{GL}(n, \mathbb{Z})$ Harder showed that the rational Euler characteristic vanishes for all $n \geq 3$, and in general the rational Euler characteristics of arithmetic groups are closely related to values of zeta functions. For mapping class groups of closed surfaces the rational Euler characteristic vanishes for surfaces of odd genus, but for even genus it alternates in sign and is basically given by the classical Bernoulli numbers, as was shown by Harer and Zagier in [14].

The rational Euler characteristic of a group can be computed by finding a contractible complex on which the group acts cocompactly with finite stabilizers, and calculating the alternating sum, over all orbits of cells σ , of the terms

$$\frac{(-1)^{\dim(\sigma)}}{|\text{stab}(\sigma)|}.$$

This was done for $\text{Out}(F_n)$ in [29] using the spine of outer space. The result was a generating function for $\chi(\text{Out}(F_n))$ built from standard generating functions for counting graphs and forests in graphs. Using this generating function, $\chi(\text{Out}(F_n))$ was computed explicitly for values of n up to 100. It is strictly negative in all cases computed and seems to grow in absolute value faster than exponentially. Smillie and Vogtmann proved that $\chi(\text{Out}(F_n))$ is non-zero for all even n , and computed the p -power of the denominator for many primes p .

A different approach to the problem of counting graphs and forests was given by Kontsevich, who produced an integral formula for the rational Euler characteristic of $\text{Out}(F_n)$ using techniques of perturbative series and Feynman diagrams. He also produced integral formulas for the rational Euler characteristic of mapping class groups which recapture the relation with Bernoulli numbers found by Harer and Zagier. In the case of $\text{Out}(F_n)$, neither the generating function nor the integral formula make it clear what the asymptotic growth rate of $\chi(\text{Out}(F_n))$ might be, or even whether it is non-zero for all n . A non-zero, quickly growing Euler characteristic would indicate the presence of a large amount of unstable homology.

7. IA automorphisms and the IA quotient of Outer space

At the beginning of this article we noted the existence of a natural map from $\text{Out}(F_n)$ onto $\text{GL}(n, \mathbb{Z})$. The kernel of this map is called the subgroup of *IA automorphisms*

because it consists of automorphisms which induce the *identity* on the *abelianization* of F_n . It is clearly a natural object to study if one is trying to understand the relation between $\text{Out}(F_n)$ and $\text{GL}(n, \mathbb{Z})$. Magnus found a finite generating set for IA_n in 1934, and asked at the same time whether IA_n was finitely presentable [26].

There is an interesting application of the non-vanishing of the rational Euler characteristic of $\text{Out}(F_n)$ to the study of IA_n . If the homology of IA_n were finitely generated then the rational Euler characteristic of IA_n would be defined, and the short exact sequence

$$1 \rightarrow IA_n \rightarrow \text{Out}(F_n) \rightarrow \text{GL}(n, \mathbb{Z}) \rightarrow 1$$

would result in the equation $\chi(IA_n)\chi(\text{Out}(F_n)) = \chi(\text{GL}(n, \mathbb{Z}))$. However, we know that $\chi(\text{GL}(n, \mathbb{Z})) = 0$ for $n \geq 3$, while $\chi(\text{Out}(F_n))$ is non-zero, at least for n even. Thus we can conclude that the homology of IA_n is not finitely generated in some dimension.

If IA_n was finitely presentable, that would imply that the second homology is finitely generated. The argument in the previous paragraph shows that the homology is not finitely generated in *some* dimension, but gives no definite conclusion about dimension 2. McCool and Krstic finally answered Magnus' question for $n = 3$ in 1997, by showing that IA_3 is *not* finitely presentable [25]. Recently Bestvina, Bux and Margalit have shown that the top-dimensional homology of IA_n vanishes, while the codimension one homology is infinitely generated. They prove this by using Morse theory to study the topology of the quotient of Outer space by IA_n , which is an aspherical space with fundamental group IA_n . This work implies the McCool–Krstic result, since it says that $H_3(IA_3) = 0$ and $H_2(IA_3)$ is infinitely generated, but still leaves open the question of finite presentability for $n \geq 4$.

8. Further reading

In this article I have focused on cohomological properties of automorphism groups of free groups, but there are many other areas in which our knowledge of these groups is rapidly expanding. These include, for example, the subgroup structure, metric theory and rigidity properties. I wrote two other survey articles which address some of these advances. The first paper [30] gives a more detailed introduction to Outer space and related spaces and mentions other powerful techniques such as Bestvina–Handel's train tracks, as well as many applications to the study of $\text{Out}(F_n)$ and $\text{Aut}(F_n)$. It also contains a fairly extensive bibliography and more thorough references for work on automorphisms of free groups. The focus of the more recent paper [4], which is joint with Martin Bridson, is a discussion of open problems in the field.

References

- [1] Armstrong, Heather, Forrest, Brad, and Vogtmann, Karen, A presentation for $\text{Aut}(F_n)$. Preprint, 2006.
- [2] Bestvina, Mladen, and Feighn, Mark, The topology at infinity of $\text{Out}(F_n)$. *Invent. Math.* **140** (3) (2000), 651–692.
- [3] Billera, Louis J., Holmes, Susan P., and Vogtmann, Karen, Geometry of the space of phylogenetic trees. *Adv. in Appl. Math.* **27** (4) (2001), 733–767.
- [4] Bridson, Martin R., and Vogtmann, Karen, Automorphisms of free groups, surface groups and free abelian groups. Preprint, 2005; math.GR/0507602.
- [5] Brown, Kenneth S., Presentations for groups acting on simply-connected complexes. *J. Pure Appl. Algebra* **32** (1) (1984), 1–10.
- [6] Charney, Ruth M., Homology stability of GL_n of a Dedekind domain. *Bull. Amer. Math. Soc. (N.S.)* **1** (2) (1979), 428–43.
- [7] Conant, James, and Vogtmann, Karen, The Morita classes are stably trivial. Preprint, 2006.
- [8] Conant, James, and Vogtmann, Karen, On a theorem of Kontsevich. *Algebr. Geom. Topol.* **3** (2003), 1167–1224.
- [9] Conant, James, and Vogtmann, Karen, Morita classes in the homology of automorphism groups of free groups. *Geom. Topol.* **8** (2004), 1471–1499.
- [10] Conant, Jim, and Vogtmann, Karen, Infinitesimal operations on complexes of graphs. *Math. Ann.* **327** (3) (2003), 545–573.
- [11] Culler, Marc, Finite groups of outer automorphisms of a free group. In *Contributions to group theory*, Contemp. Math. 33, Amer. Math. Soc., Providence, RI, 1984, 197–207.
- [12] Culler, Marc, and Vogtmann, Karen, Moduli of graphs and automorphisms of free groups. *Invent. Math.* **84** (1) (1986), 91–119.
- [13] Galatius, Soren, in preparation.
- [14] Harer, J., and Zagier, D., The Euler characteristic of the moduli space of curves. *Invent. Math.* **85** (3) (1986), 457–485.
- [15] Harer, John L., Stability of the homology of the moduli spaces of Riemann surfaces with spin structure. *Math. Ann.* **287** (2) (1990), 323–334.
- [16] Hatcher, Allen, Homological stability for automorphism groups of free groups. *Comment. Math. Helv.* **70** (1) (1995), 39–62.
- [17] Hatcher, Allen, and Vogtmann, Karen, Tethers and homology stability. Preprint, 2006.
- [18] Hatcher, Allen, and Vogtmann, Karen, Cerf theory for graphs. *J. London Math. Soc.* (2) **58** (3) (1998), 633–655.
- [19] Hatcher, Allen, and Vogtmann, Karen, Rational homology of $\text{Aut}(F_n)$. *Math. Res. Lett.* **5** (6) (1998), 759–780.
- [20] Hatcher, Allen, and Vogtmann, Karen, Homology stability for outer automorphism groups of free groups. *Algebr. Geom. Topol.* **4** (2004), 1253–1272.
- [21] Hatcher, Allen, Vogtmann, Karen, and Wahl, Nathalie, Erratum to: Homology stability for outer automorphism groups of free groups. *Algebr. Geom. Topol.* **6** (2006), 573–579.
- [22] Igusa, Kiyoshi, *Higher Franz-Reidemeister torsion*. AMS/IP Stud. Advanced Math. 31, Amer. Math. Soc., Providence, RI, 2002.

- [23] Kontsevich, Maxim, Formal (non)commutative symplectic geometry. In *The Gel'fand Mathematical Seminars, 1990–1992*, Birkhäuser Boston, Boston, MA, 1993, 173–187.
- [24] Kontsevich, Maxim, Feynman diagrams and low-dimensional topology. In *First European Congress of Mathematics* (Paris, 1992), Vol. II, Progr. Math. 120, Birkhäuser, Basel 1994, 97–121.
- [25] Krstić, Sava, and McCool, James, The non-finite presentability of $IA(F_3)$ and $GL_2(\mathbf{Z}[t, t^{-1}])$. *Invent. Math.* **129** (3) (1997), 595–606.
- [26] Magnus, Wilhelm, Über n -dimensionale Gittertransformationen. *Acta Math.* **64** (1934), 353–367.
- [27] Morita, Shigeyuki, Cohomological structure of the mapping class group and beyond. Preprint, 2005; math.GT/0507308.
- [28] Morita, Shigeyuki, Structure of the mapping class groups of surfaces: a survey and a prospect. In *Proceedings of the Kirbyfest* (Berkeley, CA, 1998), Geom. Topol. Monogr. 2, Geom. Topol. Publ., Coventry, 1999, 349–406.
- [29] Smillie, John, and Vogtmann, Karen, A generating function for the Euler characteristic of $Out(F_n)$. *J. Pure Appl. Algebra* **44** (1–3) (1987), 329–348.
- [30] Vogtmann, Karen, Automorphisms of free groups and outer space. *Geom. Dedicata* **94** (2002), 1–31.

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