On the local Langlands and Jacquet–Langlands correspondences

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Abstract. Let F be a locally compact non Archimedean field, and D a division algebra with centre F and finite dimension d^2 over F. Fix an integer $r \geqslant 1$, and let $G = \mathrm{GL}_n(F)$, $G' = \mathrm{GL}_r(D)$, where $n = rd$. Smooth irreducible representations of G' are related to those of G via the Jacquet–Langlands correspondence, whereas the Langlands correspondence relates such representations of G to degree n representations of the absolute Galois group of F. We review some recent results on those correspondences, in particular on their explicit description.

Résumé. Soient F un corps commutatif localement compact non archimédien, et D un corps gauche de centre F et de dimension finie d^2 sur F. Fixons un entier $r \geqslant 1$ et posons $n = rd$, $G' = \mathrm{GL}_r(D)$, $G = \mathrm{GL}_n(F)$. Les représentations lisses irréductibles de G' sont reliées à celles de G par la correspondance de Jacquet–Langlands, tandis que la correspondance de Langlands relie celles de G aux représentations de dimension n du groupe de Galois absolu de F. Nous passons en revue quelques résultats récents concernant ces correspondances, et en particulier leur description explicite.

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1. Smooth representations

Let F be a locally compact non-Archimedean field, and p its residue characteristic. So F is a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers, or of the field of Laurent power series $\mathbb{F}_p((X))$ in one variable X.

If $\mathbb{H}$ is a reductive linear algebraic over F, then the group $H = \mathbb{H}(F)$, with its natural topology coming from F, is locally profinite: there is a basis of neighbourhoods of identity consisting of open compact subgroups. We shall be interested only in the following special cases. We let D be a division algebra with centre F and finite dimension over F; that dimension is necessarily a square d^2, $d \geqslant 1$. We fix an integer $r \geqslant 1$, put $n = rd$, and write G'_r or simply G' for $\mathrm{GL}_r(D)$ and G_n or simply G for $\mathrm{GL}_n(F)$. Both are examples of groups H as above, and G is the special case of G' where $d = 1, n = r$.

A representation of a locally profinite group H, on a complex vector space V, is *smooth* if every vector v in V has open stabilizer in H. With obvious morphisms smooth representations of H form an abelian category.

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Simple objects in that category are called *irreducible*: a smooth representation (π, V) of H is irreducible if V is non-zero and contains no subspace invariant under H, other than $\{0\}$ and V. We write $\mathcal{A}(H)$ for the set of isomorphism classes of irreducible smooth representations of H.

A *character* of H is a group homomorphism into $\mathbb{C}^\times$, with open kernel. Characters of H parametrize isomorphism classes of smooth 1-dimensional representations.

We consider only locally profinite groups H such that for one – and hence for all – open compact subgroups K, H/K is *countable*. In that case Schur's lemma is valid for an irreducible smooth representation (π, V) of H, and in particular the centre $Z(H)$ of H acts on V via a character ω_π called the *central character* of π. In our case the centre of G' or G is made out of scalar matrices, and will be identified with $F^\times$.

2. Coefficients

If (π, V) is a smooth representation of a locally profinite group H, the group H acts on $V^* = \mathrm{Hom}(V, \mathbb{C})$ by $g \mapsto {}^t\pi(g^{-1})$. The subspace $V^\vee$ of V^* of linear functionals which are *smooth*, i.e. stabilized by an open subgroup of H, carries a smooth representation $(\pi^\vee, V^\vee)$ of V, called the *contragredient* of (π, V). The natural embedding $V \to V^{\vee\vee}$ is an isomorphism if and only if (π, V) is *admissible*, which means that for all open compact subgroups K of H, the subspace V^K of fixed points under K is finite-dimensional; in that case $(\pi^\vee, V^\vee)$ is also admissible. When H is reductive over F, a smooth representation of H of finite length is admissible.

A *coefficient* of (π, V) is a function c on H of the form $c(g) = \lambda(\pi(g)v)$ where $v \in V$, $\lambda \in V^\vee$. If (π, V) has a central character ω_π, then $c(zg) = \omega_\pi(z)c(g)$ for $z \in Z(H)$, $g \in H$.

Let (π, V) be *irreducible*. We say that it is *cuspidal* if the support of any coefficient is compact modulo $Z(H)$; we say it is *square integrable* (L^2) if ω_π is unitary and for any coefficient c of π, its absolute value $|c|$ is square integrable on $H/Z(H)$, for any Haar measure on that quotient group. We let $\mathcal{A}^o(H)$ (resp. $\mathcal{A}^2(H)$) be the subset of $\mathcal{A}(H)$ made out of cuspidal (resp. L^2) representations.

3. The Jacquet–Langlands correspondence

That correspondence is a special case of Langlands' functoriality principle, which very roughly speaking says that two groups sharing many conjugacy classes should share a large part of their representation theory. That is precisely the case for our groups G' and G.

An element g' of G' has a reduced characteristic polynomial $P(g')$, which is a monic polynomial of degree $n = rd$ in $F[T]$. We say that g' is regular semisimple if $P(g')$ has no repeated root in an algebraic closure of F, and that g' is regular elliptic if moreover $P(g')$ is irreducible in $F[T]$. Two regular semisimple elements of G' are conjugate in G' if and only if they have the same characteristic polynomial. The set G'^s of regular semisimple elements in G' is open and dense, and the set G'^e of regular elliptic elements is open.

All that applies to G as a special case. For $g' \in G'^s$, there is $g \in G^s$, unique up to conjugation, such that $P(g) = P(g')$. For $g \in G^s$, there is a $g' \in G'^s$ such that $P(g) = P(g')$ if and only if all irreducible factors of $P(g)$ have degree a multiple of d; that is obviously the case for $g \in G^e$.

To express how G and G' can share some of their representation theory, recall that if we fix a Haar measure dg' on G', the convolution algebra $\mathcal{H}(G')$ of locally constant compactly supported functions on G' acts in any smooth representation (π', V') of G' via $\pi'(f)v' = \int_{G'} f(g')\pi(g')\, dg'$, for v' in V' and f in $\mathcal{H}(G')$. If (π', V') is admissible (in particular when π' has finite length), $\pi(f)$ has finite dimensional range hence has a trace.

Theorem 1. *Let (π', V') be a finite length smooth representation of G'. There is a unique locally constant function $\chi_{\pi'}$ on G'^s, locally integrable on G', such that for $f \in \mathcal{H}(G')$ we have* $\operatorname{tr} \pi'(f) = \int_{G'} \chi_{\pi'}(g') f(g')\, dg'$.

Note that by uniqueness $\chi_{\pi'}$ is invariant under conjugation. Also it does not depend on the choice of Haar measure dg' on G'. If f has support in G'^s, the result is due to Harish Chandra [27], [28], as is the general case when F has characteristic 0 [27]. When F has positive characteristic, it is more difficult and is due to B. Lemaire, first for G [46] and recently for G' [47].

Corollary. *Let $\pi'_1, \dots, \pi'_s$ be inequivalent smooth irreducible representations of G'. Then the functions $\chi_{\pi'_1}, \dots, \chi_{\pi'_s}$ are linearly independent.*

The Jacquet–Langlands correspondence is expressed by the following result.

Theorem 2. *There is a unique bijection $\pi \leftrightarrow \pi'$ between $\mathcal{A}^2(G)$ and $\mathcal{A}^2(G')$ such that $\chi_\pi(g) = (-1)^{n-r} \chi_{\pi'}(g')$ whenever $g \in G^e$, $g' \in G'^e$, $P(g) = P(g')$.*

When $n = 2$, where G' is $D^\times$, D the quaternion division algebra over F, the result is due to Jacquet and Langlands [39], hence the name. For $n = 3$ it is due to Flath [22] when F has characteristic 0 and to the author [33] in general. When $n > 3$, $r = 1$, it was established by Rogawski [50] in characteristic 0 and Badulescu [2] in positive characteristic.

Finally the general case $n > 3$, $r > 1$ was obtained by Deligne, Kazhdan and Vignéras [21] in characteristic 0 and Badulescu [3] in positive characteristic. Note that the method is global, it uses automorphic forms and trace formulas. Only when $n = 2$ a purely local proof is known [24], see also [14].

Remarks. 1) If $\pi \leftrightarrow \pi'$ as above, then $\omega_\pi = \omega_{\pi'}$ and $\pi^\vee \leftrightarrow \pi'^\vee$.

2) For π' a smooth representation of G' and χ a character of $F^\times$, we let $\chi\pi'$ be the smooth representation $g' \mapsto \chi \circ \operatorname{Nrd}(g')\pi'(g')$, where Nrd is the reduced norm

(the determinant in the special case of G). For $\pi \leftrightarrow \pi'$ as above, and χ unitary, then $\chi\pi \in \mathcal{A}^2(G)$, $\chi\pi' \in \mathcal{A}^2(G')$ and $\chi\pi \leftrightarrow \chi\pi'$.

4. Extending the Jacquet–Langlands correspondence

When $r > 1$, the groups G and G' share more conjugacy classes than just elliptic ones.

Theorem 3 ([4]). *If* $\pi \in \mathcal{A}^2(G)$ *and* $\pi' \in \mathcal{A}^2(G')$ *correspond as above, then*

$$\chi_\pi(g) = (-1)^{n-r}\chi_{\pi'}(g') \ \ \textit{whenever} \ \ g \in G^s, \ g' \in G'^s, \ P(g) = P(g').$$

It is natural to expect that G and G' also share more of their representation theory than just discrete series.

From Remark 2 of § 3, we readily extend the Jacquet–Langlands correspondence a bit, as follows.

A smooth representation π' of G' is called *essentially* L^2 if it is of the form $\chi\pi'_0$, where π'_0 is L^2 and χ is a character of $F^\times$. Writing $\mathcal{A}^d(G')$ for the essentially L^2 elements in $\mathcal{A}(G')$, we get a unique bijection $\pi \leftrightarrow \pi'$ between $\mathcal{A}^d(G)$ and $\mathcal{A}^d(G')$, for which the assertion of Theorem 3 is still valid, and then $\chi\pi \leftrightarrow \chi\pi'$ for all characters χ of $F^\times$, if $\pi \leftrightarrow \pi'$.

To go further, we have to use parabolic induction. For $i = 1, 2$, let r_i be a positive integer, and (π'_i, V_i) a smooth representation of G'_{r_i}. We consider the space $\mathcal{F}$ of functions $f : G'_{r_1+r_2} \to V_1 \otimes V_2$ which satisfy

$$f(mu\,g) = \delta(m)^{1/2}(\pi'_1(g_1) \otimes \pi'_2(g_2)(f(g))$$

for any $g \in G'_{r_1+r_2}$, any diagonal block matrix m in $G'_{r_1+r_2}$ with diagonal blocks $g_1 \in G'_{r_1}$ and $g_2 \in G'_{r_2}$, and any upper unipotent block matrix u with diagonal blocks 1_{r_1} and 1_{r_2}. Here δ is a specific character of $G'_{r_1} \times G'_{r_2}$ with positive real values, called the *modulus character*. The group $G'_{r_1+r_2}$ acts on $\mathcal{F}$ by right translations and the subspace $\mathcal{F}^\infty$ of functions stabilized by an open subgroup of $G'_{r_1+r_2}$ carries a smooth representation $i(\pi'_1, \pi'_2)$ of $G'_{r_1+r_2}$.

Note. The rôle of the modulus character is that if π'_1 and π'_2 are unitary – i.e. there is a G'_{r_i}-invariant hermitian positive definite form on the space V_i – then so is $i(\pi'_1, \pi'_2)$.

If π'_1 and π'_2 have finite length, the same is true of $i(\pi'_1, \pi'_2)$. Writing $\mathcal{R}'_r$ for the Grothendieck group of the category of finite length smooth representations of G'_r – this is a free $\mathbb{Z}$-module with basis $\mathcal{A}(G'_r)$ – we get a $\mathbb{Z}$-linear multiplication $\mathcal{R}'_{r_1} \times \mathcal{R}'_{r_2} \to \mathcal{R}'_{r_1+r_2}$ such that $[\pi'_1] \times [\pi'_2] = [i(\pi'_1, \pi'_2]$ where $[\pi']$ is the class in the Grothendieck group of the smooth representation π' of finite length. Putting $\mathcal{R}' = \bigoplus_{r\geqslant 1}\mathcal{R}'_r$, we get a graded ring ([61] for $d = 1$, [58]) which is *commutative*; as a special we get a graded ring $\mathcal{R} = \bigoplus_{k\geqslant 1}\mathcal{R}_k$ for the groups G_k.

Theorem (Badulescu, [4]). a) *There is a unique $\mathbb{Z}$-linear map $JL\colon \mathcal{R} \to \mathcal{R}'$ sending $\mathcal{R}_k$ to 0 if $d \nmid k$ and to ${\mathcal{R}'}_{k/d}$ if $d \mid k$, such that for $\pi \in \mathcal{R}$ and $\pi' = JL(\pi)$ we have*

$$\chi_\pi(g) = (-1)^{n-r}\chi_{\pi'}(g') \ \text{ whenever } \ g \in G^s_{rd},\ g' \in G'^s_r,\ P(g) = P(g').$$

b) *The map JL is the unique ring homomorphism which is trivial on $\mathcal{R}_k$ when $d \nmid k$ and extends the Jacquet–Langlands correspondence on $\mathcal{A}^d(G_{rd})$ for any integer $r \geqslant 1$.*

In fact $\mathcal{R}$ and $\mathcal{R}'$ are even Hopf algebras with a canonical involution, the Aubert–Bernstein–Zelevinsky involution [1], [61], and JL is a Hopf-algebra homomorphism, which preserves the involutions, up to predictable signs. The kernel of JL is the ideal generated by the $\mathcal{R}_k$ for $d \nmid k$.

It is expected that when π is a *unitary* smooth irreducible representation of G_k, then $JL([\pi])$ is either 0 – in particular if $d \nmid k$ – or plus or minus the class of a unitary smooth *irreducible* representation of $G'_{k/d}$. Badulescu [5] has proved that when π is a local component of some discrete series representation of $\mathrm{GL}_k(\mathbb{A}_E)$, where E is a number field with completion F at some finite place.

Remark. The unitary elements in $\mathcal{A}(G_k)$ have been classified by Tadić [57], and a similar classification is expected for $\mathcal{A}(G'_r)$ [58]. By recent work of Tadić, Badulescu and Renard [4], [5], [6], it is enough to prove the following conjecture, a result due to Bernstein [7] for G_k.

Conjecture. Let $\pi_i \in \mathcal{A}(G'_{r_i})$, $i = 1, 2$, be unitary. Then $\pi_1 \times \pi_2$ is irreducible.

5. The Langlands correspondence

Let $\overline{F}$ be a separable algebraic closure of F. The group $\mathrm{Gal}(\overline{F}/F)$ is then profinite, and class field theory gives a canonical bijection between characters of $\mathrm{Gal}(\overline{F}/F)$ and finite order characters of $F^\times$. Replacing $\mathrm{Gal}(\overline{F}/F)$ by a variant, the Weil group W_F [60, Appendix II], even gives a bijection between characters of W_F and characters of $F^\times = \mathrm{GL}_1(F)$.

It was Langlands' fundamental intuition that irreducible smooth representations of G_n, $n \geqslant 2$, are intimately related to degree n smooth representations of W_F, cuspidal representations of G_n corresponding to irreducible representations of W_F. Write $\mathcal{G}^o(n)$ for the set of isomorphism classes of irreducible degree n smooth representations of W_F.

Theorem. *There is a unique family of bijective maps $\mathcal{G}^o(n) \to \mathcal{A}^o(G_n)$, $\sigma \mapsto \pi(\sigma)$, such that*

1) *for $n = 1$, it is given by class field theory;*

2) *for $n, m \geqslant 1$, $\sigma \in \mathcal{G}^o(n)$, $\tau \in \mathcal{G}^o(m)$ we have*

$$L(\sigma \otimes \tau, s) = L(\pi(\sigma) \times \pi(\tau), s) \quad \text{and} \quad \varepsilon(\sigma \otimes \tau, s, \psi) = \varepsilon(\pi(\sigma) \times \pi(\tau), s, \psi)$$

for all non-trivial characters ψ of F.

Here the L-factors are of the form $P(p^{-s})^{-1}$, $P \in \mathbb{C}[X]$, $P(0) = 1$, and the ε factors are monomials in p^{-s}. The L-factor on the left is Artin's, whereas the ε-factors on the left have been defined by Langlands and Deligne [20], generalizing Tate's thesis [59] which concerns characters of $F^{\times}$, i.e. one dimensional representations of W_F. The L-factors and ε-factors on the left are those defined by Jacquet, Piatetski–Shapiro, and Shalika [40], or equivalently Shahidi [54], [55], a very different generalization of Tate's thesis. It is wonderfully ironic that both generalizations turn out to be "the same".

Remark. The central character of $\pi(\sigma)$, for $\sigma \in \mathcal{G}^o(n)$, corresponds to the character $\det\sigma$ of W_F via class field theory. Also $\pi(\sigma^\vee) = \pi(\sigma)^\vee$.

The theorem is due to Laumon, Rapoport and Stuhler [45] when F has positive characteristic, and to M. Harris and R. Taylor [32], see also [30], [31] for p-adic fields. Both proofs are global and geometric, using automorphic forms, trace formulas and the geometry of Drinfeld or Shimura modular varieties. They also rely on counting arguments due to the author [34].

The proof in positive characteristic is easier because one can exploit then the functional equation for the L function of an ℓ-adic representation of global Galois groups. Of course the theorem is now also a consequence of L. Lafforgue's result [43] establishing the Langlands conjecture for global fields of positive characteristic. In characteristic zero, the set of Galois L-functions for which one can prove a functional equation is much more restricted, and one has to use Harris' approach [29]. The method in [32] uses heavily the geometry of Shimura varieties at places of bad reduction, which has the advantage of yielding a geometric model for the Langlands correspondence – that is also the case for [45]. The author has given a simpler proof [36], which uses the same Shimura varieties but only at places of good reduction, where it is much easier to get – but one does not get the geometric model that way.

It is highly desirable to find a proof not relying on geometry, and ideally a purely local proof. However the program of C. J. Bushnell and the author [9 and references therein] faces obstacles in establishing the necessary properties of the ε-factors for pairs above. In the simplest cases $n = 2, 3$ the following can be proved ([41] for $n = 2$, [33] for $n = 3$) without geometry.

Theorem. *Let n be 2 or 3. There is a unique bijective map $\mathcal{G}^o(n) \to \mathcal{A}^o(G_n)$, $\sigma \mapsto \pi(\sigma)$, such that $\varepsilon(\chi\sigma, s, \psi) = \varepsilon(\chi\pi(\sigma), s, \psi)$ for all non-trivial additive characters ψ of F and all characters χ of $F^{\times}$.*

Here $\chi\sigma$ denotes the representation $g \mapsto \tilde{\chi}(g)\sigma(g)$ of W_F, where $\tilde{\chi}$ is the character of W_F corresponding to χ via class field theory.

The proof uses automorphic forms on global fields and trace formulas, even for $n = 2$. It is only recently that Bushnell and the author found a proof for $n = 2$, essentially local, which *does not* use *automorphic forms* [14].

Remark. The Langlands correspondence preserves more than the L and ε factors considered in the theorem. For example, the author shows in [37] that if $\sigma \in \mathcal{G}^o(n)$

then

$$L(\Lambda^2\sigma, s) = L(\tau(\sigma), \Lambda^2, s) \text{ and}$$
$$L(S^2\sigma, s) = L(\tau(\sigma), S^2, s)$$

where the L-factors on the right are those defined by Shahidi [55]. The corresponding ε-factors are also preserved, at least up to a non-zero constant (depending on σ).

6. Explicit Langlands correspondence in the tame case

All elements of $\mathcal{A}^o(G_n)$ have been constructed explicitly by Bushnell and Kutzko [15], see also [8]: for each $\pi \in \mathcal{A}^o(G_n)$ they describe an open subgroup J of G_n, which contains the centre Z and is compact mod Z, and a finite dimensional smooth irreducible representation λ of J such that π is obtained from λ by smooth induction (a simpler variant of the construction in § 4). When n is prime to p, the so-called *tame case*, the construction goes back to R. Howe [38], and when $n = p$ to Carayol [19], but the general case is much harder.

It is natural to ask for an explicit description of the Langlands correspondence in terms of such a construction. When p divides n it is out of reach at present: only the case $n = p = 2$ is reasonably settled [42], [14]. In the tame case Howe parametrized both $\mathcal{G}^o(n)$ and $\mathcal{A}^o(G_n)$ in terms of *admissible pairs* $(E/F, \theta)$, where E/F is a degree n extension, θ a character of $E^\times$ not factorizing through an intermediate norm $N_{F/E'}$ $F \subset E' \subset E$, $E' \neq E$, and such that if θ restricted to principal units of E does factorize, then E/E' is unramified. There is then a canonical map $(E/F, \theta) \mapsto \pi(E/F, \theta)$ giving a bijection between admissible pairs up to isomorphism and $\mathcal{A}^o(G_n)$; see [49] for a precise construction, or [12].

On the other hand, if $(E/F, \theta)$ is an admissible pair, θ can be seen as a character of the Weil group W_E, which is an open subgroup of index n of the Weil group W_F. We can then form $\sigma(E/F, \theta)$, the degree n smooth representation of W_F induced from the character θ of W_E. The map $(E/F, \theta) \mapsto \sigma(E/F, \theta)$ gives a bijection between admissible pairs, up to isomorphism, and $\mathcal{G}^o(n)$.

However, the determinant of $\sigma(E/F, \theta)$, seen as a character of $F^\times$, differs in general from the central character of $\pi(E/F, \theta)$, so $\pi(E/F, \theta)$ cannot be $\pi(\sigma(E/F, \theta))$. Recently C.J. Bushnell and the author obtained the following result, in the more general *essentially tame* situation where n is not necessarily prime to p but E/F is still tamely ramified of degree n.

Theorem ([12]). *Let $(E/F, \theta)$ be an admissible pair with E/F tamely ramified of degree n. Then there is a unique tamely ramified character $\mu = \mu(E/F, \theta)$ of $E^\times$ such that $\pi(\sigma(E/F, \theta)) = \pi(E/F, \mu\theta)$.*

The twisting character μ has been computed, at least when E/F is totally ramified [13]. The answer is easy to state only when n is odd; moreover when n is even, the answer *does not* coincide in general with the recipe conjectured in [49].

7. Explicit Jacquet–Langlands correspondence, $r = 1$

An explicit construction of $\mathcal{A}(G'_r)$ or even its cuspidal part $\mathcal{A}^o(G'_r)$, is not known in general (see § 8). However, when $r = 1$, the group $G'_1 = D^\times$ is compact modulo its centre Z and all its irreducible smooth representations are cuspidal and finite-dimensional. They have been constructed by E.-W. Zink [62] and, in terms closer to [15], by P. Broussous [16]. In the tame case, when d is prime to p, the construction goes back to R. Howe [38], see also [48], [49]: there is a natural construction $(E/F, \theta) \mapsto \pi'(E/F, \theta)$ which parametrizes $\mathcal{A}(G'_1) = \mathcal{A}(D^\times)$ via isomorphism classes of admissible pairs of degree $[E : F]$ dividing d.

In the case where E/F is totally ramified of degree n, there is an unramified quadratic character η of $E^\times$ such that $\pi(E/F, \theta)$ corresponds to $\pi'(E/F, \eta\theta)$ via the Jacquet–Langlands correspondence [35, when n is prime]. A similar answer is expected in general but does not seem to be available yet.

More interesting, because the construction is much more involved, is the *totally wild* case, when n is power of p and we consider $\pi \in \mathcal{A}^o(G_n)$ such that there is no unramified character χ of $F^\times$ for which $\chi\pi$ is isomorphic to π. In that case [15] π is constructed from a so-called "*simple*" pair (β, θ) where β is an element of G generating a totally ramified extension E/F of degree n; to such an element β are attached two open subgroups $J = J(\beta)$ and $H^1 = H^1(\beta)$, θ is a character of H^1 of a very specific shape, and π is induced from a representation λ of J with restriction to H^1 isotypic of type θ. There certainly exists $\beta' \in G'$ with $P(\beta) = P(\beta')$ and there is a natural procedure to deduce from θ a similar character θ' of some open compact subgroup $H^1(\beta')$ of $D^\times$ and then from λ a similar representation λ' of an open compact subgroup $J(\beta')$ which induces to $\pi' \in \mathcal{A}(D^\times)$. The main result of [11] (see [10] when $n = p$) states that, at least when p is odd, π corresponds to π' under the Jacquet–Langlands correspondence.

8. Construction and explicit Jacquet–Langlands correspondence, $1 < r < n$

Generalizing the above constructions to intermediate cases $1 < r < n$ is not easy, even when restricting to tame or totally wild situations! Moreover one wants to deal with L^2 representations and not only with cuspidal ones.

This is already visible in the so-called "*level-zero*" case, where one considers the Jacquet–Langlands correspondence $\pi \leftrightarrow \pi'$ for π corresponding to a tamely ramified representation of W_F: in $G'_r = \mathrm{GL}_r(D)$, this translates into the property that in the space of π' there is a non-zero vector fixed under the compact open subgroup $1 + P_D M_r(O_D)$ where O_D is the ring of integers of D and P_D its maximal ideal. We say that such $\pi' \in \mathcal{A}^d(G'_r)$ have *level zero*. Level zero elements of $\mathcal{A}^d(G'_r)$ or $\mathcal{A}^d(G_n)$, $n = rd$, are parametrized by admissible pairs $(E/F, \theta)$, where E/F is unramified of degree dividing n, and θ is a tamely ramified character of $E^\times$. However it is hard to label exactly the representation corresponding to $(E/F, \theta)$, and even more difficult to get the Jacquet–Langlands correspondence explicit in that level zero case. That was recently accomplished by Grabitz, Silberger and Zink [26], [56].

The higher level case is even more difficult, and has been subject to investigations of P. Broussous and M. Grabitz [17], [18], [25] and V. Sécherre [51], [52], [53]. By analogy with [61] and [15], they construct "*simple*" *pairs* (β, θ) as above in G'_r. Sécherre then follows the procedure of [15], and constructs [51], [52] from a simple pair (β, θ) a *simple type* (J, λ) in G'_r as in § 7, and controls its intertwining, so that in particular it is known when λ gives rise by induction to cuspidal smooth irreducible representations of G'_r; in the contrary case, Sécherre computes the intertwining algebra, which in particular tells how many elements in $\mathcal{A}^d(G'_r)$ one can get from λ [53]. It remains to prove that all of $\mathcal{A}^d(G'_r)$ has been obtained: this is the problem of *exhaustion*. Grabitz has obtained some partial results in that direction [25].

This is only for the explicit construction of $\mathcal{A}^d(G'_r)$ in the higher level case: describing the Jacquet–Langlands correspondence explicitly is harder.

As a final remark, let me mention that once exhaustion is proved, Sécherre's result on the intertwining algebra will imply the conjecture of Tadić mentioned in § 4.

References

[1] Aubert, A.-M., Duality in the Grothendieck group of the category of finite length smooth representations of a p-adic reductive group. *Trans. Amer. Math. Soc.* **347** (1995), 2179–2189; Erratum ibid. *Trans. Amer. Math. Soc.* **348** (1996), 4687–4690.

[2] Badulescu, A., Orthogonalité des caractères pour GL(n) sur un corps de caractéristique non nulle. *Manuscripta Math.* **101** (2000), 49–70.

[3] Badulescu, A., Correspondance de Jacquet-Langlands pour les corps locaux de caractéristique non nulle. *Ann. Sci. École Norm. Sup.* (4) **35** (2002), 695–747.

[4] Badulescu, A., Correspondance de Jacquet-Langlands étendue à toutes les représentations. Preprint, 2005.

[5] Badulescu, A., Global Jacquet-Langlands correspondence. Preprint, 2005.

[6] Badulescu, A., Renard, D., Sur une conjecture de Tadić. *Glas. Math. Sec.* III **39** (2004), 49–54.

[7] Bernstein, J. N., P-invariant distributions on GL(N) and the classification of unitary representations of GL(N) (non-Archimedean case). In *Lie groups and representations* II, Lecture Notes in Math 1041, Springer-Verlag, Berlin 1987, 50–102.

[16] Broussous, P., Extension du formalisme de Bushnell et Kutzko au cas d'une algèbre à division. *Proc. London Math. Soc.* (3) **77** (1998), 292–326.

[17] Broussous, P., Minimal strata for GL(m, D). *J. Reine Angew. Math.* **514** (1999), 199–236.

[18] Broussous, P., Grabitz, M., Pure elements and intertwining classes of simple strata in local central simple algebras. *Comm. Algebra* **28** (2000), 5405–5442.

[8] Bushnell, C. J., Smooth representations of p-adic groups: the rôle of compact open subgroups. In *Proceedings of the International Congress of Mathematicians* (Zürich, 1994), Vol. 2, Birkhäuser, Basel 1995, 770–779.

[9] Bushnell, C. J., Henniart, G., Davenport-Hasse relations and an explicit Langlands correspondence II: twisting conjectures. *J. Théor. Nombres Bordeaux* **12** (2000), 309–347.

[10] Bushnell, C. J., Henniart, G., Correspondance de Jacquet-Langlands explicite II. Le cas de degré égal à la caractéristique résiduelle. *Manuscripta Math.* **102** (2000), 211–225.

[11] Bushnell, C. J., Henniart, G., Local tame lifting for GL_n, III. Explicit base change and Jacquet-Langlands correspondence. *J. Reine Angew. Math.* **580** (2005), 39–100.

[12] Bushnell, C. J., Henniart, G., The essentially tame local Langlands correspondence I. *J. Amer. Math. Soc.* **18** (2005), 685–710.

[13] Bushnell, C. J., Henniart, G., The essentially tame local Langlands correspondence II. *Compositio Math.* **141** (2005), 979–1011.

[14] Bushnell, C. J., Henniart, G., *The local Langlands conjecture for* $\mathrm{GL}(2)$. To appear.

[15] Bushnell, C. J., Kutzko, P. C., *The admissible dual of* $\mathrm{GL}(N)$ *via compact open subgroups*. Ann. of Math. Stud. 129, Princeton University Press, Princeton, NJ, 1993.

[19] Carayol, H., Représentations cuspidales du groupe linéaire. *Ann. Sci. École Norm. Sup.* (4) **17** (1984), 191–226.

[20] Deligne, P., Les constantes des équations fonctionnelles des fonctions L. In *Modular functions of one variable* II, Lecture Notes in Math. 349, Springer-Verlag, Berlin (1973), 501–597.

[21] Deligne, P., Kazhdan, D., Vignéras, M.-F., Représentations des algèbres centrales simples p-adiques. In *Représentations des groupes réductifs sur un corps local*, Hermann, Paris 1984, 33–117.

[22] Flath, D., A comparison of the automorphic representations of GL(3) and its twisted forms. *Pacific J. Math.* **97** (1981), 373–402.

[23] Gelbart, S., Jacquet, H., Forms of GL(2) from the analytic point of view. In *Automorphic forms, representations and L-functions*, Part I, Proc. Sympos. Pure Math. 33, Amer. Math. Soc., Providence, RI, 1979, 213–251.

[24] Gérardin, P., Li, W., Fourier transforms of representations of quaternions. *J. Reine Angew. Math.* **359** (1985), 125–173.

[25] Grabitz, M., Simple characters for principal orders and their matching I. Peprint, MPI Bonn 99-117; II, preprint, MPI Bonn 03-56; III, preprint, 2005.

[26] Grabitz, M., Silberger, A., Zink, E. W., Level zero types and Hecke algebras for local central simple algebras. *J. Number Theory* **91** (2001), 92–125.

[27] Harish-Chandra, *Admissible invariant distributions on reductive p-adic groups*. Univ. Lecture Ser. 16, Amer. Math. Soc., Providence, RI, 1999.

[28] Harish-Chandra, A submersion principle and its applications. In *Geometry and analysis*, Indian Acad. Sci., Bangalore 1980, 95–102.

[29] Harris, M., The local Langlands conjecture for GL_n over a p-adic field, $n < p$. *Invent. Math.* **134** (1998), 177–210.

[30] Harris, M., On the local Langlands correspondence. *Proceedings of the International Congress of Mathematicians* (Beijing, 2002), Vol. II, Higher Ed. Press, Beijing 2002, 583–597.

[31] Harris, M., The local Langlands correspondence. In *Formes Automorphes* I, Astérisque **298** (2005), 17–145.

[32] Harris, M., Taylor, R. *The geometry and cohomology of some simple Shimura varieties*. Ann. of Math. Stud. 151, Princeton University Press, Princeton, NJ, 2001.

[33] Henniart, G., *La conjecture de Langlands locale pour GL*(3). *Mém. Soc. Math. France* (*NS*) **11/12** (1984), 186 pp.

[34] Henniart, G., La conjecture de Langlands locale numérique pour GL(n). *Ann. Sci. École Norm. Sup* (4) **21** (1988), 497–544.

[35] Henniart, G., Correspondance de Jacquet-Langlands explicite I: le cas modéré de degré premier. In *Séminaire de théorie des nombres* (Paris, 1990–1991), Progr. Math. 108, Birkhäuser, Basel 1993, 85–114.

[36] Henniart, G., Une preuve simple des conjectures de Langlands pour GL(n) sur un corps p-adique. *Invent. Math.* **139** (2000), 439–455.

[37] Henniart, G., Correspondance de Langlands et fonctions L des carrés extérieur et symétrique. Preprint, IHES/M/03/20, 2003.

[38] Howe, R., Tamely ramified supercuspidal representations of GL(n). *Pacific J. Math.* **73** (1977), 437–460.

[39] Jacquet, H., Langlands, R. P., *Automorphic forms on* GL(2). Lecture Notes in Math. 114, Springer-Verlag, Berlin 1970.

[40] Jacquet, H., Piatetski-Shapiro, I. I., Shalika, J., Rankin-Selberg convolutions. *Amer. J. Math.* **105** (1983), 367–464.

[41] Kutzko, P. C., The Langlands conjecture for GL(2) of a local field. *Ann. of Math.* (2) **112** (1980), 381–412.

[42] Kutzko, P. C., The exceptional representations of GL(2). *Compositio Math.* **51** (1984), 3–14.

[43] Lafforgue, L., Chtoucas de Drinfeld et correspondance de Langlands. *Invent. Math.* **147** (2002), 1–24.

[44] Lafforgue, L., Chtoucas de Drinfeld, formule des traces d'Arthur-Selberg et correspondance de Langlands. In *Proceedings of the International Congress of Mathematicians* (Beijing, 2002), Vol. I, Higher Ed. Press, Beijing 2002, 383–400.

[45] Laumon, G., Rapoport, M., Stuhler, U., $\mathcal{D}$-elliptic sheaves and the Langlands correspondence. *Invent. Math.* **113** (1993), 217–338.

[46] Lemaire, B., Intégrabilité locale des caractères-distributions de GL(N, F), où F est un corps local non archimédien de caractéristique quelconque. *Compositio. Math.* **100** (1996), 41–75.

[47] Lemaire, B., Intégrabilité locale des caractères tordus de $\mathrm{GL}_r(D)$. *J. Reine Angew. Math.* **566** (2004), 1–39.

[48] Moy, A., Local constants and the tame Langlands correspondence. *Amer. J. Math.* **108** (1986), 863–930.

[49] Reimann, H., Representations of tamely ramified p-adic division and matrix algebras. *J. Number Theory* **38** (1991), 58–105.

[50] Rogawski, J., Representations of GL(n) and division algebras over a local field. *Duke Math. J.* **50** (1983), 161–196.

[51] Sécherre, V., Représentations lisses de GL(m, D) I. Caractères simples. *Bull. Soc. Math. France* **132** (2004), 327–396.

[52] Sécherre, V., Représentations lisses de GL(m, D) II: β-extensions. *Compositio Math.*, to appear.

[53] Sécherre, V., Représentations lisses de GL(m, D) III: types simples. *Ann. Sci. École Norm. Sup. Paris*, to appear.

[54] Shahidi, F., Fourier transforms of intertwining operators and Plancherel measures for GL(n). *Amer. J. Math.* **106** (1984), 67–111.

[55] Shahidi, F., A proof of Langlands' conjecture on Plancherel measure, complementary series for p-adic groups. *Ann. of Math.* **132** (1990), 273–330.

[56] Silberger, A., Zink, E.-W., Weak explicit matching for level zero discrete series of unit groups of p-adic, simple algebras. *Canad. J. Math.* **55** (2003), 353–378 (and more recent preprints).

[57] Tadić, M., Classification of unitary representations in irreducible representations of general linear group (non-Archimedean case). *Ann. Sci. École Norm. Sup.* (4) **19** (1986), 335–382.

[58] Tadić, M., Induced representations of GL(n, A) for p-adic division algebras A. *J. Reine Angew. Math.* **405** (1990), 48–77.

[59] Tate, J., Fourier Analysis in Number Fields and Hecke's Zeta functions. In *Algebraic Number Theory* (ed. by J. W. S. Cassels and A. Fröhlich), Academic Press, London 1967, 305–347.

[60] Weil, A., *Basic Number Theory*. Appendix II, Grundlehren Math. Wiss. 144, Springer-Verlag, Berlin 1974.

[61] Zelevinsky, A., Induced representations of reductive p-adic groups II. On irreducible representations of GL(n). *Ann. Sci. École Norm. Sup.* (4) **13** (1980), 165–210.

[62] Zink, E.-W., Representation theory of local division algebras. *J. Reine Angew. Math.* **428** (1992), 1–44.

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