

Noncommutative counterparts of the Springer resolution

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Abstract. Springer resolution of the set of nilpotent elements in a semisimple Lie algebra plays a central role in geometric representation theory. A new structure on this variety has arisen in several representation theoretic constructions, such as the (local) geometric Langlands duality and modular representation theory. It is also related to some algebro-geometric problems, such as the derived equivalence conjecture and description of T. Bridgeland's space of stability conditions. The structure can be described as a noncommutative counterpart of the resolution, or as a t -structure on the derived category of the resolution. The intriguing fact that the same t -structure appears in these seemingly disparate subjects has strong technical consequences for modular representation theory.

Mathematics Subject Classification (2000). Primary 17B50, 18F99, 20G05; Secondary 20G42, 22E67.

Keywords. Derived categories of sheaves and modules, modular Lie algebras, local geometric Langlands duality.

1. Introduction

Springer resolution of the variety of nilpotent elements in a semi-simple Lie algebra is ubiquitous in geometric representation theory. In this article we show that, besides this well-known resolution of singularities, the variety of nilpotents, as well as some other closely related varieties, admits a particular *noncommutative resolution of singularities*, which arises in different representation theoretic and algebro-geometric constructions. Here by a noncommutative resolution of a singular variety Y we mean, following, e.g., [11], a coherent sheaf of associative \mathcal{O}_Y algebras satisfying certain natural conditions, and defined up to a Morita equivalence.

The constructions are related to such subjects as: the (local) geometric Langlands duality program and categorification of representation theory of affine Hecke algebras, representation theory of modular Lie algebras and quantum enveloping algebras at roots of unity, Bridgeland's theory of stability conditions on triangulated categories, and categorical McKay correspondence and generalizations.

Let G be a semi-simple adjoint algebraic group, \mathfrak{g} be its Lie algebra and $\mathcal{N} \subset \mathfrak{g}$ be the variety of nilpotent elements. Let \mathcal{B} be the variety of Borel subalgebras in \mathfrak{g} , also known as the flag variety of G , and $\tilde{\mathcal{N}} = T^*(\mathcal{B})$ be the cotangent bundle to \mathcal{B} . The Springer resolution is the moment map $\pi : \tilde{\mathcal{N}} \rightarrow \mathcal{N}$.

*The author is partially supported by NSF grant DMS-0505466 and Sloan Foundation grant 0965-300-L216.

Our noncommutative resolution A of \mathcal{N} comes with an equivalence between the derived category $D(A)$ of modules over A and the derived category $D(\tilde{\mathcal{N}})$ of coherent sheaves on $\tilde{\mathcal{N}}$. Thus A is determined uniquely up to Morita equivalence by the t -structure on $D(\tilde{\mathcal{N}})$ induced by the equivalence, i.e., by the image of the subcategory of A modules in $D(A)$ under the equivalence. We will call this t -structure the *exotic t -structure* and objects of its heart *exotic sheaves*. Thus an exotic sheaf is a complex of coherent sheaves on $\tilde{\mathcal{N}}$ which corresponds to an A -module under the equivalence $D(A) \cong D(\tilde{\mathcal{N}})$.

Closely related data first appeared in [2], which can be considered as a contribution to a local version of the geometric Langlands duality program [8], [37], [34]. A typical result of geometric Langlands duality is an equivalence between some derived category of constructible sheaves on a variety related to ${}^L G$ bundles on a curve C and derived category of coherent sheaves on a variety related to G local systems on C ; here G and ${}^L G$ are reductive groups, which are dual in the sense of Langlands. In the local version of the theory the curve C is a punctured formal disc \mathbb{D} . The role of the moduli stack of ${}^L G$ bundles is played by a homogeneous space for the group ${}^L G$, where ${}^L G((t))$ stands for the group of maps from \mathbb{D} to ${}^L G$ (also known as the formal loop group). An example of such a homogeneous space is the *affine flag variety* \mathcal{Fl} of ${}^L G$. For an appropriate choice of the category of constructible sheaves, the variety related to G local systems turns out to be $\tilde{\mathcal{N}}$, or rather the quotient stack $\tilde{\mathcal{N}}/G$ of $\tilde{\mathcal{N}}$ by the natural action of G . An equivalence between the derived category of G -equivariant coherent sheaves on $\tilde{\mathcal{N}}$ and a certain triangulated category of constructible sheaves on \mathcal{Fl} is proved in [2]. The image of the subcategory of perverse sheaves on \mathcal{Fl} under this equivalence turns out to consist of *equivariant exotic sheaves*, which are closely related to exotic sheaves (see Section 2.2 below).

Another construction leading to exotic sheaves is related to modular representation theory.

In the second half of the 20th century various geometric methods for representation theory of semi-simple Lie algebras over characteristic zero fields have been developed. One of the culminating points is the Localization Theorem [5], [30], motivated by a conjecture by Kazhdan and Lusztig, which provides an equivalence between the category of modules over a semi-simple Lie algebra \mathfrak{g} with a fixed (integral regular) central character and the category of D -modules on the flag variety \mathcal{B} . In the paper [19], motivated by Lusztig's extension [45] of Kazhdan–Lusztig conjectures to the modular setting, we provide a similar result for semi-simple Lie algebras over algebraically closed fields of positive characteristic. More precisely, we establish a derived localization theorem, which is an equivalence between the derived category of appropriately defined D -modules (called crystalline, or PD D -modules) on a flag variety and the derived category of Lie algebra modules, where a part of the center, the so-called Harish-Chandra center, acts by a fixed character.

Furthermore, in the case of positive characteristic there is a close relationship between crystalline D -modules on a smooth variety X and coherent sheaves on the cotangent space T^*X [19], [48]. The algebra of crystalline differential operators has a

huge center provided by invariant polynomials of the p -curvature of a D -module. This allows one to view the differential operators as a sheaf of algebras on the cotangent bundle. This algebra turns out to be an Azumaya algebra. In the case of the flag variety this Azumaya algebra splits on the formal neighborhood of each Springer fiber. Thus the derived localization theorem yields a full embedding from the category of finite dimensional \mathfrak{g} -modules with a fixed (integral regular) action of the Harish-Chandra center into the derived category of coherent sheaves on $\tilde{\mathcal{N}}$. It turns out that the image of this embedding consists precisely of exotic sheaves with proper support. A similar relation is expected between exotic sheaves over a field of characteristic zero and representations of the quantum Kac–De Concini enveloping algebra at a root of unity [32], and also with some class of ${}^L\mathfrak{g}$ modules at the critical level (cf. [4] and [35] respectively); here ${}^L\mathfrak{g}$ stands for the affine Kac–Moody algebra corresponding to the Langlands dual algebra ${}^L\mathfrak{g}$.

Thus exotic sheaves are related, on the one hand, to perverse sheaves on the affine flag variety for the dual group, and on the other hand, to modular Lie algebra representations. Comparison of these two connections allows one to apply the known deep results about weights of Frobenius acting on Ext's between irreducible perverse sheaves to numerical questions about modular representations, thereby providing a strategy for a proof of Lusztig's conjectures from [45]. The conjectures relate the classes of irreducible \mathfrak{g} -modules to elements of the *canonical basis* in the Borel–Moore homology of a Springer fiber; thus our work provides a categorification of the canonical bases in (co)homology of Springer fibers. See also Remark 2.21 for an application to representations of quantum groups.

I also would like to point out some parallels between exotic sheaves and objects arising in the work of algebraic geometers studying derived categories of coherent sheaves on algebraic varieties. Exotic sheaves can be described in terms of a certain action of the affine braid group B_{aff} of ${}^L G$ on $D(\tilde{\mathcal{N}})$. This description can be reformulated in terms of a map from the set of alcoves (connected components of the complement to affine coroot hyperplanes in the dual space to the Cartan algebra of \mathfrak{g} over \mathbb{R}) to the set of t -structures on $D(\tilde{\mathcal{N}})$. Similar data have been used by Bridgeland in [25]–[27] to construct a component in the space of stability conditions [24], on the derived categories of coherent sheaves on certain varieties. See also Examples 2.8, 2.9 below.

The appearance of the affine braid group, which can be interpreted as the fundamental group of the set of regular semisimple conjugacy classes in the dual group ${}^L G(\mathbb{C})$, suggests a possibility that the structures described above admit a natural interpretation via homological mirror duality, which would identify our derived category of coherent sheaves with a certain Fukaya type category, where the action of the affine braid group arises from monodromy of some family over the space of regular semisimple conjugacy classes in ${}^L G(\mathbb{C})$.

Another connection to algebraic geometry is provided by [17] and [41]. As has been noted above, the derived localization theorem can be interpreted as a construction of a noncommutative resolution of the nilpotent cone \mathcal{N} using crystalline differential

operators in positive characteristic. It turns out that for more general resolutions of singularities, which carry an algebraic symplectic form, a non-commutative resolution can be constructed by a similar procedure. The construction involves quantizing the algebraic symplectic variety in characteristic p , and relating modules over the quantization to coherent sheaves. It has been carried out in [17] for crepant resolutions of quotients V/Γ , where V is a vector space equipped with a symplectic form, and Γ is a finite subgroup in $\mathrm{Sp}(V)$; this yields a particular case of the so-called *categorical McKay correspondence*. The particular case when Γ is the symmetric group on n letters acting on $V = (\mathbb{A}^2)^n$ is related to representations of the rational Cherednik algebra [16]. In Kaledin's work [41] the construction is generalized to more general symplectic resolutions of singularities.

In the remainder of the text we explain some of these contexts (in the order which is roughly inverse to the above) in some detail.

This text is a mixture of an exposition of published results and announcement of yet unpublished ones; statements for which no reference is provided, and which are not well-known, are to appear in a future publication.

Notations and conventions. Throughout the text we work over an algebraically closed field k ; when a semi-simple group G is involved, we assume that the characteristic of k is zero or exceeds the Coxeter number of G .

For an algebraic variety X we let \mathcal{O}_X denote the structure sheaf, and $D(X) = D^b(\mathrm{Coh}_X)$ be the bounded derived category of coherent sheaves on X . Given an action of an algebraic group H on X we write $\mathrm{Coh}^H(X)$ for the category of H -equivariant coherent sheaves; given a coherent sheaf of associative \mathcal{O}_X algebras we let $\mathrm{Coh}(X, \mathcal{A})$ be the category of sheaves of coherent \mathcal{A} modules; if \mathcal{A} is H -equivariant for an algebraic group H acting on X , we let $\mathrm{Coh}^H(X, \mathcal{A})$ be the category of H -equivariant sheaves of coherent \mathcal{A} -modules. We write $D(X)$, $D^H(X)$, $D(\mathcal{A})$, $D^H(\mathcal{A})$ for the bounded derived category of $\mathrm{Coh}(X)$, $\mathrm{Coh}^H(X)$, $\mathrm{Coh}(X, \mathcal{A})$, $\mathrm{Coh}^H(X, \mathcal{A})$ respectively, and $K(X)$, $K^H(X)$, $K(\mathcal{A})$, $K^H(\mathcal{A})$ for the corresponding Grothendieck groups. In particular, these notations apply for an algebra A finite over the center of finite type.

The functors of pull-back, push-forward etc. between categories of sheaves are understood to be the derived functors.

Acknowledgements. I thank my coauthors S. Arkhipov, V. Ginzburg, D. Kaledin, I. Mirković, V. Ostrik and D. Rumynin for their contribution to the joint results, without which they would have never been accomplished. The project described here was conceived during IAS Special Year in Representation Theory (98/99) led by G. Lusztig. Most of the results have been obtained by unraveling the formulas in Lusztig's papers, thus they owe their existence to him. I have learned Lusztig's results and many other things from M. Finkelberg. I have also benefitted a lot from ideas of I. Mirković and his generosity in sharing them. Conversation with many people were very helpful, the incomplete list includes A. Beilinson, V. Drinfeld, D. Gaitsgory, V. Ginzburg, V. Ostrik. I am very grateful to all these people. Finally, I thank

M. Finkelberg, J. Humphreys, D. Kazhdan and I. Mirković for reading a preliminary version of this text and making helpful comments and suggestions.

2. Noncommutative resolutions and braid group actions

2.1. Braid group actions and noncommutative Springer resolution. Though the motivation for the study of our main object comes from applications to representation theory, we first describe it in the language of algebraic geometry. We briefly recall some ideas from [22], [23], [11].

Let Z be a singular algebraic variety. We refer, e.g., to [11] for the notion of a *crepant* resolution; it is easy to see that resolutions $\pi, \tilde{\pi}$ described above are crepant.

By a *noncommutative resolution* [11]¹ one means a coherent torsion free sheaf A of associative \mathcal{O}_Z algebras, which is generically a sheaf of matrix algebras and has finite homological dimension. There exists also a notion of a noncommutative crepant resolution, see [11]. It has been conjectured in *loc. cit.* that any two crepant resolutions, commutative or not, are derived equivalent, in particular, for any crepant resolution $X \rightarrow Z$ and any noncommutative resolution A of Z we have an equivalence $D(X) \cong D(A)$.

2.1.1. The set-up. The notations $G, \mathfrak{g}, \mathcal{B}, \pi: \tilde{\mathcal{N}} \rightarrow \mathcal{N}$ have been defined in the Introduction. Recall that $\tilde{\mathcal{N}} = T^*(\mathcal{B})$ parametrizes pairs (\mathfrak{b}, x) , where $\mathfrak{b} \in \mathcal{B}$ is a Borel subalgebra, and x is the element in the nilpotent radical of \mathfrak{b} . The Springer map $\pi: \tilde{\mathcal{N}} \rightarrow \mathcal{N}$ is given by $\pi: (\mathfrak{b}, x) \mapsto x$. It is embedded in the *Grothendieck simultaneous resolution* $\tilde{\pi}: \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$, where $\tilde{\mathfrak{g}}$ is the variety of pairs (\mathfrak{b}, x) , $\mathfrak{b} \in \mathcal{B}$, $x \in \mathfrak{b}$, and $\tilde{\pi}: (\mathfrak{b}, x) \mapsto x$. The variety $\tilde{\mathfrak{g}}$ is smooth, and the map $\tilde{\pi}$ is proper and generically finite of degree $|W|$, where W is the Weyl group. It factors as the composition of a resolution of singularities $\tilde{\pi}': \tilde{\mathfrak{g}} \rightarrow \mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}$ and the finite projection $\mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h} \rightarrow \mathfrak{g}$; here \mathfrak{h} is the Cartan algebra of \mathfrak{g} . Let $\mathfrak{g}^{\text{reg}} \subset \mathfrak{g}$ denote the subspace of regular (not necessarily semi-simple) elements, and $\tilde{\mathfrak{g}}^{\text{reg}}$ be the preimage of $\mathfrak{g}^{\text{reg}}$ in $\tilde{\mathfrak{g}}$; then $\tilde{\pi}'$ induces an isomorphism $\tilde{\mathfrak{g}}^{\text{reg}} \cong \mathfrak{g}^{\text{reg}} \times_{\mathfrak{h}/W} \mathfrak{h}$.

Much of the representation theory of G or \mathfrak{g} is in one way or another related to the geometry of these spaces and maps.

2.1.2. Affine braid group action. For a characterization of our noncommutative resolution we need to introduce some more notation.

Let Λ be the root lattice of G . For $\lambda \in \Lambda$ we will write $\mathcal{O}(\lambda)$ for the corresponding G -equivariant line bundle on \mathcal{B} , and we set $\mathcal{F}(\lambda) = \mathcal{F} \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}(\lambda)$ if $\mathcal{F} \in D(X)$ for some X mapping to \mathcal{B} .

Let W be the Weyl group, and set $W_{\text{aff}} = W \ltimes \Lambda$. Then W, W_{aff} are Coxeter groups. Notice that W_{aff} is the affine Weyl group of the *Langlands dual* group ${}^L G$.

¹The definition in *loc. cit.* is wider, we use a version convenient for our exposition.

It was mentioned above that $\tilde{\mathfrak{g}}^{\text{reg}} \cong \mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}$; thus W acts on this space via its action on the second factor. The formulas $\Lambda \ni \lambda: \mathcal{F} \mapsto \mathcal{F}(\lambda)$, $W \ni w: \mathcal{F} \mapsto w^*(\mathcal{F})$ are easily shown to define an action² of W_{aff} on the category of coherent sheaves on $\tilde{\mathfrak{g}}^{\text{reg}}$.

The characterization of our “noncommutative Springer resolution” relies on the possibility to extend this action to a weaker structure on the whole of $\tilde{\mathfrak{g}}$. To describe this weaker structure recall that to each Coxeter group one can associate an Artin braid group; let B_{aff} denote the group corresponding to W_{aff} . It admits a topological interpretation, as the fundamental group of the space of regular semi-simple conjugacy classes in the universal cover of the dual group ${}^L G(\mathbb{C})$. For $w \in W_{\text{aff}}$ consider the minimal decomposition of w as a product of simple reflection, and take the product of corresponding generators of B_{aff} . This product is well known to be independent of the choice of the decomposition of w , thus we get a map $W_{\text{aff}} \rightarrow B_{\text{aff}}$ which is a one-sided inverse to the canonical surjection $B_{\text{aff}} \rightarrow W_{\text{aff}}$. We denote this map by $w \mapsto \tilde{w}$. The map is not a homomorphism, however, we have $\tilde{u}\tilde{v} = \tilde{u} \cdot \tilde{v}$ for any $u, v \in W_{\text{aff}}$ such that $\ell(uv) = \ell(u) + \ell(v)$, where $\ell(w)$ denotes the length of the minimal decomposition of w . Let $B_{\text{aff}}^+ \subset B_{\text{aff}}$ be the sub-monoid generated by \tilde{w} , $w \in W_{\text{aff}}$.

For a simple reflection $s_\alpha \in W$ let $S_\alpha \subset \tilde{\mathfrak{g}}^2$ be the closure of the graph of s_α acting on $\tilde{\mathfrak{g}}^{\text{reg}}$. We let S_α denote the intersection of S_α with $\tilde{\mathcal{N}}^2$. Let $\text{pr}_i^\alpha: S \rightarrow \tilde{\mathfrak{g}}$, $\text{pr}_i^\alpha: S \rightarrow \tilde{\mathcal{N}}$, where $i = 1, 2$, be the projections.

Let $\Lambda^+ \subset \Lambda$ be the set of dominant weights in Λ .

For a scheme Y over \mathfrak{g} we set $\tilde{Y} = \tilde{\mathcal{N}} \times_{\mathfrak{g}} Y$, $\tilde{Y} = \tilde{\mathfrak{g}} \times_{\mathfrak{g}} Y$.

Theorem 2.1. (a) *There exists an (obviously unique) action of B'_{aff} on $D(\tilde{\mathfrak{g}})$, $D(\tilde{\mathcal{N}})$ such that for $\lambda \in \Lambda^+ \subset \Lambda \subset W'_{\text{aff}}$ we have $\tilde{\lambda}: \mathcal{F} \mapsto \mathcal{F}(\lambda)$ and for a simple reflection $s_\alpha \in W$ we have $\tilde{s}_\alpha: \mathcal{F} \mapsto (\text{pr}_1^\alpha)^*(\text{pr}_2^\alpha)_* \mathcal{F}$ (respectively, $\tilde{s}_\alpha: \mathcal{F} \mapsto (\text{pr}_1^\alpha)^*(\text{pr}_2^\alpha)_* \mathcal{F}$).*

(b) *This action induces an action on $D(\tilde{Y})$, $D(\tilde{Y})$ for any scheme Y over \mathfrak{g} such that $\text{Tor}_i^{\mathcal{O}_{\tilde{\mathfrak{g}}}}(\mathcal{O}_{\tilde{\mathfrak{g}}}, \mathcal{O}_Y) = 0$, respectively $\text{Tor}_i^{\mathcal{O}_{\tilde{\mathcal{N}}}}(\mathcal{O}_{\tilde{\mathcal{N}}}, \mathcal{O}_Y) = 0$, for $i > 0$.*

Comment on the proof. The theorem can be deduced from material of either Section 3 or 4 below.

Remark 2.2. An example of Y satisfying the assumptions of the theorem is given by a transversal slice to a nilpotent orbit. In particular, if Y is a transversal slice to a subregular orbit, then $\tilde{\mathcal{N}} \times_{\mathfrak{g}} Y$ is well known to be the minimal resolution of a simple surface singularity. The affine braid group action in this case coincides with the one constructed by Bridgeland in [27].

Remark 2.3. The induced action of B_{aff} on the Grothendieck group $K(\tilde{\mathcal{N}})$ factors through W_{aff} . If one passes to the category of sheaves equivariant with respect to

²Throughout the paper by an action of a group on a category I mean a weak action, i.e., a homomorphism to the group of isomorphism classes of autoequivalences. I believe that in all the examples in this text a finer structure can be established, though I have not studied this question.

the multiplicative group, acting by dilations in the fibers of the projection $\tilde{\mathcal{N}} \rightarrow \mathcal{B}$, then the induced action factors through the *affine Hecke algebra* \mathcal{H} , cf. discussion after Theorem 4.2. Furthermore, this construction yields an action of \mathcal{H} on the Grothendieck group $K(\pi^{-1}(e))$ for each $e \in \mathcal{N}$; these \mathcal{H} modules are called the standard \mathcal{H} -modules. Thus the theorem provides a *categorification* of the standard modules for the affine Hecke algebra.

The next result, which plays an important technical role in the proofs, is a categorical counterpart of the quadratic relation in the affine Hecke algebra, see discussion after Theorem 4.2.

Proposition 2.4. *For every simple reflection $s_\alpha \in W_{\text{aff}}$ and every $\mathcal{F} \in \mathfrak{D}$ we have a (canonical) isomorphism in the quotient category $\mathfrak{D}/\langle \mathcal{F} \rangle$*

$$\tilde{s}_\alpha(\mathcal{F}) \cong \tilde{s}_\alpha^{-1}(\mathcal{F}) \pmod{\langle \mathcal{F} \rangle}.$$

Here $\langle \mathcal{F} \rangle$ denotes the full triangulated subcategory generated by \mathcal{F} .

2.1.3. The t -structure and the noncommutative resolution. We will describe certain noncommutative resolutions A, \mathbf{A} of $\mathcal{N}, \mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}$ respectively, together with equivalences $D(A) \cong D(\tilde{\mathcal{N}}), D(\mathbf{A}) \cong D(\tilde{\mathfrak{g}})$, and show how they appear in representation theory. Such data is uniquely determined by the t -structures on $D(\tilde{\mathcal{N}}), D(\tilde{\mathfrak{g}})$, which are the images of the tautological t -structures on $D(A), D(\mathbf{A})$.

Definition 2.5. Let D be a triangulated category equipped with an action of B_{aff} . A t -structure $(D^{<0}, D^{\geq 0})$ on D will be called *braid right exact* if any $b \in B_{\text{aff}}^+$ sends $D^{<0}$ to $D^{<0}$.

Theorem 2.6. (a) *Let X be either \tilde{Y} or \tilde{Y} , where $Y \rightarrow \mathfrak{g}$ is as in Theorem 2.1.*

The category $D(X)$ admits a unique t -structure which is

- (i) *braid right exact, and*
- (ii) *compatible with the standard t -structure on the derived category of vector spaces under the functor of derived global sections $R\Gamma$.*

(b) *There exists a vector bundle \mathcal{E}_X on X , such that the functor $\mathcal{F} \mapsto R\text{Hom}(\mathcal{E}, \mathcal{F})$ is an equivalence between $D(X)$ and $D(A_X)$, sending the t -structure described in (a) to the tautological t -structure on $D(A_X)$; here $A_X = \text{End}(\mathcal{E}_X)^{\text{op}}$, where the upper index denotes the opposite ring.*

Moreover, there exists a vector bundle $\mathcal{E} = \mathcal{E}_{\tilde{\mathfrak{g}}}$ on $\tilde{\mathfrak{g}}$, such that for any X we can take \mathcal{E}_X to be the pull-back of \mathcal{E} to X .

Remark 2.7. It is clear from the definitions that if X is smooth, then A_X is a noncommutative resolution of $Y \times_{\mathfrak{g}} \mathcal{N}$ or $Y \times_{\mathfrak{h}/W} \mathfrak{h}$. In particular, for $Y = \mathfrak{g}$ we get $A = A_{\tilde{\mathcal{N}}}, \mathbf{A} = A_{\tilde{\mathfrak{g}}}$, which are the promised noncommutative resolutions of $\mathcal{N}, \mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}$.

We will call the t -structures described in Theorem 2.6 the *exotic t -structures*, the objects of their heart will be called exotic sheaves.

Example 2.8. Let $G = \mathrm{SL}(2)$, thus $\tilde{\mathcal{N}}$ is the total space of the line bundle $\mathcal{O}(-2)$ on \mathbb{P}^1 , and $\tilde{\mathfrak{g}}$ is the total space of the vector bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. In this case we can set $\mathcal{E} \cong \mathcal{O}_{\tilde{\mathfrak{g}}} \oplus \mathcal{O}_{\tilde{\mathfrak{g}}}(1)$.

This t -structure on $D(\tilde{\mathfrak{g}})$ appeared in Bridgeland’s proof of the derived equivalence conjecture for varieties of dimension three [28]. More precisely, for a flop of threefolds $X, X' \mapsto Y$ Bridgeland constructs some noncommutative resolution of Y which is derived equivalent to both X and X' . The simplest example of a three-fold flop is as follows: $X = X' = \tilde{\mathfrak{g}}, Y = \mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}$ and the two maps $X, X' \rightarrow Y$ are $\tilde{\pi}'$ and $\tilde{\pi}'' = \iota \circ \tilde{\pi}'$, where ι is an involution of $\mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}$ given by $(x, h) \mapsto (x, -h)$. The t -structure on $D(\tilde{\mathfrak{g}})$ given by Bridgeland’s construction applied to this flop turns out to coincide with the t -structure provided by Theorem 2.6.

Example 2.9. Let Y be a transversal slice to the subregular orbit. Thus Y is isomorphic to the quotient \mathbb{A}^2/Γ for some finite subgroup $\Gamma \subset \mathrm{SL}(2)$. The fiber product $X = \tilde{\mathcal{N}} \times_{\mathfrak{g}} Y$ is the minimal resolution of Y . It is well known that there exists a natural equivalence $D(X) \cong D^\Gamma(\mathbb{A}^2)$. The exotic t -structure coincides with the one induced from the tautological t -structure on $D^\Gamma(\mathbb{A}^2)$. Thus A_X is Morita equivalent to the smash product algebra $\Gamma \# \mathcal{O}(\mathbb{A}^2)$. This t -structure appears also in [26].

2.1.4. Parabolic version. One can also consider the partial flag varieties $\mathcal{P} = G/P$, where $P \subset G$ is a parabolic subgroup; thus \mathcal{P} parametrizes parabolic subalgebras $\mathfrak{p} \subset \mathfrak{g}$ of a given type. There exist parabolic versions of the Grothendieck–Springer spaces: $\tilde{\mathfrak{g}}_{\mathcal{P}} = \{\mathfrak{p} \in \mathcal{P}, x \in \mathfrak{p}\}$ and $\tilde{\mathcal{N}}_{\mathcal{P}} = T^*(\mathcal{P})$. We have a proper map $\pi_{\mathcal{P}}: \tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}}_{\mathcal{P}}, (gB, x) \mapsto (gP, x)$. Also, the projection $G/B \rightarrow G/P$ induces a closed embedding $\iota_{\mathcal{P}}: \mathcal{B} \times_{\mathcal{P}} \tilde{\mathcal{N}}_{\mathcal{P}} \hookrightarrow \tilde{\mathcal{N}}$; we let $\mathrm{pr}_{\mathcal{B}}^{\mathcal{P}}$ denote the projection $\mathcal{B} \times_{\mathcal{P}} \tilde{\mathcal{N}}_{\mathcal{P}} \rightarrow \tilde{\mathcal{N}}_{\mathcal{P}}$.

The following result easily follows from the results of [20].

Theorem 2.10. (a) *There exists a unique t -structure on $D(\tilde{\mathfrak{g}}_{\mathcal{P}})$, whose heart contains the image of exotic sheaves under the functor $R\pi_{\mathcal{P}*}: D(\tilde{\mathfrak{g}}) \rightarrow D(\tilde{\mathfrak{g}}_{\mathcal{P}})$.*

(b) *There exists a unique t -structure on $D(\tilde{\mathcal{N}}_{\mathcal{P}})$, such that for any object \mathcal{F} in its heart the object $(\iota_{\mathcal{P}*}(\mathrm{pr}_{\mathcal{B}}^{\mathcal{P}})^*\mathcal{F})(\rho)$ is an exotic sheaf.*

One also has induced nice t -structures on $D(Y \times_{\mathfrak{g}} \tilde{\mathfrak{g}}_{\mathcal{P}}), D(Y \times_{\mathfrak{g}} \tilde{\mathcal{N}}_{\mathcal{P}})$ for Y satisfying a Tor vanishing condition; we omit the details to save space.

Example 2.11. Let $G = \mathrm{SL}(n + 1)$ and $\mathcal{P} = \mathbb{P}^n$. The heart of the t -structure on $\tilde{\mathcal{N}}_{\mathcal{P}} = T^*\mathbb{P}^n$ has a projective generator $\bigoplus_{i=0}^n \mathcal{O}_{T^*\mathbb{P}^n}(-i)$. The heart of the t -structure on $\tilde{\mathfrak{g}}_{\mathbb{P}^n}$ has a projective generator $\bigoplus_{i=0}^n \mathcal{O}_{\tilde{\mathfrak{g}}_{\mathcal{P}}}(i)$.

2.1.5. Reformulation in terms of t -structure assigned to alcoves. A connected component of the complement to the coroot hyperplanes H_α in the dual space to the real Cartan algebra $\mathfrak{h}_{\mathbb{R}}^*$ is called an alcove; in particular, the *fundamental alcove* A_0 is the locus of points where all positive coroots take value between zero and one. Let Alc be the set of alcoves. For $A_1, A_2 \in \mathrm{Alc}$ we will say that A_1 lies above A_2 if for any

positive coroot $\check{\alpha}$ and $n \in \mathbb{Z}$, such that the affine hyperplane $H_{\check{\alpha},n} = \{\lambda \mid \langle \check{\alpha}, \lambda \rangle = n\}$ separates A_1 and A_2 , A_1 lies above $H_{\check{\alpha},n}$, while A_2 lies below $H_{\check{\alpha},n}$, i.e. for $\mu \in A_2$, $\lambda \in A_1$ we have $\langle \alpha, \mu \rangle < n < \langle \alpha, \lambda \rangle$.

Lemma 2.12. *There exists a unique map $\text{Alc} \times \text{Alc} \rightarrow B_{\text{aff}}$, $(A_1, A_2) \mapsto b_{A_1, A_2}$, such that*

- (i) $b_{A_2, A_3} b_{A_1, A_2} = b_{A_1 A_3}$ for any $A_1, A_2, A_3 \in \text{Alc}$;
- (ii) $b_{A_1, A_2} = \tilde{w}$, provided that A_2 lies above A_1 . Here $w \in W_{\text{aff}}$ is such that $w(A_1) = A_2$.

The following result is equivalent to Theorem 2.6.

Theorem 2.13. *Let $X = \tilde{Y}$ or $\tilde{\bar{Y}}$, where Y is as in Theorem 2.1. There exists a unique collection of t -structures indexed by alcoves, $(D_A^{\leq 0}(X), D_A^{> 0}(X))$ such that:*

- (1) (Normalization) *The derived global sections functor $R\Gamma$ is t -exact with respect to the t -structure corresponding to A_0 .*
- (2) (Compatibility with the braid action) *The action of the element b_{A_1, A_2} sends the t -structure corresponding to A_1 to the t -structure corresponding to A_2 .*
- (3) (Monotonicity) *If A_1 lies above A_2 , then $D_{A_1}^{> 0}(X) \supset D_{A_2}^{> 0}(X)$.*

Remark 2.14. The exotic t -structure described in Theorem 2.6 is the one attached to the fundamental alcove A_0 by the construction of Theorem 2.13.

Remark 2.15. The data described in Theorem 2.13 resemble the ones obtained by Bridgeland in the course of description of the manifold of stability conditions on some derived categories of coherent sheaves. To enhance this point we mention a positivity property of the t -structure $(D_A^{\leq 0}(X), D_A^{> 0}(X))$; such properties play a role in the definition of stability conditions [24].

It is easy to show that each of the above t -structures induces a t -structure on the full subcategory $D^f(X) \subset D(X)$ consisting of complexes whose cohomology sheaves have proper support. Let $\mathcal{A}_A = D_A^{\leq 0}(X) \cap D_A^{\geq 0}(X)$ be the heart of the t -structure, and set $\mathcal{A}_A^f = \mathcal{A}_A \cap D^f(X)$. It is easy to show that \mathcal{A}_A^f consists of objects of finite length in \mathcal{A}_A .

Assume that $k = \mathbb{C}$ and X is smooth. Recall that for a smooth complex variety X we have the Chern character map $K(D^f(X)) \rightarrow H_*^{\text{BM}}(X)$, where H_*^{BM} stands for the Borel–Moore homology of the corresponding complex variety endowed with the classical topology. We have a perfect pairing between cohomology and Borel–Moore homology.

We have a well-known identification $\mathfrak{h}^* = H^2(\mathcal{B})$.

Proposition 2.16. *For $A \in \text{Alc}$, $\mathcal{F} \in \mathcal{A}_A^f$, $\mathcal{F} \neq 0$ and $x \in A \subset \mathfrak{h}_{\mathbb{R}}^* \subset H^2(\mathcal{B})$ we have*

$$\langle \text{ch}(\mathcal{F}), \text{pr}^*(\exp(x)) \rangle > 0,$$

where $\exp(x) = 1 + x + \frac{x^2}{2} + \dots + \frac{x^{\dim \mathcal{B}}}{(\dim \mathcal{B})!}$, and pr stands for the projection $X \rightarrow \mathcal{B}$.

Finally, we describe compatibility of our t -structures with duality.

Let \mathcal{S} denote the Grothendieck–Serre duality functor.

Proposition 2.17. \mathcal{S} sends \mathcal{A}_A^f to \mathcal{A}_{-A}^f , where $-A$ denotes the alcove opposite to A .

2.2. Equivariant category and mutations of exceptional sets. The categories $D(\tilde{\mathcal{N}})$, $D(\tilde{\mathfrak{g}})$ have equivariant versions $D^G(\tilde{\mathcal{N}})$, $D^G(\tilde{\mathfrak{g}})$. It turns out that these equivariant categories carry t -structures which are, on the one hand, closely related to the above t -structures on non-equivariant categories, and, on the other hand, admit a direct description in terms of generating exceptional sets in a triangulated category.

Until the end of 2.3.2 we assume that $\text{char}(\mathbf{k}) = 0$.

2.2.1. Exceptional sets and mutations. Recall that an ordered set of objects $\nabla = \{\nabla^i, i \in I\}$ in a triangulated category is called *exceptional* if we have $\text{Hom}^\bullet(\nabla^i, \nabla^j) = 0$ for $i < j$; $\text{Hom}^n(\nabla^i, \nabla^i) = 0$ for $n \neq 0$, and $\text{End}(\nabla^i) = \mathbf{k}$. A set $\Delta = \{\Delta_i, i \in I\}$ of objects is called *dual* to ∇ if $\text{Hom}^\bullet(\Delta_i, \nabla^i) = \mathbf{k}$, and $\text{Hom}^\bullet(\Delta_i, \nabla^j) = 0$ for $i \neq j$; it is exceptional provided ∇ is, where the order on Δ is defined to be opposite to that on ∇ . Let ∇, Δ be two dual exceptional sets which generate a triangulated category \mathcal{D} ; assume that $\{j \mid j \leq i\}$ is finite for every $i \in I$. Then there exists a unique t -structure $(\mathcal{D}^{\geq 0}, \mathcal{D}^{< 0})$ on \mathcal{D} , such that $\nabla \subset \mathcal{D}^{\geq 0}$; $\Delta \subset \mathcal{D}^{< 0}$. This construction is closely related to the definition of a perverse sheaf, see [14] for details.

Let (I, \leq) be an ordered set, and $\nabla^i \in \mathcal{D}, i \in I$ be an exceptional set. Let \leq be another order on I ; we assume that $\{j \mid j \leq i\}$ is finite for every $i \in I$. We let $\mathcal{D}_{\leq i}$ be the full triangulated subcategory generated by $\nabla^j, j \leq i$, and similarly for $\mathcal{D}_{< i}$. Then for $i \in I$ there exists a unique (up to a unique isomorphism) object ∇_{mut}^i such that $\nabla_{\text{mut}}^i \in \mathcal{D}_{\leq i} \cap \mathcal{D}_{< i}^\perp$, and $\nabla_{\text{mut}}^i \cong \nabla^i \text{ mod } \mathcal{D}_{< i}$ (see e.g. [14]). The objects ∇_{mut}^i form an exceptional set indexed by (I, \leq) .

We will say that the exceptional set (∇_{mut}^i) is the \leq mutation of (∇^i) . This construction is related, cf. [14], to the action of the braid group on the set of exceptional sets in a given triangulated category constructed in [21], this action is also called the action by mutations.

2.2.2. Exceptional sets in $D^G(\tilde{\mathcal{N}})$. Recall the standard partial order \leq on the set Λ of weights of G , which is given by: $\lambda \leq \mu$ if $\mu - \lambda$ is a sum of positive roots. Then line bundles $\mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)$ generate $D^G(\tilde{\mathcal{N}})$, and we have $\text{Hom}^\bullet(\mathcal{O}(\lambda), \mathcal{O}(\mu)) = 0$ unless $\mu \leq \lambda$ and $\text{Hom}^\bullet(\mathcal{O}(\lambda), \mathcal{O}(\lambda)) = \mathbf{k}$ [14]. Thus for any complete order on Λ compatible with the partial order \leq , the set of objects $\mathcal{O}(\lambda)$ indexed by Λ with this order is an exceptional set generating $D^G(\tilde{\mathcal{N}})$.

We now introduce another partial ordering \leq on Λ . To this end, recall the 2-sided Bruhat partial order on the affine Weyl group W_{aff} . For $\lambda \in \Lambda$ let w_λ be the minimal length representative of the coset $W\lambda \subset W_{\text{aff}}$. We set $\mu \leq \lambda$ if w_μ precedes w_λ in the Bruhat order.

We fix a complete order \leq_{compl} on Λ compatible with \leq ; we assume that $\{\mu \mid \mu \leq_{\text{compl}} \lambda\}$ is finite for any λ . We define the exceptional set ∇_λ to be the \leq_{compl} mutation of the set $\mathcal{O}(\lambda)$. It follows from the above that ∇^λ is an exceptional set generating $D^G(\tilde{\mathcal{N}})$. We define the *equivariant exotic t -structure* to be the t -structure of the exceptional set ∇^λ , the objects in the heart will be called equivariant exotic sheaves.

We now state compatibility between exotic and equivariant exotic t -structures. Roughly speaking, over an orbit in \mathcal{N} of codimension $2d$ they differ by a shift by d . To state this property more precisely, we need to recall the *perverse coherent t -structure* [13]. Let H be an algebraic group (assumed for simplicity of statements connected) acting on an algebraic variety X . Let \mathbf{p} be a function, called the perversity function, from the set of H -invariant points of the scheme X to \mathbb{Z} . We assume that \mathbf{p} is *strictly monotone and comonotone*, i.e. for points x, y , such that x lies in the closure of y we have $\mathbf{p}(y) < \mathbf{p}(x) < \mathbf{p}(y) + \dim(y) - \dim(x)$. Then one can define the perverse t -structure on $D^H(X)$, which shares some properties with perverse t -structure on the derived category of constructible sheaves [7]. For example, each perverse coherent sheaf (i.e., object in the heart of the t -structure) has finite length, and irreducible objects are in bijection with pairs (O, \mathcal{L}) , where $O \subset X$ is an H -orbit, and \mathcal{L} is an irreducible H -equivariant vector bundle on O . In particular, if the action is such that all orbits have even dimension, then the perversity function $\mathbf{p}(x) = \frac{\text{codim } x}{2}$, called the middle perversity, is strictly monotone and comonotone. It is well-known that the adjoint action of a semi-simple group G on the nil-cone \mathcal{N} has even dimensional orbits.

This construction works also for the category $D^H(A)$, where A is a coherent sheaf of associative \mathcal{O}_X algebras equivariant under H .

Proposition 2.18. *There exists a G -equivariant vector bundle \mathcal{E} on $\tilde{\mathcal{N}}$, such that \mathcal{E} , with the G -equivariant structure forgotten, is a projective generator for the heart of the exotic t -structure.*

We have an equivalence $\mathcal{F} \mapsto R \text{Hom}(\mathcal{E}, \mathcal{F})$ between $D^G(\tilde{\mathcal{N}})$ and $D^G(A)$, where $A = \text{End}(\mathcal{E})^{\text{op}}$. Under this equivalence the equivariant exotic t -structure corresponds to the perverse coherent t -structure of the middle perversity.

2.3. Grading on exotic sheaves and canonical bases

2.3.1. Graded equivariant category and positivity by Frobenius weights. We proceed to state a deep property of exotic sheaves related to an additional grading on the Ext spaces between them. Recall the current assumption that $\text{char}(\mathbf{k}) = 0$.

Consider the category $D^{G \times \mathbb{G}_m}(\tilde{\mathcal{N}})$, where \mathbb{G}_m acts on $\tilde{\mathcal{N}}$ by $t: x \mapsto t^2x$. For $d \in \mathbb{Z}$ let $\mathcal{F} \mapsto \mathcal{F}(d)$ denote twisting by the d -th power of the tautological character of \mathbb{G}_m . We refer to [14] for an elementary description of a canonical lifting $\tilde{\Delta}_\lambda, \tilde{\nabla}^\lambda$ of $\Delta_\lambda, \nabla^\lambda$ to $D^{G \times \mathbb{G}_m}$. This also fixes a lifting \tilde{L} of each irreducible equivariant exotic sheaf L to $D^{G \times \mathbb{G}_m}$.

Theorem 2.19. *For irreducible exotic equivariant sheaves L_1, L_2 we have*

$$\mathrm{Ext}^i(\tilde{L}_1, \tilde{L}_2(d)) = 0$$

for $d \leq 0$ and all i .

Remark 2.20. The theorem follows from results of [12] on relation between exotic sheaves and perverse sheaves on the affine flag manifold of the dual group, see also Proposition 4.5 below. They allow to deduce the theorem from Gabber's Theorem [7] on positivity of weights of Frobenius action on Ext's between pure perverse sheaves of the same weight. Thus it is the least elementary of the results mentioned so far in this text.

The motivation for the theorem is its consequence below, which shows (in most cases) that classes of exotic sheaves form a *canonical basis* in the Grothendieck group. This is parallel to the proof of the Kazhdan–Lusztig conjecture: according to Soergel, cf. [50], the latter is equivalent to the statement that for a certain explicitly defined graded version of Bernstein–Gel'fand–Gel'fand category \mathcal{O} the grading on Ext^1 between irreducible objects has vanishing components of non-positive degrees. The only known way to prove this vanishing is to identify category \mathcal{O} with a category of perverse sheaves or Hodge D -modules, and use deep information about purity of Frobenius or Hodge weights.

Remark 2.21. Another application of Theorem 2.19 is explained in [14]. Together with the *Koszul duality* formalism of [9] it allows one to show that equivariant exotic sheaves control cohomology of quantum groups at a root of unity with coefficients in a tilting module.

2.3.2. Non-equivariant graded category and canonical bases. We fix $X = \tilde{\mathcal{N}}$. Recall the category $\mathcal{A}^f = \mathcal{A}_{A_0}^f \subset \mathcal{A}$ of exotic sheaves of finite length. It is easy to see that $\mathcal{A}^f = \bigoplus_{e \in \mathcal{N}} \mathcal{A}_e$, where $\mathcal{A}_e = \mathcal{A} \cap \mathcal{D}_e$, and $\mathcal{D}_e \subset D(\tilde{\mathcal{N}})$ is the full subcategory of complexes whose cohomology sheaves are set-theoretically supported on $\mathcal{B}_e = \pi^{-1}(e)$. We have $K(\mathcal{A}_e) \cong K(\mathcal{B}_e)$. Furthermore, the Chern character map provides an isomorphism $K(\mathcal{B}_e)_F \cong H_*^{\mathrm{BM}}(\mathcal{B}_e)_F$, where F denotes a coefficient field of characteristic zero (\mathbb{C} or $\overline{\mathbb{Q}_l}$), see, e.g. [19].

The classes of irreducible objects form a basis in $K(\mathcal{A}_e)$. We proceed to explain the properties of the category, which are needed to relate this basis to *the canonical bases* in $H_*^{\mathrm{BM}}(\mathcal{B}_e)$. The definition of the latter is due to Lusztig [45], and follows the example of Kashiwara's characterization of crystal bases [42]. More precisely, Lusztig suggested a way to characterize a basis in $H_*^{\mathrm{BM}}(\mathcal{B}_e)$, and conjectured that a basis satisfying his axioms exists; he showed that it is then unique (up to a sign). We will not recall Lusztig's characterization in detail; instead we describe its structure and explain the properties of exotic sheaves, which imply (modulo a technicality, which is easy to check in many cases) that Lusztig's axioms are satisfied by the basis of irreducible exotic sheaves.

One can find a homomorphism $\varphi: \mathrm{SL}(2) \rightarrow G$, such that $d\varphi$ sends the standard upper triangular generator of $\mathfrak{sl}(2)$ to e . Then we get an action a_φ of the multiplicative group \mathbb{G}_m on \mathfrak{g} given $a_\varphi(t): x \mapsto t^2 \cdot \mathrm{ad}(\varphi(\mathrm{diag}(t^{-1}, t)))x$. This action fixes e .

We let $\mathcal{D}_e^{\mathbb{G}_m} \subset D^{\mathbb{G}_m}(\tilde{\mathcal{N}})$ be the full subcategory of complexes, which are set theoretically supported on $\pi^{-1}(e)$. Twisting by the tautological character of \mathbb{G}_m defines an auto-equivalence of this category, which we denote by $\mathcal{F} \mapsto \mathcal{F}(1)$. The exotic t -structure is inherited by the \mathbb{G}_m -equivariant category; we let $\mathcal{A}_e^{\mathrm{gr}}$ denote the heart of the latter. It is easy to see that the forgetful functor $\mathcal{D}_e^{\mathbb{G}_m} \rightarrow \mathcal{D}_e$ sends $\mathrm{Irr}(\mathcal{A}_e^{\mathrm{gr}})$ to $\mathrm{Irr}(\mathcal{A}_e)$, where Irr stands for the set of isomorphism classes of irreducible objects. This gives a bijection $\mathrm{Irr}(\mathcal{A}_e^{\mathrm{gr}})/\mathbb{Z} \cong \mathrm{Irr}(\mathcal{A}_e)$, where \mathbb{Z} acts on $\mathrm{Irr}(\mathcal{A}_e^{\mathrm{gr}})$ by $\mathcal{F} \mapsto \mathcal{F}(n)$. We also have $K(\mathcal{A}_e^{\mathrm{gr}}) \cong K^{\mathbb{G}_m}(\mathcal{B}_e) \cong K(\mathcal{B}_e)[v, v^{-1}]$, where multiplication by v corresponds to twisting by the tautological character of \mathbb{G}_m .

The canonical basis in $K(\mathcal{B}_e)[v, v^{-1}]$ is characterized (up to a sign) by two properties: *invariance under an involution* and *asymptotic orthogonality* [45]. These are reflected, respectively, in categorical properties (i) and (ii) in the next theorem.

Notice that the action of B_{aff} on \mathcal{D}_e is inherited by $\mathcal{D}_e^{\mathbb{G}_m}$. Recall that \mathcal{S} is the Grothendieck–Serre duality.

In view of Theorem 2.13 and Proposition 2.17, the contravariant auto-equivalence $\tilde{w}_0 \circ \mathcal{S}$ is t -exact with respect to the t -structure corresponding to the fundamental alcove A_0 , hence it permutes irreducible objects of \mathcal{A}_{A_0} ; here $w_0 \in W$ is the long element.

Theorem 2.22. *There exists a canonical section of the map $\mathrm{Irr}(\mathcal{A}_e^{\mathrm{gr}}) \rightarrow \mathrm{Irr}(\mathcal{A}_e)$, $L \mapsto \tilde{L}$, such that*

(i) *The image of the section is invariant under every automorphism of G which is identity on the image of φ_e , and also under $\tilde{w}_0 \circ \mathcal{S}$.*

(ii) $\mathrm{Ext}_{\mathcal{A}_e^{\mathrm{gr}}}^1(\tilde{L}_1, \tilde{L}_2(i)) = 0$ for $i \leq 0$ and any $L_1, L_2 \in \mathrm{Irr}(\mathcal{A}_e)$; here $\bar{\mathcal{A}}_e^{\mathrm{gr}} \subset \mathcal{A}_e$ is the full subcategory of objects where the ideal of the point e in $\mathcal{O}(\mathfrak{g})$ acts by zero.

Comments on the proof. The theorem can be deduced formally from Theorem 2.19 and Proposition 2.18. Thus its proof relies on ideas of geometric Langlands duality used in [2], and on Gabber’s Theorem (see comments after Theorem 2.19).

Corollary 2.23. *Suppose that the involution $\tilde{\beta}$ defined in [45], §5.11 induces identity on the specialization at $q = 1$. Then Conjecture 5.12 of loc. cit., except, possibly, 5.12 (g), holds; moreover, the signed basis $\mathbf{B}_{\mathcal{B}_e}^\pm$, whose existence is conjectured in loc. cit., is formed by the classes of the objects \tilde{L} , where L runs over irreducible objects in \mathcal{A}_e .*

Remark 2.24. The assumptions of the corollary are easy to check in many cases, e.g., if the nilpotent element e is regular in a Levi subalgebra.

Remark 2.25. In fact, in [45] Lusztig works with sheaves which are also equivariant under a maximal torus in the centralizer of e . We omit this version here to simplify notations, treating this set-up does not involve new ideas.

Remark 2.26. Validity of Conjecture 5.12 (g) of [45] is related to the following question. Let $Y \subset \mathfrak{g}$ be a (Slodowy) transversal slice to a G orbit in \mathcal{N} , and $X = \tilde{\mathcal{N}} \times_{\mathfrak{g}} Y$. Let A_X be as in Theorem 2.6. One can show that A can be endowed with a natural grading; moreover, Theorem 2.22 is equivalent to the fact that this grading can be chosen so that the graded components of negative degree vanish, while the component of degree zero is semi-simple. The question is whether the resulting graded algebra is Koszul. If e is subregular, then the positive answer is easy to prove.

2.3.3. Independence of the (large) prime. It is not hard to show that (co)homology of the Springer fiber is independent of the ground field k , i.e. we have canonical isomorphisms $H_{\bullet}^{\text{BM}}(\mathcal{B}_e^k) \cong H_{\bullet}^{\text{BM}}(\mathcal{B}_e^{\mathbb{C}})$, where the upper index denotes the ground field, and H_{\bullet}^{BM} stands for l -adic Borel–Moore homology, $l \neq \text{char}(k)$.

The definition of the exotic t -structure is not specific to a particular ground field. This allows one to prove the following.

Proposition 2.27. *For all but finitely many prime numbers p the following is true. The classes in $H_{\bullet}^{\text{BM}}(\mathcal{B}_e^k) = H_{\bullet}^{\text{BM}}(\mathcal{B}_e^{\mathbb{C}})$ of irreducible exotic sheaves over k of characteristic p coincide with the classes of irreducible exotic sheaves over \mathbb{C} .*

3. D -modules in positive characteristic and localization theorem

3.1. Generalities on crystalline D -modules in positive characteristic

3.1.1. Definition and description of the center. Let X be a smooth variety over the field k .

The sheaf $\mathcal{D} = \mathcal{D}_X$ of *crystalline differential operators* (or differential operators without divided powers, or PD differential operators) on X is defined as the enveloping of the tangent Lie algebroid, i.e., for an affine open $U \subset X$ the algebra $\mathcal{D}(U)$ contains the subalgebra \mathcal{O} of functions, has an \mathcal{O} -submodule identified with the Lie algebra of vector fields $\text{Vect}(U)$ on U , and these subspaces generate $\mathcal{D}(U)$ subject to relations $\xi_1 \xi_2 - \xi_2 \xi_1 = [\xi_1, \xi_2] \in \text{Vect}(U)$ for $\xi_1, \xi_2 \in \text{Vect}(U)$, and $\xi \cdot f - f \cdot \xi = \xi(f)$ for $\xi \in \text{Vect}(U)$ and $f \in \mathcal{O}(U)$.

If $\text{char}(k) = 0$, then \mathcal{D}_X is the familiar sheaf of differential operators. From now on assume that k is of characteristic $p > 0$. Then \mathcal{D}_X shares some features with the characteristic zero case; for example, \mathcal{D}_X carries an increasing filtration “by order of a differential operator”, and the associated graded $\text{gr}(\mathcal{D}_X) \cong \mathcal{O}_{T^*X}$ canonically. On the other hand, some phenomena are special to the characteristic p setting. We have an action map $\mathcal{D}_X \rightarrow \text{End}(\mathcal{O}_X)$, which is not injective, unlike in the case of characteristic zero. For example, if $X = \mathbb{A}^1 = \text{Spec}(k[x])$, the section $\partial_x^p \neq 0$ of \mathcal{D}_X acts by zero on \mathcal{O} . Also, \mathcal{D}_X has a huge center; for example, if $X = \mathbb{A}^n = \text{Spec}(k[x_1, \dots, x_n])$, then x_i^p and $\partial_{x_i}^p$ are readily seen to generate the center $Z(\mathcal{D}_{\mathbb{A}^n})$ freely. More generally, for any X the center $Z(\mathcal{D}_X)$ is freely generated by elements of the form f^p , $f \in \mathcal{O}_X$

and $\xi^p - \xi^{[p]}$, $\xi \in \text{Vect}_X$, where $\xi^{[p]}$ is the *restricted power* of the vector field ξ ; it is characterized by $\text{Lie}_{\xi^{[p]}}(f) = \text{Lie}_{\xi}^p(f)$ for $f \in \mathcal{O}_X$, where Lie stands for the Lie derivative. The center $Z(\mathcal{D}_X)$ is canonically isomorphic to the sheaf of rings $\mathcal{O}_{T^*X^{(1)}}$ where the super-index (1) stands for Frobenius twist.³ Thus \mathcal{D}_X can be considered as a quasi-coherent sheaf of algebras on $T^*X^{(1)}$.

3.1.2. Azumaya property. Recall that an *Azumaya algebra* on a scheme X is a locally free sheaf \mathcal{A} of associative \mathcal{O}_X algebras, such that the fiber of \mathcal{A} at every geometric point is isomorphic to a matrix algebra. The following fundamental observation is due to Mirković and Rumynin, though a weak form of it can be traced to an earlier work [40].

Theorem 3.1 ([19]). \mathcal{D}_X is an Azumaya algebra of rank $p^{2 \dim(X)}$ on $T^*X^{(1)}$.

See [48], [10] for generalizations and applications.

Recall that two Azumaya algebras $\mathcal{A}, \mathcal{A}'$ are called *equivalent* (we then write $\mathcal{A} \sim \mathcal{A}'$) if they are Morita equivalent, i.e. if there exists a coherent locally projective sheaf \mathcal{M} of $\mathcal{A} - \mathcal{A}'$ bimodules, such that $\mathcal{A}' \xrightarrow{\sim} \text{End}(\mathcal{M})^{\text{op}}$; we will then say that \mathcal{M} provides an equivalence between \mathcal{A} and \mathcal{A}' . In particular, an Azumaya algebra \mathcal{A} is *split* if $\mathcal{A} \sim \mathcal{O}_X$; this happens iff $\mathcal{A} \cong \text{End}(\mathcal{E})$ for a vector bundle \mathcal{E} . For two equivalent Azumaya algebras $\mathcal{A}, \mathcal{A}'$ we have an equivalence of categories of modules $\text{Coh}(X, \mathcal{A}) \cong \text{Coh}(X, \mathcal{A}')$, depending on the choice of a bimodule providing the equivalence between \mathcal{A} and \mathcal{A}' ; in particular, for a split Azumaya algebra we have $\text{Coh}(X, \mathcal{A}) \cong \text{Coh}(X)$.

For a smooth variety X over a positive characteristic field, the Azumaya algebra \mathcal{D}_X is not split unless $\dim(X) = 0$. However, it is split on the zero section, see [48] for more information.

We will also need a twisted version of differential operators. If \mathcal{L} is a line bundle on X , then one can consider the sheaf $\mathcal{D}^{\mathcal{L}} = \mathcal{D}_X^{\mathcal{L}}$ of differential operators in \mathcal{L} . A similar argument shows that this is also an Azumaya algebra over $T^*X^{(1)}$; moreover, we have a canonical equivalence

$$\mathcal{D}_X \sim \mathcal{D}_X^{\mathcal{L}}, \tag{1}$$

given by the bimodule $\mathcal{D}_X \otimes_{\mathcal{O}(X)} \mathcal{L}^{-1}$.

Remark 3.2. Notice that if $\mathcal{L} = \mathcal{L}_0^p = \text{Fr}^*(\mathcal{L}_0)$ for some line bundle \mathcal{L}_0 , then we have a canonical isomorphism $\mathcal{D}_X^{\mathcal{L}} \cong \mathcal{D}_X$; however, the above equivalence $\mathcal{D}_X^{\mathcal{L}} \cong \mathcal{D}_X$ is not identity, but rather tensor product *over the ring* $\mathcal{O}_X^p = \mathcal{O}_{X^{(1)}}$ with the line bundle $\mathcal{L}_0^{(1)}$.

³Recall that Frobenius twist of a variety X over a perfect field k is defined to be isomorphic to X as an abstract scheme, with the k -linear structure twisted by Frobenius. Not only $X \cong X^{(1)}$ as abstract schemes, but also $X \cong X^{(1)}$ as k -schemes, provided that X is defined over \mathbb{F}_p . For this reason we will sometimes identify X with $X^{(1)}$ and omit Frobenius twist from notation.

3.2. Crystalline operators on \mathcal{B} . We now consider $X = \mathcal{B}$. We abbreviate $\mathcal{D}_{\mathcal{B}}^{\mathcal{O}(\lambda)} = \mathcal{D}^{\lambda}$.

3.2.1. Splitting the Azumaya algebra. It was mentioned above that $\mathcal{D}_{\mathcal{B}}$ splits on the zero section. In fact, we have the following stronger statement.

Theorem 3.3. (a) *There exists an Azumaya algebra \mathcal{A} on $\mathcal{N}^{(1)}$, such that $\mathcal{D}^{-\rho} \cong \pi^*(\mathcal{A})$.*

(b) *For any λ we have an equivalence of Azumaya algebras on $\mathcal{N}^{(1)}$: $\mathcal{D}^{\lambda} \sim \pi^*(\mathcal{A})$.*

(c) *$\mathcal{D}^{\lambda}(\mathcal{B})$ is split on the formal neighborhood of every fiber of π .*

Sketch of proof. (a) reduces to irreducibility of baby Verma modules with highest weight $-\rho$, which follows from [29]. It implies, moreover, that the statement holds for \mathcal{A} being the quotient of the enveloping algebra $U(\mathfrak{g})$ by the central ideal corresponding to $-\rho$. (b) follows from (a) in view of the equivalence (1). Finally, (c) follows from (b), since every Azumaya algebra over a complete local ring with an algebraically closed residue field splits. \square

Let $\mathcal{D}^{\lambda}\text{-mod}^f \subset \text{Coh}(\tilde{\mathcal{N}}^{(1)}, \mathcal{D}^{\lambda})$ the full subcategory of sheaves, whose support (which is a subvariety in $\tilde{\mathcal{N}}^{(1)}$) is proper. Let $\text{Coh}^f(\tilde{\mathcal{N}}) \subset \text{Coh}(\tilde{\mathcal{N}})$ be the full subcategory of sheaves with proper support.

Corollary 3.4. *For every $\lambda \in \Lambda$ we have an equivalence $\mathcal{D}^{\lambda}\text{-mod}^f \cong \text{Coh}^f(\tilde{\mathcal{N}})$.*

Proof. Since the target of π is affine, a subscheme Z in $\tilde{\mathcal{N}}^{(1)}$ is proper iff it lies in a finite union of nilpotent neighborhoods of Springer fibers. Thus the claim follows from Theorem 3.3(b). \square

For each $\lambda \in \Lambda$ and $e \in \mathcal{N}$ we fix the splitting bundle \mathcal{E}_e^{λ} for \mathcal{D}^{λ} on the formal neighborhood of $\pi^{-1}(e)$ as follows. For $\lambda = -\rho$ we let \mathcal{E}_e^{λ} be the pull-back under π of a splitting bundle for the Azumaya algebra \mathcal{A} on the formal neighborhood of e in $\mathcal{N}^{(1)}$. For a general λ we get \mathcal{E}^{λ} from $\mathcal{E}^{-\rho}$ by applying the canonical equivalence (1) between $\mathcal{D}^{-\rho}$ and \mathcal{D}^{λ} ; thus $\mathcal{E}^{\lambda} = \mathcal{E}^{-\rho} \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}(\lambda + \rho)$.

We let F_{λ} denote the resulting equivalence between $\mathcal{D}^{\lambda}\text{-mod}^f$ and $\text{Coh}^f(\tilde{\mathcal{N}})$. Notice that for $\lambda' = \lambda + p\mu$ the sheaves of algebras \mathcal{D}^{λ} and $\mathcal{D}^{\lambda'}$ are canonically identified; however, the equivalences F_{λ} and $F_{\lambda'}$ are different, cf. Remark 3.2.

3.2.2. Derived localization in positive characteristic. Let $U = U(\mathfrak{g})$ be the enveloping algebra.

Assume first that $\text{char}(\mathbf{k}) = 0$. Recall the famous Localization Theorem [5], [30], which provides an equivalence $U^{\lambda}\text{-mod} \cong \mathcal{D}^{\lambda}\text{-mod}(\mathcal{B})$, where λ is a dominant integral weight, $\mathcal{D}^{\lambda}\text{-mod}$ denotes the corresponding twisted D -modules category, and $U^{\lambda}\text{-mod}$ is the category of \mathfrak{g} -modules with central character corresponding to λ . For two integral weights λ, μ the categories $\mathcal{D}^{\mu}\text{-mod}$ and $\mathcal{D}^{\lambda}\text{-mod}$ can be identified by means of the equivalence $T_{\mu}^{\lambda}: \mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{O}(\lambda - \mu)$. If λ, μ are dominant, then

the global sections functors intertwine this equivalence with the translation functor, which provides an equivalence $U^\lambda\text{-mod} \cong U^\mu\text{-mod}$.

Assume now that μ is integral regular, thus $\mu = w(\lambda + \rho) - \rho$ for some dominant integral λ , $w \in W$. Then the functor of global sections on $D_\mu\text{-mod}$ is no longer exact; however, it follows from [6] that the derived functor $R\Gamma = R\Gamma_\mu : D^b(\mathcal{D}^\mu\text{-mod}) \rightarrow D^b(U^\lambda\text{-mod})$ is still an equivalence. The triangle formed by the three equivalences $R\Gamma_\mu, T_\mu^\lambda, R\Gamma_\lambda$ does not commute. Thus we get an auto-equivalence R_w of $D^b(\mathcal{D}^\lambda\text{-mod})$, $R_w = R\Gamma_\lambda^{-1} \circ R\Gamma_\mu \circ T_\lambda^\mu$. In [6] it is shown that R_w can be described by an explicit correspondence, which makes it natural to call R_w the *Radon transform*, or *the intertwining functor*. Moreover, the assignment $\tilde{w} \mapsto R_w$ extends to an action of the Artin braid group B attached to G on $D^b(U^\lambda\text{-mod})$.

A part of this picture can be generalized to characteristic p .

The obvious characteristic p analogue of the above equivalence of abelian categories does not hold for any integral λ . Indeed, it is well known that for any coherent sheaf \mathcal{F} on the (Frobenius twist of) a smooth variety the sheaf $\text{Fr}^*(\mathcal{F})$ carries a flat connection; in particular, so does the sheaf $\text{Fr}^*(L) = L^{\otimes p}$, where L is a line bundle. Thus for $\mathcal{F} \in \mathcal{D}^\lambda\text{-mod}$ we have $\mathcal{F} \otimes L^p \in \mathcal{D}^\lambda\text{-mod}$. If L is anti-ample and the support of \mathcal{F} is projective of positive dimension, then some of the higher derived functors $R^i\Gamma(\mathcal{F} \otimes L^{dp}) \neq 0$ for large d .

However, we do have an analogue of the “derived” localization theorem. From now on assume that $\text{char}(\mathbf{k}) = p > 0$.

The center Z of U contains the subalgebra $Z_{\text{HC}} = U^G \cong \text{Sym}(\mathfrak{h})^W$, which we call the Harish-Chandra center. We have a natural map $\Lambda/p\Lambda \rightarrow \mathfrak{h}^*/W, \lambda \mapsto d\lambda \text{ mod } W$. Thus every $\lambda \in \Lambda$ defines a maximal ideal of Z_{HC} . We let $U^\lambda = U \otimes_{Z_{\text{HC}}} \mathbf{k}$ denote the corresponding central reduction. Notice that the set of weights $\mu \in \Lambda$, such that the quotients U^λ and U^μ of U coincide, is precisely the W_{aff} -orbit of λ with respect to the action $w \bullet \lambda = p w(\frac{\lambda + \rho}{p}) - \rho$. We will say that $\lambda \in \Lambda$ is p -regular if the stabilizer in W of $\lambda + p\Lambda \in \Lambda/p\Lambda$ is trivial.

We also have another central subalgebra $Z_{\text{Fr}} \subset U$, called the Frobenius center. It is generated by expressions of the form $x^p - x^{[p]}, x \in \mathfrak{g}$, where the *restricted power map* $x \mapsto x^{[p]}$ is characterized by $\text{ad}(x^{[p]}) = \text{ad}(x)^p$. Thus maximal ideals of Z_{Fr} are in bijection with points of $\mathfrak{g}^* \cong \mathfrak{g}$.

Let $U^\lambda\text{-mod}$ denote the category of finitely generated U^λ -modules, and let $U^\lambda\text{-mod}^f \subset U^\lambda\text{-mod}$ be the full subcategory of finite length modules.

For a pair $\lambda \in \Lambda, e \in \mathfrak{g}^*$ let $U_e^\lambda\text{-mod}$ be the category of finitely generated U^λ -modules, which are killed by some power of the maximal ideal of e in Z_{Fr} . This category is zero unless $e \in \mathcal{N}$. We also have $U^\lambda\text{-mod}^f = \bigoplus_{e \in \mathcal{N}} U_e^\lambda\text{-mod}$.

Let $\mathcal{D}_e^\lambda\text{-mod} \subset \mathcal{D}^\lambda\text{-mod}$ be the full subcategory of objects which are supported on a nilpotent neighborhood of $\pi^{-1}(e)$; here we think of \mathcal{D}^λ modules as sheaves on $\tilde{\mathcal{N}}^{(1)}$ with an additional structure.

Theorem 3.5 ([19]). (a) *We have a natural isomorphism $\Gamma(\mathcal{D}^\lambda) \cong U^\lambda$ for every $\lambda \in \Lambda$.*

(b) If $\lambda \in \Lambda$ is p -regular, then the derived global sections functor provides an equivalence $R\Gamma_\lambda: D(\mathcal{D}^\lambda) \xrightarrow{\sim} D(U^\lambda)$. It restricts to equivalences $D^b(\mathcal{D}^\lambda\text{-mod}^f) \cong D^b(U^\lambda\text{-mod}^f)$, $D^b(\mathcal{D}_\epsilon^\lambda\text{-mod}) \cong D^b(U_\epsilon^\lambda\text{-mod})$.

Remark 3.6. This theorem has several versions and generalization. One can work with the more general categories of twisted D -modules, thereby obtaining a category of modules over an Azumaya algebra on the formal neighborhood of $\tilde{\mathcal{N}}$ in $\tilde{\mathfrak{g}}$, or more general subschemes or formal completions of $\tilde{\mathfrak{g}}$. For singular weights λ there is a version of the theorem that relates derived categories of modules to sheaves on (the neighborhoods of) *parabolic Springer fibers* [20]. Another construction works with differential operators on a partial flag variety G/P for a parabolic subgroup $P \subset G$, *loc. cit.*, cf. also subSection 2.1.4 above.

For a scheme Y mapping to \mathfrak{g} and satisfying the Tor vanishing conditions of Theorem 2.1 we have an equivalence between the derived category of modules over Azumaya algebras on \tilde{Y}, \tilde{Y} obtained as pull-back of the algebra of (twisted) differential operators and derived category of modules over the algebra of global sections.

If Y is a transversal slice to a nilpotent orbit, then the algebra of global sections of the Azumaya algebra on \tilde{Y} is probably related to Premet’s quantization of Slodowy slices and generalized Whittaker D -modules, see [49], [38].

There exists a generalization of this result for Y not satisfying the Tor vanishing condition. It involves coherent sheaves on the *differential graded* scheme, which is the derived fiber product of Y and $\tilde{\mathfrak{g}}$ over \mathfrak{g} . The particular case $Y = \{0\}$ is closely related to the description of the derived category of the principal block in representations of a quantum group at a root of unity provided by [3].

Proposition 3.7. (a) A weight $\lambda \in \Lambda$ is p -regular iff $\frac{\lambda+\rho}{p}$ lies in some alcove.

(b) The t -structure on $\mathcal{D}^0\text{-mod}$ induced by the equivalence $R\Gamma^\lambda \circ T_0^\lambda$ for a p -regular λ depends only on the alcove of $\frac{\lambda+\rho}{p}$ (see the beginning of Section 3.2.2 for notation).

Thus we get a collection of t -structure on $\mathcal{D}^0\text{-mod}$ indexed by alcoves; we denote the t -structure attached to $A \in \text{Alc}$ by $D_A^{<0}(\mathcal{D}), D_A^{\geq 0}(\mathcal{D})$. The following properties of the collection follow from [19], [20].

Theorem 3.8. (a) Let A_1, A_2 be two alcoves. If A_1 lies above A_2 , then $D_{A_1}^{>0}(\mathcal{D}) \supset D_{A_2}^{>0}(\mathcal{D})$.

(b) There exists an action of B_{aff} on $D(\mathcal{D})$, such that the following holds. Let $\lambda \in \Lambda$, $\frac{\lambda+\rho}{p} = w\left(\frac{\rho}{p}\right)$ for $w \in W_{\text{aff}}$, and let A be the alcove of $\frac{\lambda+\rho}{p}$. Then $R\Gamma_\lambda \cong R\Gamma_0 \circ b_{A_0, A_1}$. Thus b_{A_0, A_1} sends the t -structure $D_{A_0}^{<0}(\mathcal{D}), D_{A_0}^{\geq 0}(\mathcal{D})$ to the t -structure $D_A^{<0}(\mathcal{D}), D_A^{\geq 0}(\mathcal{D})$.

(c) The restriction of the B_{aff} action to $D^b(\mathcal{D}^\lambda\text{-mod}^f) \cong D^b(\text{Coh}^f(\tilde{\mathcal{N}}))$ coincides with the restriction of the action from Theorem 2.1.

(d) *The derived global sections functor $R\Gamma$ on $D^b(\text{Coh}^f(\tilde{\mathcal{N}}))$ is t -exact with respect to the t -structure induced from $(D_{A_0}^{<0}(\mathcal{D}), D_{A_0}^{\geq 0}(\mathcal{D}))$ via the equivalence F_0 (see the end of Section 3.2.1 for notation).*

Corollary 3.9. *The t -structure on $\text{Coh}^f(\tilde{\mathcal{N}})$ induced by the exotic t -structure coincides with the one induced from $(D_{A_0}^{<0}(\mathcal{D}), D_{A_0}^{\geq 0}(\mathcal{D}))$ via the equivalence F_0 (see the end of 3.2.1).*

In particular, we have a Morita equivalence $\mathcal{A}_e \sim U\text{-mod}_e^\lambda$ for every p -regular $\lambda \in \Lambda$.

The corollary follows by comparing Theorem 3.8 with Theorem 2.13.

Corollary 3.10. (a) *Let U_e^λ denote the specialization of the enveloping algebra $U(\mathfrak{g})$ at the central character corresponding to $e \in \mathcal{N}$ and a regular integral weight λ . Then we have a canonical isomorphism $K(U_e^\lambda)_F \cong H_\bullet^{\text{BM}}(\mathcal{B}_e)_F$, where F is a field of characteristic zero.*

(b) *The image of the set of classes of irreducible modules under this isomorphism is independent of the base field \mathbf{k} , except for a finite number of values of the characteristic.*

Part (a) of the corollary follows directly from Theorem 3.5 (cf. the discussion preceding Proposition 2.16), while part (b) follows from Corollary 3.9 and Proposition 2.27.

Remark 3.11. For $e = 0$ part (a) of the proposition is standard, and part (b) can be deduced from [1]. Our method uses the principal tool of [1], namely, the reflection functors, in the disguise of the braid group action; geometry of the Springer map is the new ingredient.

In fact, we have the following stronger, though more difficult statement. We will say that a basis in $H_\bullet^{\text{BM}}(\mathcal{B}_e)$ is canonical if it is the image of a basis in the equivariant Grothendieck group $K^{\text{G}_m}(\mathcal{B}_e)$, satisfying Lusztig’s axioms [45], under forgetting the equivariance composed with the Chern character map. According to a result of [45] such a basis is unique up to multiplication of some of its elements by -1 , if it exists.

Corollary 3.12. *Enforce the assumption of Corollary 2.23. Then for almost all $p = \text{char}(\mathbf{k})$ the isomorphism $K^0(U\text{-mod}_e^0)_F \cong H_\bullet^{\text{BM}}(\mathcal{B}_e)_F$ of Corollary 3.10 (a) sends classes of irreducible objects to elements of a canonical basis. Thus Conjecture 17.2 of [45] holds in this case.*

This corollary is immediate from Corollary 2.23 together with Theorem 3.8(d). Thus its proof, unlike the proof of Corollary 3.10, relies on Gabber’s Theorem [7] and ideas of local geometric Langlands duality, on which the results of [2] are based.

Remark 3.13. For large p the particular case $e = 0$ of the Conjecture 17.2 of [45] is well known to imply the previous Lusztig conjectures [46], which describe characters of algebraic groups in finite characteristics. Lusztig’s program for a proof of these

conjectures for indefinitely large p has been carried out by several authors. An alternative proof is given in [3].

None of the available methods gives a proof of the conjectures for some particular fixed value of p .

Notice that the strategy of proof of this conjecture (for indefinitely large p) outlined above does not use quantum groups.

4. Perverse sheaves on affine flags of the dual group (local geometric Langlands)

4.1. Generalities on geometric Langlands duality. Recall that ${}^L G$ is the group dual to G in the sense of Langlands. Several good surveys of geometric Langlands duality program has appeared recently [33], [34], [37], so I will only briefly recall the set-up.

The geometric Langlands duality is a categorification of the classical Langlands duality for function fields. The latter seeks to attach an automorphic form to a homomorphism from a version of the Galois group to ${}^L G$. In other words, the problem is to provide a spectral decomposition for Hecke operators acting in the space of automorphic functions, and relate the space of spectral parameters to homomorphisms of the Galois group to the dual group. As was probably first observed by A. Weil, in the case of a function field the automorphic space in question is the set of isomorphism classes of G -bundles, possibly with an additional level structure, on an algebraic curve over \mathbb{F}_q . Thus it is the set of \mathbb{F}_q points of the corresponding moduli space (stack).

Passage to the geometric duality theory is based on the following variation of Grothendieck's sheaf-function correspondence principle. The variation says that for an algebraic variety (or stack) over \mathbb{F}_q a natural categorification of the space of functions on the set $X(\mathbb{F}_q)$ is the derived category of l -adic sheaves on X . Thus the objective of the geometric duality theory is a spectral decomposition of the derived category of l -adic sheaves on a moduli space of G -bundles, where the space of spectral parameters is identified with the space of ${}^L G$ local systems. It is a non-trivial, and not completely solved, problem to assign a formal meaning to the previous sentence; however, in some cases it amounts to an equivalence between the l -adic derived category of the moduli stack and the derived category of coherent sheaves on a stack mapping to the stack of local systems.

The above formulations referred to a more developed *global* version of the theory. However, the classical Langlands conjectures have both a global and a local version. The global one provides a conjectural classification of automorphic representations of the group of adèle points of a reductive group over a global field, i.e. either a number field, or the field of rational functions on a curve over a finite field. The local one describes all irreducible representations of a reductive group over a local field; recall that a local function field is a field of formal Laurent series $\mathbb{F}_q((t))$. The geometric theory studies the derived category of l -adic sheaves on homogeneous spaces of the

formal loop group ${}^L G((t))$ of the dual group ${}^L G$. It is a group ind-scheme over \mathbb{F}_q , whose group of \mathbb{F}_q points is identified with ${}^L G(\mathbb{F}_q((t)))$. The results are expected to link such l -adic derived categories to coherent sheaves on spaces related to G -local systems on the punctured formal disc, cf. [35].

4.2. Results of [2], [12], [15]. Some particular results of this type have been achieved in *loc. cit.*

4.2.1. Statement of a result. Recall that the Iwahori subgroup $I \subset {}^L G((\mathbb{F}_q(t)))$ consists of those maps from the punctured formal disc to ${}^L G$, which can be extended to a map from the whole disc, so that the image of the closed point is contained in a fixed Borel subgroup ${}^L B \subset {}^L G$.

We have a group subscheme (pro-algebraic group) $I \subset {}^L G((t))$, such that $I(\mathbb{F}_q) = I$. The affine flag space \mathcal{Fl} of ${}^L G$ is the homogeneous space ${}^L G((t))/I$. It is an ind-algebraic variety such that $\mathcal{Fl}(\mathbb{F}_q) = {}^L G(\mathbb{F}_q((t)))/I$. The group I acts on \mathcal{Fl} . The orbits of this action, called affine Schubert cells, are in bijection with the affine Weyl group W_{aff} .

Let \mathcal{P} denote the category of perverse sheaves on \mathcal{Fl} , which are equivariant with respect to the prounipotent radical of I . Let $\mathcal{P}^I \subset \mathcal{P}$ be the full subcategory of I equivariant sheaves.

Let ${}^{nf}\mathcal{P}^I \subset \mathcal{P}^I$ be the Serre subcategory generated by irreducible objects, corresponding to those $w \in W_{\text{aff}}$, which are not the minimal length representatives of a left W coset. Let ${}^f\mathcal{P} = \mathcal{P}/{}^{nf}\mathcal{P}$ be the Serre quotient category.

Remark 4.1. To clarify the definition of ${}^f\mathcal{P}$ we remark that this category can be also described as the category of Iwahori–Whittaker sheaves [2]. Thus it is related to the Whittaker model, which is one of the main tools in representation theory of reductive groups over local and global fields.

Theorem 4.2. (a) ([15]) *We have a canonical equivalence $D^b(\mathcal{P}) \cong D^G(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathcal{N}})$.*

(b) ([2]) *We have a canonical equivalence $D^b({}^f\mathcal{P}) \cong D^G(\tilde{\mathcal{N}})$. The image of ${}^f\mathcal{P}$ under this equivalence consists of equivariant exotic sheaves.*

The theorem is motivated by the known isomorphisms of Grothendieck groups; the question of possibility of such (or similar) equivalence has been raised, e.g., by V. Ginzburg, see Introduction to [31]. More precisely, the Grothendieck groups of the two categories appearing in Theorem 4.2 (a) are isomorphic to the group algebra of the affine Weyl group of ${}^L G$. A more interesting version of the isomorphism is obtained by replacing the categories by their graded version: $D^G(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathcal{N}})$ is replaced by $D^{G \times \mathbb{G}_m}(\tilde{\mathfrak{g}} \times \tilde{\mathcal{N}})$, while the definition of the graded version of \mathcal{P} is more subtle (cf. Proposition 4.5 below and also [9]). The corresponding Grothendieck groups turn out to be isomorphic to the affine Hecke algebra, see [31], [44].

Similarly, the Grothendieck groups of both categories appearing in Theorem 4.2 (b) are identified with the anti-spherical module over the extended affine Weyl group,

while in the graded version of the theory we get the anti-spherical module over the affine Hecke algebra. Here by the anti-spherical module we mean the induction of the sign representation from the finite Weyl group (respectively, Hecke algebra) to the affine one.

I would like to emphasize that this isomorphism of the two realizations of the affine Hecke algebra is the key step in the proof of classification of its representations due to Kazhdan and Lusztig [43] (see also [31]), which establishes a particular case of the local Langlands conjecture. This is another illustration of the relation of Theorem 4.2 to local Langlands duality.

The proof of Theorem 4.2 builds on previously known constructions of categories related to G in terms of perverse sheaves on homogeneous spaces for ${}^L G((t))$. The first important result is *the geometric Satake isomorphism* [39], [47], [8], which identifies the tensor category $\text{Rep}(G)$ of algebraic representations with the category of perverse sheaves on the affine Grassmannian $\mathcal{G}_r = {}^L G((t))/{}^L G_O$ equivariant with respect to ${}^L G_O$. Here ${}^L G_O \subset {}^L G((t))$ is the group subscheme, such that ${}^L G_O(\mathbb{F}_q)$ consists of maps which extend to the non-punctured disc. Furthermore, Gaitsgory [36] used this result to provide a categorification of the description of the *center* of the affine Hecke algebra. Using some ideas of I. Mirković we observe that the so-called Wakimoto sheaves provide a categorification of the maximal abelian subalgebra in the affine Hecke algebra due to Bernstein, see, e.g., [31], [44]. The maximal projective object in the category of sheaves on the finite dimensional flag variety of ${}^L G$ smooth along the Schubert stratification, which plays a central role in Soergel's description of category O , cf. [50], is a categorification of the q anti-symmetrizer (an element of the finite Hecke algebra, which acts by zero in all irreducible representation except for the sign representation). Under the equivalence of Theorem 4.2 (a) it corresponds to the structure sheaf of $\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathcal{N}}$. A combination of these ingredients yields a proof of the theorem.

4.2.2. Possible generalizations. It is natural to ask if the multiplication in the affine Hecke algebra corresponds to a monoidal structure on the derived categories of coherent sheaves and constructible sheaves appearing in Theorem 4.2 (a). In order to get such a monoidal structure, we need to replace the categories defined above by closely related ones with the same Grothendieck group. One way to do it is as follows. Let I' be the pro-unipotent radical of I . Let \mathcal{P}' be the category of perverse sheaves on “the basic affine space” ${}^L G((t))/I'$, which are I' -monodromic with unipotent monodromy. Then convolution provides the derived category $D^b(\mathcal{P}')$ with a monoidal structure. Notice that this monoidal category does not have a unit object, though this can be repaired by adding some pro-objects to the category, the unit object is then the free pro-unipotent local system on $I/I' \subset {}^L G((t))/I'$.

On the dual side we consider the category $D^G(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})$. One can show that convolution provides this category with a monoidal structure. Let $\text{Coh}^G(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})' \subset \text{Coh}^G(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})$ denote the full subcategory of complexes, whose cohomology sheaves are set-theoretically supported on the preimage of $\mathcal{N} \subset \mathfrak{g}$. A standard

argument shows that it yields a full embedding of derived categories $D^G(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})' := D^b(\text{Coh}^G(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})')$ into $D^G(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})$. The full subcategory $D^G(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})' \subset D^G(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})$ is closed under the convolution product, though it does not contain the unit object $\delta_*(\mathcal{O})$, where $\delta: \tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$ is the diagonal embedding.

It is easy to see that the push-forward (respectively, pull-back) functors $\text{Coh}^G(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathcal{N}}) \rightarrow \text{Coh}^G(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})', \mathcal{P} \rightarrow \mathcal{P}'$ induce isomorphisms of Grothendieck groups.

Theorem 4.3 ([15]). *We have a natural monoidal equivalence $D(\mathcal{P}') \cong D^G(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})'$.*

Remark 4.4. Another version of Theorem 4.2 links the monoidal I equivariant derived category to the monoidal derived category of G -equivariant coherent sheaves on the fiber square of $\tilde{\mathcal{N}}$ over \mathfrak{g} . An additional subtlety in this case is related to nonvanishing of $\text{Tor}_{>0}^{\mathfrak{g}}(\mathcal{O}_{\tilde{\mathcal{N}}}, \mathcal{O}_{\tilde{\mathcal{N}}})$. One actually has to take these Tor groups into account by working with the *derived fiber product*, which is a differential-graded scheme, rather than an ordinary scheme. This issue does not arise in the other settings mentioned above, because $\text{Tor}_{>0}^{\mathfrak{g}}(\mathcal{O}_{\tilde{\mathfrak{g}}}, \mathcal{O}_{\tilde{\mathfrak{g}}}) = 0 = \text{Tor}_{>0}^{\mathfrak{g}}(\mathcal{O}_{\tilde{\mathfrak{g}}}, \mathcal{O}_{\tilde{\mathcal{N}}})$. However, one has to work with differential graded schemes in order to define the convolution product on $D^G(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})$.

4.2.3. Relation to the material of Section 2. Many of the constructions from Section 2 are motivated by the equivalences of Theorem 4.2.

For example, the categories $D(\mathcal{P}), D(\mathcal{P}')$ carry a natural B_{aff} action by *Radon transforms*, cf. beginning of Section 3.2.2, where a similar structure for a finite dimensional flag variety is mentioned. To define the action we recall that the ${}^L G((t))$ orbits on $\mathcal{F}l^2$ are indexed by the affine Weyl group. If $\mathcal{F}l_w^2$ is the orbit corresponding to $w \in W_{\text{aff}}$, and $\text{pr}_i^w: \mathcal{F}l_w^2 \rightarrow \mathcal{F}l$ are the projections, where $i = 1, 2$ then we define a functor $R_w: D(\mathcal{P}) \rightarrow D(\mathcal{P})$ by $R_w(\mathcal{F}) = \text{pr}_{2*}^w \text{pr}_1^{w*}(\mathcal{F})$. Then we have an action of B_{aff} on $D(\mathcal{P}), D({}^f \mathcal{P})$, such that $\tilde{w} \mapsto R_w$. Under the equivalences of Theorem 4.2 (b) this action corresponds to the action described in Section 2.

Finally, I would like to quote the statement that allows to link the grading on Ext spaces appearing in Theorem 2.19 to Frobenius weights, thus providing a way to prove Theorem 2.19. To state it we introduce the following notation. Let Φ be either of the two equivalences appearing in Theorem 4.2. Let Fr be the autoequivalence of the corresponding category of constructible sheaf, sending a sheaf to its pull-back under the Frobenius morphism. Let q be an automorphism of either $\tilde{\mathcal{N}}$ or $\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathcal{N}}$ given by $(\mathfrak{b}, x) \mapsto (\mathfrak{b}, qx)$ or $(\mathfrak{b}_1, \mathfrak{b}_2, x) \mapsto (\mathfrak{b}_1, \mathfrak{b}_2, qx)$ respectively; here q stands for the cardinality of the base finite field \mathbb{F}_q .

Proposition 4.5 (cf. [2]). *We have a canonical isomorphism $\Phi \circ q^* \cong \text{Fr} \circ \Phi$.*

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