The algebraization of Kazhdan's property (T)

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Abstract. We present the surge of activity since 2005, around what we call the algebraic (as contrasted with the geometric) approach to Kazhdan's property (T). The discussion includes also an announcement of a recent result (March 2006) regarding property (T) for linear groups over arbitrary finitely generated rings.

Keywords. Property (T), spectral gap, cohomology of unitary representations, bounded generation, stable range, expanders, sum–product phenomena, finite simple groups.

1. Introduction

I. The objectives and setting. A discrete group is said to have *property* (T) if every isometric action of it on a Hilbert space has a global fixed point. This property (in an equivalent characterization), was introduced by Kazhdan in 1967 [62], as a means to establish its two consequences: being *finitely generated* and having *finite abelianization*, for lattices in "higher rank" simple algebraic groups. While originally property (T) appeared unexpectedly, during the 70s–80s it found various surprising applications, e.g., to the first explicit construction of expander graphs (Margulis), the solution to the so-called Banach–Ruziewicz problem (Rosenblatt, Margulis, Sullivan), and in operator algebras, to the first constructions of type II_1 factors with a countable fundamental group (Connes). Since the 90s, and particularly during this decade, the study of property (T) has seen further rapid developments, both in theory and in applications, and its perception has substantially been transformed. It is now a fundamental notion and a powerful tool in diverse areas of mathematics, ranging from representation theory (where it was born), ergodic theory and geometric group theory, to operator algebras and descriptive set theory.

An excellent account of the 70s–80s theory can be found in de la Harpe–Valette's influential book [50]; the developments of the 90s are presented in Valette's Bourbaki [104], and a comprehensive up-to-date exposition of the subject can be found in the outstanding forthcoming book by Bekka, de la Harpe and Valette [11]. Consequently, our purpose is not to present another general exposition. Rather, we discuss two different, rather opposite trends in the study of property (T), *geometric* and *algebraic* which, we believe, can be detected quite clearly in retrospect. Following a brief historical account of the former, we focus our attention here mainly on recent exciting developments of the latter, and announce the following new result:

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Theorem 1.1. *Let R be any finitely generated commutative ring with* 1*. Denote by* $\mathrm{EL}_n(R) < \mathrm{GL}_n(R)$ *the group generated by the elementary matrices over* R*. Then for all* $n \geq 2 + \mathrm{Krull}\dim R$*, the group* $\mathrm{EL}_n(R)$ *has Kazhdan's property* (T)*.*

In particular, it follows from a non-trivial "$\mathrm{EL}_n = \mathrm{SL}_n$" result of Suslin [101], that for any $m \geq 0$, the group $\mathrm{SL}_n(\mathbb{Z}[x_1, \ldots, x_m])$ has property (T) when $n \geq m + 3$. These groups are the first known *linear* Kazhdan groups outside the family of lattices. In fact a better result holds, covering non commutative rings as well, in which Bass' ring theoretic notion of the *stable range* of R replaces the Krull dimension. The proof of the theorem also reduces to a new treatment of the "classical" case, $R = \mathbb{Z}$. The reader interested primarily in Theorem 1.1 may wish to skip to Section 4 on first reading, where a sketch of its proof is presented.

II. The geometrization of property (T). As is well known by now, a lattice Γ in a simple algebraic group G over a local field k has property (T), unless k-rank $G = 1$, but including $G = F_4^{-20}$, $\mathrm{Sp}(n, 1)$ when $k = \mathbb{R}$ – cf. [11], [50]. Due to the celebrated results of Margulis (rank > 1) and Corlette and Gromov–Schoen (rank 1), these simple algebraic groups turn out to be also the ones whose lattices enjoy *superrigidity*, hence a fortiori, these lattices are *arithmetic groups* (with $\Gamma = \mathrm{SL}_{n \geq 3}(\mathbb{Z})$ serving as outstanding examples). For a long time, the simultaneous appearance of property (T) and superrigidity–arithmeticity was only empirical, having very different origin. However during the 90s, beginning with the breakthrough of Corlette [31], it has become evident that the theory of harmonic maps should provide a unified explanation for the two phenomena, even though in practice this has been carried out primarily for archimedean k and co-compact Γ. Be that as it may, for two decades following Kazhdan's discovery, essentially no new constructions of Kazhdan groups were found, and the proofs of property (T) for the *arithmetic groups* were depending crucially on their being *lattices*. As the latter are intimately related with the special geometry of symmetric spaces or Bruhat–Tits buildings, as well as with arithmetic-algebraic objects, this highly rigid framework was naturally projected back to the general perception of property (T).

The first new constructions of Kazhdan groups were put forward in Gromov's seminal work [46], as quotients of co-compact lattices in the rank one Kazhdan Lie groups. Although this geometric method gave rise to a continuum of Kazhdan groups, it is above all an achievement of Gromov's hyperbolic group theory; from the point of view of property (T), the implicit constructions are "deformations" of existing ones. It is only in 1994 that the first explicit constructions of entirely new Kazhdan groups appeared, in a remarkable work of Cartwright, Młotkowski and Steger [26]. The groups constructed there act simply transitively on the vertices of certain "exotic $\tilde{A}_2$ buildings" introduced in [25], and for a natural generating subset, the *best* Kazhdan constant was computed. At the time, these groups seemed to form a "singular" class of "cousins" of standard lattices, and their original treatment was quite algebraic. However, it is now understood that they are outstanding representatives of the "geometrization of property (T)", an approach going back to Garland's seminal paper [44].

In 1973, Garland [44] established the first general results in what has later developed to become the "vanishing below the rank" principle. Loosely speaking, this asserts that for a simple algebraic group G over a local field k, the cohomology $H^i(G, \pi)$ vanishes for a wide class of representations π, as long as $i < k - \operatorname{rank} G$. A similar statement is inherited by the discrete co-compact subgroups $\Gamma < G$. Although soon a complete algebraic theory had been established in this setting (cf. [13], [27], [64], [107]), it is the geometric approach of Garland that has lent itself to broad generalizations outside the linear framework, thereby giving birth, a decade ago, to the "geometrization of property (T)". At the heart of Garland's approach lies the idea that an appropriate bound on the norm of a *local* Laplacian, defined on the links of a complex on which Γ acts, leads to vanishing of cohomology. Since the above definition of property (T) is tautologically equivalent to the vanishing of $H^1(\Gamma, \pi)$ for any unitary Γ-representation π, this can be used to give an extremely useful, geometric criterion for the presence of property (T). An illuminating account of the remarkable path from the classical Hodge theory and Matsushima and Bochner type formulae in the theory of harmonic maps, through Garland's work, to Kazhdan's property (T), can be found in Pansu's [84]. As was pointed out to us by Lior Silberman, one can now present a particularly simple proof of the following resulting local criterion for property (T) (due to Ballmann–Świątkowski [8], Pansu [84], Żuk [111]), using the most basic Poincaré type inequality for Hilbert space valued functions on finite graphs:

Theorem 1.2. *Let X be a 2-dimensional simplicial complex on which the group Γ acts properly and co-finitely by automorphisms. Assume that each vertex and each edge of X is contained in some triangle. If for any vertex x, its link is a connected graph whose first positive eigenvalue is $> 1/2$, then Γ has property* (T).

See also [11, Ch. 5], [82], [108], and particularly [53, Theorem 6.4]. All of the aforementioned works, as well as [35], [36], furnish us with a rich and wild family of Kazhdan groups, well beyond the original distinguished class of arithmetic groups. Last, but not least in this direction, is Gromov's "random groups" paper [47], in which the geometrization approach to fixed point properties of groups culminates in the construction of remarkable groups, having property (T) among other important features; see also the related elaborations [45], [81], [100].

III. From geometrization to algebraization of property (T). The geometrization of property (T), when it applies, is a powerful tool. It is so sweeping that it typically yields a much stronger fixed point property, covering at least all isometric actions on non-positively curved manifolds (as in Theorem 1.2 above – cf. [53, Theorem 6.4]). Thus, it *intrinsically* cannot apply when dealing with such interesting groups as $\mathrm{SL}_n(\mathbb{Z})$. Moreover, so far it has not produced a single example of a *linear* group which is not a standard arithmetic lattice. Related to this, the crude scissors of the geometric approach are currently helpless in dealing with delicate questions regarding expanding properties of *infinite families of finite groups*. It is exactly for these problems that we shall see the advantages of the recent *algebraic approach* to property (T). Unlike the geometric one, which looks at the group "locally", and essentially as a purely geometric object, the algebraic approach relies heavily on precise global algebraic structure. It applies a finer spectral analysis, and generally offers a more individual, less collective treatment. Our main purpose in this exposition is to describe its recent developments and achievements.

Trying to "geometrize the algebraization" – a failure report. Before proving Theorem 1.1 above, an attempt was made, together with Donald Cartwright, Lior Silberman, and Tim Steger, to find a computer assisted proof of some cases treated by Theorem 1.1, using Theorem 1.2 applied to Cayley complexes associated with the group. More precisely, going over $\sim 10^6$ generating subsets, the computer tested Żuk's-"$\lambda_1 > 1/2$" criterion for property (T) in an improved version, taking generating subsets invariant under conjugation by a finite subgroup, and applying a corrected variation of Żuk's Theorem 8 in [112]. While as explained, the condition inherently cannot hold over rings like $\mathbb{Z}$ or $\mathbb{Z}[x]$, a priori there seems to be no reason why it should not be satisfied when working with $R = \mathbb{F}_p[x, y]$, let alone with $R = \mathbb{F}_p[x]$ (for which $\mathrm{SL}_3(R)$ is a lattice on a $\tilde{A}_2$ building, and has property (T)). The attempts (with $p = 2, 3$) failed. There indeed seems to be little intersection between the two approaches to property (T).

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2. First algebraization results: bounded generation and general rings

I. Kazhdan constants. One driving force in the study of property (T) is the determination of *explicit Kazhdan constants*. Recall (the well known Delorme–Guichardet theorem, cf. [50]) that a finitely generated group Γ has property (T) iff for every finite generating subset $S \subseteq \Gamma$ there exists some $\varepsilon > 0$, such that the following is satisfied: *If* $\pi : \Gamma \to U(V)$ *is any unitary* Γ*-representation on a Hilbert space* V*, for which there is some* $v \in V$ *with* $\|\pi(g)v - v\| < \varepsilon\|v\|$ (*such* v *is called* (S, ε)*-invariant*)*, then there is some* $0 \neq u \in V$ *which is* Γ*-invariant.* Any $\varepsilon > 0$ is referred to as a *Kazhdan constant* for Γ, with respect to (and usually depending on) S.

Besides serving as a natural challenge, determining explicit Kazhdan constants makes quantitative many of the applications of property (T). It also makes available the next qualitative problem of *uniformity* of property (T) (more precisely, of the Kazhdan constant) over a family of groups, which reduces to the important theme of expander Cayley graphs for finite groups (discussed in Section 3 below). The particularly natural, intriguing case of $\mathrm{SL}_{n\geq 3}(\mathbb{Z})$, with its "canonical" generating subset of $n^2 - n$

unit elementary matrices, was a problem raised by Serre in the 80s – see also the problem list in [50].

In [94] it was realized that Kazhdan's original proof of property (T) for higher rank algebraic groups, actually is (or can be made) effective. Thus, explicit and even optimal Kazhdan constants can be obtained for these groups (see also [10], [12], [80] in this direction). From there, by making quantitative Kazhdan's original argument that any lattice in a Kazhdan group is Kazhdan, explicit Kazhdan constants for each lattice can be obtained, based on some (soft) information on a fundamental domain of it. The latter being handled by standard reduction theory, this settles the issue *in principle*. In practice, however, matters are not quite as simple, and many basic questions regarding the asymptotic behavior of the Kazhdan constants are difficult to understand from this viewpoint. Although this is the solution one can hope for when dealing with the *family* of all lattices, for many individual arithmetic groups, such as $\mathrm{SL}_{n\geq 3}(\mathbb{Z})$, it yields Kazhdan constants for generating sets of "geometric" rather than "algebraic" nature. From the more general perspective we are trying to pursue here, this is far from being the "right" solution, as it continues to treat the arithmetic groups as *lattices*, rather than approaching them as independent groups. Such an approach was indeed accomplished in [96], providing a solution to Serre's question above for $\mathrm{SL}_{n\geq 3}(\mathbb{Z})$. In that paper a systematic use of two tools was initiated: the group theoretic notion of *bounded generation*, and the passage from standard rings like $\mathbb{Z}$, to *arbitrary* finitely generated rings, like $\mathbb{Z}[x]$, via a general relative property (T) result. The next two subsections introduce these two ingredients, and some of their earlier (yet still quite recent) combined applications. Further recent developments follow in subsequent sections.

II. Bounded generation

Definition 2.1. Let G be a group, and $\{H_i\}$ be a finite family of subgroups. We say that G is *boundedly generated* by $\{H_i\}$, if there exists some $M < \infty$, such that every $g \in G$ is a product of at most M elements, each belonging to some H_i. If the H_i are cyclic subgroups, we simply say that G is boundedly generated.

This notion, and the first non-trivial examples of it, came with the work of Carter–Keller [24], who showed that for the ring of integers $\mathcal{O}$ of any number field k, $\mathrm{SL}_{n\geq 3}(\mathcal{O})$ is boundedly generated. More precisely, they showed that this group is boundedly generated by the family of its $(n^2 - n)$ *elementary subgroups*, a property which makes sense when working over *any* ring R, and is then termed *bounded elementary generation*. Carter–Keller's result uses Dirichlet's theorem on primes in arithmetic progressions, and gives an explicit bound on M in terms of n and the discriminant ($2n^2 + 50$ works for $\mathrm{SL}_{n\geq 3}(\mathbb{Z})$ – see also the friendlier account in [1], still not avoiding Dirichlet's theorem, where matters stand as of today). Bounded generation has since been studied further, especially for arithmetic groups and in relation to the congruence subgroup property (cf. [85], [109] and the references therein). Although it certainly deserves more attention, we shall not be able to discuss it outside the framework of property (T), except to mention the fundamentally important problem of deciding whether it is shared by any single *co-compact* lattice in a higher rank simple Lie group.

The relevance of bounded generation to questions around property (T) was first demonstrated by Colin de Verdière (cf. [30, Theorem 3.9]), and independently in [96]. The idea can be easily explained through the following simple, yet remarkably useful observation:

Lemma 2.2 (Bounded Generation Lemma). *Assume that G is boundedly generated by $\{H_i\}$. If an isometric G-action on a Hilbert space admits a fixed point for each H_i separately, then it admits a global fixed point.*

Proof. The existence of a fixed point for one H_i, implies that all its orbits are norm bounded (as the action is isometric). Therefore the G-action has bounded orbits, and the unique circumcenter of one such orbit is fixed by all of G. □

In reality, one often tries to argue more quantitatively, at the level of the *unitary representation*, in order to get an *explicit Kazhdan constant* for the group G. The general scheme goes as follows:

(i) Show that for any unitary G-representation with almost invariant vectors, there is an H_i-invariant vector for each i. Quantitatively, any ε-invariant vector v (with respect to a fixed generating set), is ε' close to an H_i-invariant vector, for each i separately.

(ii) Deduce from (i) that *the same* v is $2\varepsilon'$-invariant under *all* of H_i.

(iii) By choosing $\varepsilon > 0$ small enough, make $2\varepsilon' < 1/M$, where M is the bounded generation constant. Hence all of G moves the unit vector v by less than unit distance, and the circumcenter of Gv is a *non-zero* vector, invariant under all of G.

As we shall see, this scheme (with small variations) turns out to be extremely useful even when all the groups involved are *finite*, enabling one to "lift" Kazhdan constants from "smaller" to "larger" groups, upon having a precise structural algebraic information. The main spectral analysis lies in part (i), and it is this *relative property* (T) (of G with respect to H_i), to which we now turn our attention.

III. The relative property (T) over general rings. The relative property (T) is an important variant which is implicit in Kazhdan's original paper, and was first introduced by Margulis (cf. [74], [75]). In analogy with Delorme–Guichardet's equivalent characterization of property (T), Julissaint [54] established the following (see also de Cornulier's extension of this notion in [32]):

Definition 2.3. Let Γ be a discrete group and $N < \Gamma$ a subgroup. We say that the pair (Γ, N) has the *relative property* (T), if either one of the following equivalent conditions is satisfied:

(i) Any isometric Γ-action on a Hilbert space admits a fixed point for N.

(ii) There exists a finite subset $S \subseteq \Gamma$ and $\varepsilon > 0$ ("Kazhdan constants"), so that any unitary Γ-representation containing a (S, ε)-invariant vector, admits a non-zero vector invariant under N.

The outstanding example, used by Margulis in his first explicit construction of expanders [74], is the semi-direct product $\Gamma = \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$, with $N = \mathbb{Z}^2$. In the course of computing explicit Kazhdan constants for the finite representations of $\mathrm{SL}_3(\mathbb{Z})$, Burger [21] found explicit Kazhdan constants for this pair. While his method used, by means of unitary induction, the co-compact embedding $\mathbb{Z}^2 < \mathbb{R}^2$, a variant avoiding it was found in [96]. This seemingly technical issue turned out to be of importance, as it triggered the passage to working with *general finitely generated commutative rings*, thereby releasing a part of the theory from the burden of an ambient locally compact group. Kassabov observed that the commutativity of the ring multiplication operation is not required in the proof, and consequently we have [58], [96]:

Theorem 2.4. *Let R be any finitely generated ring with* 1. *Then* $(\mathrm{EL}_2(R) \ltimes R^2, R^2)$ (*and* $(\mathrm{SL}_2(R) \ltimes R^2, R^2)$ *when R is commutative*), *has the relative property* (T), *with explicit Kazhdan constants available, depending only on the number of generators of R.*

The main tool in the proof of the result is the spectral theorem for representations of abelian groups. By taking the spectral measure on the Pontrjagin dual $\widehat{R^2}$, corresponding to almost $\mathrm{EL}_2(R)$-invariant vectors, one gets a sequence of almost $\mathrm{EL}_2(R)$-invariant measures with respect to the dual action on $\widehat{R^2}$. It is then shown that such a sequence of measures cannot exist when they have "most" of their support "close" to (but excluding) $0 \in \widehat{R^2}$. That, however, would have been the case if the vectors in consideration were taken to be also *almost R^2-invariant*. The whole proof can be made quantitative, leading to explicit Kazhdan constants for the relative property (T). Without getting more technical, we note (anticipating the sum–product results to be discussed in Section 3.III below), the tension used here between the two algebraic operations of the ring.

IV. Bounded generation + relative (T) approach: first applications. It was shown in [96] that when $R = \mathbb{Z}$ in Theorem 2.4, one can take $\varepsilon = 1/10$ in Definition 2.3 (ii), for the generating set S consisting of the unit elementary matrices of $\mathrm{SL}_2(\mathbb{Z})$ and the standard basis of $\mathbb{Z}^2$. It is easy to see that every elementary subgroup $H_i \cong \mathbb{Z}$ of $\mathrm{SL}_{n\geq 3}(\mathbb{Z})$ can be placed in a copy of $\mathbb{Z}^2$, which is normalized by some embedding of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{SL}_n(\mathbb{Z})$. Thus, the relative property (T) for $(\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ gives the explicit $\varepsilon = 1/10$ in step (i) of the quantitative scheme described following Lemma 2.2, for the generating set of $n^2 - n$ unit elementary matrices. Taking the Carter–Keller bounded generation estimate, one completes the last two steps of the scheme to get explicit Kazhdan constants for $\mathrm{SL}_{n\geq 3}(\mathbb{Z})$, which decrease quadratically in n. Later, Kassabov realized in [56] that with additional effort, one can execute this scheme "more efficiently" for large n, to obtain Theorem 2.5 below. It is quite remarkable that while the unitary dual of $\mathrm{SL}_n(\mathbb{Z})$ (or any non-virtually abelian group for that matter), is entirely out of reach, ultimately these ideas give rise to an *explicit* determination of the *precise asymptotic behavior* of the Kazhdan constants of these groups, over all n:

Theorem 2.5. *Let ε_n denote the largest Kazhdan constant of the group* $\mathrm{SL}_{n\geq 3}(\mathbb{Z})$ *w.r.t. the set of $n^2 - n$ unit elementary matrices. Then* $10^{-3}n^{-\frac{1}{2}} < \varepsilon_n < 2n^{-\frac{1}{2}}$.

A more careful analysis reduces the ratio between the two bounds to 60 as $n \to \infty$.

The successful treatment of $\mathrm{SL}_{n\geq 3}(\mathbb{Z})$, together with the generality of Theorem 2.4, suggests that one should aim higher, at property (T) for other groups of similar type, notably $\mathrm{SL}_{n\geq 3}(\mathbb{Z}[x_1, \dots, x_m])$. These groups are called "universal lattices" in [96], as they naturally surject onto many standard arithmetic lattices over both zero and positive characteristic local fields. Property (T) for them would thus account in a uniform manner for Kazhdan's property of very different arithmetic groups. Notice that if we knew that for $n \geq 3$ they were *boundedly elementary generated*, then the same strategy as for $\mathrm{SL}_n(\mathbb{Z})$ would establish property (T) for them as well. It was conjectured in [96] that property (T) should indeed be present among these groups, even though the question of bounded elementary generation has been open since its introduction in the context of K-theory by van der Kallen, in 1982 [55] (where it was also shown that bounded elementary generation over the ring $\mathbb{C}[x]$ does not hold). As will be seen in Section 4 below, the proof of Theorem 1.1 circumvents this delicate, still unsettled issue, by using a different cohomological proof which requires a more modest bounded generation property (which shows up when n is larger than the dimension of the ring $+1$).

The following variation on the same general scheme enables one to get other interesting consequences, where bounded elementary generation is less elusive. It is implicit in the proof of [96, Corollary 4], and subsumes the previous discussion as well, when the ring is taken with its discrete topology.

Theorem 2.6. *Fix $n \geq 3$. Assume that for some finitely generated ring R with* 1*, the group* $\mathrm{EL}_n(R)$ *is embedded* densely *in a topological group G, and the group G is boundedly generated by the* closure *of the embeddings of the elementary subgroups of* $\mathrm{EL}_n(R)$*. Then every* continuous *unitary G-representations with almost invariant vectors for* $\mathrm{EL}_n(R)$*, has a non-zero vector invariant under* all *of G. Moreover, an explicit bound on the Kazhdan constant and size of the Kazhdan set is available, depending only on the number of generators of R as a ring, and on the bounded generation estimate.*

We conclude this section with two rather different applications of this result.

Theorem 2.7. *For $n \geq 3$, the infinite dimensional loop group $G = L(\mathrm{SL}_n(\mathbb{C}))$ of all continuous maps from the circle to* $\mathrm{SL}_n(\mathbb{C})$*, has property* (T)*. More precisely, there is a finite subset $S \subseteq G$ and $\varepsilon > 0$, so that every continuous unitary G-representation with an (S, ε)-invariant vector, admits a non-zero invariant vector.*

This result from [96] is the first construction of an infinite dimensional Lie group, and in fact of any *non locally compact* topological group, with property (T) (more recent ones can be found in [9], [33], [78]). The second application of Theorem 2.6 is the following deeper result of Kassabov and Nikolov [61], giving the first positive result towards Theorem 1.1 above. The relevant ambient topological group G here, is the profinite completion.

Theorem 2.8. *Fix $n \geq 3$. Then for any finitely generated commutative ring R, the group* $\mathrm{EL}_n(R)$ *has property* (τ)*, namely, the family of all its* finite *representations not containing an invariant vector, does not contain any* (S, ε)*-invariant vector* (*for an explicit finite subset S and $\varepsilon > 0$*). *In particular, property* (τ) *holds for the groups* $\mathrm{SL}_{n\geq 3}(\mathbb{Z}[x_1, \ldots, x_m])$.

The bounded generation result needed for Theorem 2.8 controls the "failure of the congruence subgroup property" over higher dimensional rings. This is obtained by proving a result of independent interest, on the bounded generation of $K_2(R)$ by products of Steinberg symbols $\{a, b\}$.

3. The algebraization of property (T) for finite groups: expanders

The general theme of expanders, which has attracted much attention and interest in computer science, combinatorics and group theory, needs by now no introduction (cf. [69], [90] and references therein, for more information, particularly in the directions we shall follow). When dealing with Cayley graphs, the relation to property (T) is fundamental: *A family of finite groups, each equipped with a generating set of uniformly bounded size, is an* expander*, iff their Kazhdan constants are all uniformly bounded away from* 0. For our purposes, and for the benefit of the interested non-specialist, we may simply regard the latter as our definition of expanders in the framework of Cayley graphs. The main general group theoretic problems in this setting can be put into two related, yet independent directions:

1. Existence. Given a family of finite groups G_i, can one find for each i a generating set $S_i \subseteq G_i$, making $\mathrm{Cay}(G_i, S_i)$ an expander family?

2. Independence. Given a family G_i for which 1 is answered positively, is it an expander family with respect to *all* generating subsets? *Random* ones?

Implicit here is the natural question (due to Lubotzky–Weiss [73]), settled *positively* only in recent work of Alon–Lubotzky–Wigderson [3] (following [86]), of whether being an expander family *depends in general* on the choice of generating subsets. This makes any positive results towards 2 of substantial interest. For completeness, we mention that Alon–Roichman [4] showed that any family of finite groups can be made expander using generating subsets of logarithmic size, and that some families, e.g. consisting of abelian groups, *cannot* be made expanders. A completely positive answer to 2 is still unknown for any infinite family of groups; an empirical support towards it for the family $\{\mathrm{SL}_2(\mathbb{F}_p)\}$ was provided by Lafferty–Rockmore in [65], [66].

In this section we describe the remarkable progress made on these two problems *during the last year* (2005), which turns out to be very closely related to the previously discussed algebraization approach to property (T). Before getting into more details, we discuss briefly some relevant background, which shows striking parallelism to the early slow developments in constructing new Kazhdan groups (as discussed in the introduction), and may help to put this exciting progress in perspective.

I. Some background. Margulis' 1973 first explicit construction of expander graphs gives rise to a general class of so-called *"mother group expanders"*: Take a finitely generated ("mother"-)group Γ generated by a finite (symmetric) subset $S \subseteq \Gamma$, an infinite sequence of finite index normal subgroups $N_i < \Gamma$, and consider $\mathrm{Cay}(\Gamma/N_i, \overline{S})$ (where $\overline{S}$ is the canonical projection of S). This automatically yields expanders when Γ has property (T) (interestingly, Gromov's recent random constructions [47] mark a way back from expanders to property (T)). However different, even "better" constructions can be obtained, by using other mother groups such as free groups. The necessary spectral gap property is always a highly non-trivial matter, and for more than two decades after Margulis' construction, the only known approach besides property (T) relied on deep number theoretic tools, around Selberg ("$\lambda_1 \geq 3/16$")–Ramanujan type estimates. All the groups Γ which were known to become mother groups for expanders, were *arithmetic lattices* (with subgroups $N_i < \Gamma$ taken to be *congruence*). As outstanding examples, one should keep in mind the two families $\mathrm{SL}_2(\mathbb{F}_p)$ and $\mathrm{SL}_{n\geq 3}(\mathbb{F}_p)$, *always taken with the projection of a generating set of the corresponding mother group:* $\mathrm{SL}_2(\mathbb{Z})$ (the number theoretic approach), or $\mathrm{SL}_{n\geq 3}(\mathbb{Z})$ (the property (T) approach).

As will become clear, it is not a coincidence that the prolonged lack of progress here was so reminiscent of the one described in the Introduction, concerning new constructions of Kazhdan groups. Besides the little flexibility available by choosing the generators for the quotients Γ/N_i as projections of a subset $S_0 \subseteq \Gamma$ generating a *finite index* subgroup $\Gamma_0 < \Gamma$, no single other construction was known until a decade ago. See Lubotzky's "frustrated account" of this state of affairs in [70]. There the "1–2–3" test case problem was suggested, of proving that $\mathrm{SL}_2(\mathbb{F}_p)$ are expanders with respect to the projection of the set S_0 of elementary 2×2 matrices with ± 3 off the diagonal (while the usual mother group generators yield expanders only for ± 1 or ± 2).

In [92] and [93] appeared the first new constructions of subsets S_0 of the mother group Γ, generating an *infinite index* subgroup, whose projections to Γ/N_i remain expanders. The general principle put forward there, was that one can retain the expanding property when $\langle S_0 \rangle = \Gamma_0 < \Gamma$ has *infinite index*, as long as Γ_0 is "*close enough*" to Γ, in terms of a comparison between the spectral gaps of $(\ell^2-)\Gamma/\Gamma_0$ and $(\ell_0^2-)\Gamma/N_i$. The spectral gap here can be measured by means of norms of convolution operators in $\mathbb{C}[\Gamma]$, or by comparing the Riemannian λ_0 vs λ_1 eigenvalues in geometric settings. For example, one can always remain with expanders, when using $S_0 \subseteq \Gamma$ generating a *co-amenable* subgroup $\Gamma_0 < \Gamma$ (e.g. Γ_0 is normal, and $\Gamma/\Gamma_0 \cong \mathbb{Z}$, cf. the concrete example in [93], computationally analyzed in [67]). A more interesting implementation of this principle can be obtained when Γ is a (necessarily) free group for which Γ/N_i are *Ramanujan graphs* à la Lubotzky–Phillips–Sarnak–Margulis. Their optimal spectral gap property alone, implies that one can find (non-constructively, though) in *every non-trivial normal subgroup* $\Gamma_0 < \Gamma$, a finite subset S_0 whose projection to the finite quotients Γ/N_i (whatever groups they are), is an expander. The approach to these results is functional analytic, applying such tools as compactness in weak topologies, and the Krein–Milman theorem. It is based on an intimate connection with the Banach–Ruziewicz type problem on the profinite completion of Γ. Similarly to Gromov's constructions of Kazhdan groups as quotients of hyperbolic Kazhdan groups, this approach suffers from the fundamental drawback of introducing only a "deformation" of previously existing constructions. It is in itself incapable of providing expanders independently.

A second class of new, self contained constructions, came later in Gamburd's [40], where the same direction of taking $S_0 \subseteq \mathrm{SL}_2(\mathbb{Z})$ generating an infinite index "large" subgroup, is pursued. Here "largeness" is interpreted by the Hausdorff dimension (at least $5/6$) of the limit set on the boundary. The result still fell short of dealing with the "1–2–3" question of Lubotzky mentioned earlier. Gamburd's work relied on previous ideas of Sarnak and Xue [91] (see also [34]), which, as explained in Subsection III below, play a fundamental role in the recent far reaching work of Bourgain–Gamburd [15]. The latter, which concerns the independence problem 2, together with the work discussed in the next subsection regarding the existence problem 1, changed dramatically the poor progress made around the group theoretical aspects of expanders, by the end of the last century.

II. Making the finite simple groups into expanders. The first "new generation" Cayley graph expanders, i.e., ones not obtained via the previously discussed "mother group" approach, were established in [96]: *For any fixed $n \geq 3$ and $m > 0$, when R varies over all the commutative finite rings generated by at most m elements, the family $\mathrm{SL}_n(R)$ forms an expander family*. This immediately follows from Theorem 2.6, as it is a simple matter to give a uniform bounded elementary generation estimate for these rings, depending only on n. For example, all finite fields are generated as a ring by one element, hence $\{\mathrm{SL}_n(F)\}$, for any fixed $n \geq 3$, is an expander.

A major bridge was, however, yet to be crossed: obtaining a uniform Kazhdan constant for SL_n over a fixed finite field, but with *n growing*. At first glance, this seems quite unapproachable by the methods discussed earlier. However, using a fundamental clever variation, Kassabov showed [58] that one can extend the existing technique to encompass the latter as well, by taking appropriate *non-commutative rings R* in Theorem 2.6. More precisely, if F is a finite field, and if we let $R_d = \mathrm{Mat}_d(F)$ be the standard $d \times d$ matrix ring, then it can be shown that R_d is generated by 3 elements for all d, that $\mathrm{EL}_3(R_d)$ have a uniform bounded elementary generation property over all d, and that $\mathrm{EL}_3(R_d) = \mathrm{EL}_3(\mathrm{Mat}_d(F)) \cong \mathrm{SL}_{3d}(F)$. Hence, by Theorem 2.6 the latter form an expander family. Since it is easy to see that for all n and all finite fields F, $\mathrm{SL}_n(F)$ is uniformly boundedly generated by embeddings of $\mathrm{SL}_{3d}(F)$ for $d = [n/3]$, the quantitative version of the Bounded Generation Lemma 2.2 establishes the following result of Kassabov [58]:

Theorem 3.1. $\{\mathrm{SL}_n(F) \mid n \geq 3,\ F \text{ is a finite field}\}$ *can be made an expander.*

We shall next formulate a considerably more general statement (whose proof uses this result), however we mark Theorem 3.1 as the first significant indication that one might hope to cover essentially all finite simple groups (at least those of Lie type). This was accomplished very recently as an accumulation of works by Kassabov, Lubotzky and Nikolov (cf. [60] and its references):

Theorem 3.2. *Excluding the Suzuki groups, the family of all finite (non-abelian) simple groups can be made an expander.*

The result was conjectured in [6] without the exception of the Suzuki groups. By the classification of finite simple groups, the proof amounts to dealing with the family of alternating groups A_n, and with the groups of Lie type (the finitely many sporadic groups are of course negligible in such asymptotic questions). An illuminating account of the work towards the proof of this theorem can be found in the joint announcement [60] of the three authors; we shall only present here some highlights, emphasizing the intimate relations with the main theme of this exposition.

Finite simple groups of Lie type. To complete first the family $\{\mathrm{SL}_n(\mathbb{F}_q)\}$ for all n and $q = p^k$, we are left, by Theorem 3.1 above, with the case $n = 2$. This is done by Lubotzky [71] in the following way: in [72] these groups are made Ramanujan with respect to sets $S_k^{(p)} \subseteq \mathrm{SL}_2(\mathbb{F}_{p^k})$ of (unbounded !) size $p+1$. The Ramanujan spectral gap enters only is showing that they yield uniform Kazhdan constants for $\mathrm{SL}_2(\mathbb{F}_{p^k})$, over all k and p (yet with unbounded size of generating sets). However the specific construction in [72] is of use, in showing that for any p and k there is an element $g_k^{(p)} \in \mathrm{SL}_2(\mathbb{F}_{p^k})$ so that $S_k^{(p)} \subseteq \mathrm{SL}_2(\mathbb{F}_p) \cdot g_k^{(p)} \cdot \mathrm{SL}_2(\mathbb{F}_p)$. Since all the $\mathrm{SL}_2(\mathbb{F}_p)$'s can be made uniformly expanders with two generators, adding to those the $g_k^{(p)}$ and using the quantitative version of the Bounded Generation Lemma 2.2, yields expanding generating sets of three elements.

Once the case of $\{\mathrm{SL}_n(\mathbb{F}_q)\}$ has been settled, the quantitative version of the Bounded Generation Lemma 2.2 completes the treatment of all finite simple groups of Lie type, excluding the Suzuki groups, using the following two results:

Theorem 3.3. (1) (Nikolov [79]). *Every finite simple group G of* classical *type is a product of at most $M = 200$ conjugates of a subgroup H which is a (central) quotient of* $\mathrm{SL}_n(\mathbb{F}_q)$*, for some n and q.*

(2) (Lubotzky [71]). *Excluding the Suzuki family, for any family $X_r(q)$ of finite simple groups associated with a group X of* Lie type *(twisted or untwisted) and fixed rank r, there exists a constant M such that the statement in* (1) *holds (with $H \cong (P)\,\mathrm{SL}_2(\mathbb{F}_q)$).*

Since there are only finitely many families of finite simple groups not covered by (1), together with (2) Theorem 3.2 follows, excluding the alternating groups. The difficulty in treating the Suzuki groups arises from the fact that the only simple groups they contain belong to that same family. The proof of (1) involves a detailed analysis of the subgroup structure of these groups. Although it seems that a more delicate treatment in this spirit should cover (2) as well, Lubotzky appeals instead to a model theoretic approach developed by Hrushovski and Pillay [52], which enables one to deduce the result by a kind of dimension argument, as if working over an algebraically closed field. However, we note that one can prove this result with less sophisticated model theoretic tools, by appealing to the *first order logic compactness theorem*. See [109] for a different useful relation between the latter theorem and bounded generation.

Symmetric groups. For the proof of Theorem 3.2 we are left with the family A_n (or equivalently S_n), which is the most intriguing and challenging among the groups covered in Theorem 3.2. One reason is that unlike the other families $X_r(q)$, it is easy to find (natural) bounded generating subsets for S_n, which make them *non-expanders*. Additionally, a simple cardinality computation, coupled with some basic knowledge of the subgroup structure of S_n, shows that a similar bounded generation strategy, using embeddings of $\mathrm{SL}_n(F)$, will not work here. Unfortunately, in this confined exposition we cannot do justice to the brilliant (and quite technical) work of Kassabov [57], who showed that S_n can be made an expander. Besides the original paper [57], see also the announcement [59] (an elaborate account), or the previously mentioned [60] (a less technical one). Below are only some highlights.

Kassabov proves his theorem by dividing the irreducible representations of S_n into two classes, according to whether the corresponding partition of n has the first row "small" (first class), or "large" (second class), and showing the uniform spectral gap in each class independently. This method is inspired by similar previous ideas of Roichman [89], whose work [88] is also applied in the analysis of the first class, to show that the Kazhdan constant of large (unbounded) sets $F_n \subseteq S_n$, consisting of "nearly all" elements in a suitable conjugacy class of S_n, is uniform. Although the sizes of these F_n are unbounded, Kassabov is able to confine them in a bounded product of uniformly expanding subgroups ($\cong$ products of $\mathrm{SL}_d(\mathbb{F}_2)$). Thus, using the quantitative version of the Bounded Generation Lemma 2.2, he obtains bounded Kazhdan sets for the representations in the first class. The argument for the second is entirely different: those representations are contained in $\ell^2(S_n/S_m)$, for appropriate m which is "close enough" to n, in order to imply strong transitivity (or fast mixing) for the action in that space. The precise argument yields uniform Kazhdan constants only for the family S_k with $k \sim 2^{18l}$, $l = 1, 2 \dots$. The general case follows by showing that one can *uniformly boundedly generate* each S_n by embeddings of S_k for k of this type.

III. Uniform expansion over different generating sets. In a recent impressive achievement [15], Bourgain and Gamburd established the following:

Theorem 3.4. (1) *For any subset* $S \subseteq \mathrm{SL}_2(\mathbb{Z})$ *not generating a virtually cyclic subgroup,* $\mathrm{Cay}(\mathrm{SL}_2(\mathbb{F}_p), \bar{S})$ *is an expander family* (*for* p *large enough*).

(2) *Fix $k \geq 2$. As $p \to \infty$, an independent uniform random choice of k elements in each* $\mathrm{SL}_2(\mathbb{F}_p)$ *makes with probability* $\to 1$ *the (undirected) Cayley graphs into an expander.*

(3) *Fix any $c > 0$. If for every p a symmetric generating set $S_p \subseteq \mathrm{SL}_2(\mathbb{F}_p)$ is chosen, so that girth* $\mathrm{Cay}(\mathrm{SL}_2(\mathbb{F}_p), S_p) \geq c \cdot \log p$*, then these graphs form an expander.*

The heart of the result lies in part (3), which easily implies (1), and using the "random logarithmic girth" result established in [41], immediately implies (2). The proof of the theorem borrows key ideas from the approach introduced by Sarnak and Xue [91], incorporating two main ingredients: I. High multiplicity of the (bad) eigenvalues in the regular representation of $\mathrm{SL}_2(\mathbb{F}_p)$, stemming from Frobenius' classical result that the smallest dimension of a non-trivial representation of this group ($\frac{p-1}{2}$), is large relative to its size ($\sim p^3$) – i.e., there is a uniformly positive logarithmic ratio between the two (unlike the symmetric groups, for example). II. An upper bound on the number of returns to the identity for random walks of length up to logarithmic order of the group.

While previously, the upper bound in II was obtained by translating the problem into a Diophantine one, the generality of the Bourgain–Gamburd result is made possible by using instead tools from *additive combinatorics*. These include a non-commutative version of the Balog–Szemeredi–Gowers Lemma due to Tao [102], and notably Helfgott's recent breakthrough, discussed below, which capitalizes on *sum–product* results. Other aspects of Bourgain–Gamburd's work involve algebraic inputs, such as Frobenius' result above, and the precise subgroup structure of $\mathrm{SL}_2(\mathbb{F}_p)$. However, it is actually in the fascinating theme of sum–product phenomena, that one finds a rather striking similarity with previously discussed algebraization methods for property (T), and a conceptual explanation of how rich algebraic structure may lead to expansion properties. We shall try to shed some light on this fundamental ingredient, beginning with the recent pioneering result of Helfgott [51]:

Theorem 3.5. *Let $S \subseteq \mathrm{SL}_2(\mathbb{F}_p)$ be any generating set. Then* $\mathrm{Cay}(\mathrm{SL}_2(\mathbb{F}_p), S)$ *has diameter $\leq K(\log p)^c$, where the constants K, c are absolute.*

This result is weaker than "expansion" (in which case $c = 1$), and a similar statement (with $\log p$ replaced by $\log |G|$) is conjectured by Babai [5] to hold uniformly over all finite simple groups. However, it is here that for the first time, the barrier of handling uniformly independent generating subsets is crossed. The proof of Theorem 3.5 is a direct consequence of the following:

Key Proposition. *Let p be a prime and $A \subseteq \mathrm{SL}_2(\mathbb{F}_p)$. Then:*

(a) (Small sets) *If A is not contained in a proper subgroup, and $|A| < p^{3-\delta}$ with $\delta > 0$, then $|A \cdot A \cdot A| > c|A|^{1+\varepsilon}$, where $c, \varepsilon > 0$ depend only on δ.*

(b) (Large sets) *Assume A is not contained in any proper subgroup, and $|A| > p^{\delta}$, $\delta > 0$. Then there is an integer k depending only on δ, such that $(A \cup A^{-1})^k =$* $\mathrm{SL}_2(\mathbb{F}_p)$.

Theorem 3.5 follows immediately by first applying a constant number of set multiplications (depending only on c, ε), so that (a) starts giving exponential growth, and then applying (a) followed by (b). The latter is a major ingredient in Bourgain–Gamburd's work. Its proof when δ is close to 3 (e.g. $\delta > 8/3$) requires soft Fourier analysis, hence matters rest primarily on (a). We next discuss it, remarking first that the appearance of $|A \cdot A \cdot A|$ rather than $|A \cdot A|$ is necessary; consider e.g. $A = H \cup \{g_0\}$, where H is a subgroup.

Sum–product phenomena and expansion. Besides standard (by now) tools from additive combinatorics, such as the Balog–Szemeredi–Gowers theorem, and properties of Ruzsa distances, the proof of (a) in the Key Proposition makes crucial use of powerful *sum–product phenomena*. These arise in works of Bourgain, Glibichuk, Katz, Konyagin, Tao, and originally involved also subtle arithmetic techniques originated from (Stepanov's elementary proof of) Weil's work on the Riemann hypothesis over finite fields (the latter part is relevant only to Helfgott's and not to Bourgain–Gamburd's work, due to their logarithmic girth assumption). See [14], [18], [19], [63], Section 2.8 in [103], and the references therein, for further details, including the interesting intimate connections with work (notably by Bourgain), on the ring conjecture, Kakeya problem, and exponential sum estimates. For our purposes, it suffices to state the following (cf. [51]):

Theorem 3.6 (Sum–Product). *Fix $\delta > 0$. Then for any subset $A \subseteq \mathbb{F}_p - \{0\}$ with $C < |A| < p^{1-\delta}$, we have*

$$\max\{|A \cdot A|, |A + A|\} > |A|^{1+\varepsilon}$$

where $C, \varepsilon > 0$ depend only on δ.

An analogous result over the integers was first established by Erdös–Szemeredi [37]. Very recently, a simplified proof of the theorem was found by Tao (see Theorem 2.52, Corollary 2.55 in [103]). A similar statement holds for an arbitrary finite field F (taking into account the presence of subfields). Results accounting for a small growth of a set under applying internal arithmetic operations by an obvious algebraic structure (arithmetic progression, subring, subfield) capturing most of its mass, go back to Friemann's classical theorem (cf. [103, Ch. 2]). In fact, under the same heading one can also include the recent far reaching uniform exponential growth results of Eskin–Mozes–Oh [38], and Breuillard–Gelander [20], for infinite, finitely generated linear groups. In the latter, as in the proof of Theorem 3.6, one first shows that some algebraic operation on the set A yields a set with the desired growth property, and then deduces the result back for A itself. However, quite surprisingly, in the proof of Theorem 3.6 it is actually the *latter* step which is *trickier* (for instance, it can happen that $|A \cdot A + A \cdot A| \sim |A|^2$, but $|A \cdot A|, |A + A| < 2|A|$). Returning to Helfgott's Theorem 3.5 above, since matrix multiplication in $\mathrm{SL}_2(\mathbb{F}_p)$ encodes the addition and multiplication in $\mathbb{F}_p$ together, the relevance of sum–product results to Theorem 3.5 is not a surprise *a posteriori*. In practice, the proof applies Theorem 3.6

to the traces of the elements of a set, showing that the sizes of a set and its set of traces "keep track of one another".

If one wants to pin down the source of the expansion in both Bourgain–Gamburd and Helfgott results, it is the sum–product phenomenon which gives the best quick answer. It is quite interesting to examine in this light the proof of relative property (T) for general rings (Theorem 2.4 above), which is the departure point for most results in the algebraization of property (T). Its proof capitalizes on the same ring theoretic phenomenon, where sets (or measures) which are "almost invariant" under the combined ring operations, must be "degenerate" (e.g., must assign mass to 0).

The Archimedean spectral gap analogue. In this exposition we can only mention the second companion work by Bourgain–Gamburd [16], which also deals with establishing a uniform spectral gap, and involves some similar ingredients, this time working over $\mathbb{R}$ or $\mathbb{C}$. It is shown there that a certain "non-commutative Diophantine condition" on (the group generated by) a finite set $S \subseteq \mathrm{SU}(2)$, implies that its induced action on the zero mean functions $L_0^2(\mathrm{SU}(2))$ has a spectral gap. This Diophantine condition was introduced in previous (weaker) results in this direction by Gamburd–Jakobson–Sarnak [42], who also showed that it is automatically satisfied if all elements in S have algebraic traces. See these two papers, as well as [69], [90], for more on the history of the problem, which may be viewed as a quantitative version of the (positive solution to the) Banach–Ruziewicz problem, pertaining to a certain uniqueness property of the Lebesgue measure on the 2-sphere.

The bulk of this companion work by Bourgain–Gamburd consists in establishing a "statistical" analogue of Helfgott's Key Proposition above [16, Proposition 1]. Its proof replaces the sum–product Theorem 3.6 by an approach originating from (and improving on) Bourgain's work [14, Theorem 0.3], towards the ring problem. This work of Bourgain–Gamburd turns out to be more involved than the one on expanders, a fact which may seem surprising in view of past experience with these parallel problems. There are, however, two conceptual explanations why, when dealing with more "generic" (or less "special") finite subsets, one should expect more difficulties here. Firstly, while in the real topology "bad" sets can be continuously approached by "good" ones, such phenomenon cannot occur in the non-archimedean case. Even more importantly, the right analogue of the (ideal) SU(2) result, would be showing a uniform expansion for (topologically generating) finite subsets of the compact group $\mathrm{SL}_2(\mathbb{Z}_p)$ (p-adic integers), which, in return, is *equivalent* (see [92]) to showing that for any fixed k, *all* choices of k-generator subsets, independently in each $\mathrm{SL}_2(\mathbb{Z}/p^i\mathbb{Z})$, form expanders. Now, it may be expected that the latter property turns out more subtle than making $\mathrm{SL}_2(\mathbb{F}_p)$ with varying p into expanders. In the latter case, it is not a priori clear if and how the questions over different primes p relate. One may hope (perhaps naively though), that as it often happens, the "right" solution (spectral gap here) for one prime, would work uniformly over all primes. In contrast, in the case of $\mathrm{SL}_2(\mathbb{Z}/p^i\mathbb{Z})$ with i increasing, establishing a spectral gap bound for a given i automatically yields the same bound for smaller i's (as the canonical quotient map induces an inclusion at the L^2 level). Hence one faces more difficult tasks as i grows.

In fact, remarking on the preceding paragraph (in a previous draft of this paper), Jean Bourgain has informed us of substantial progress achieved recently towards a generalization of Theorem 3.4, covering uniformly essentially all $\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$. He noted that the method when $q = p^i$ is indeed closer to the one in the $\mathrm{SU}(2)$ case, and may be viewed as its p-adic analogue – see below, and [17], where this generalization is involved. The higher rank cases $\mathrm{SL}_{n\geq 3}$ should naturally be addressed as well. Whether they turn out to be as demanding, remains to be seen.

Finally, we remark that Theorem 3.4 (with its extension mentioned above) has very recently been applied by Bourgain–Gamburd–Sarnak, to develop a combinatorial sieve method for primes and almost primes, on orbits of various subgroups of $\mathrm{GL}_n(\mathbb{Z})$ as they act on $\mathbb{Z}^n$. Unlike the more familiar case of sieving in $\mathbb{Z}^n$, in this setting the expander property plays a critical role. See the announcement [17] for further details.

4. Reduced cohomology and property (T) for elementary linear groups

I. Definition and basic properties of reduced cohomology. The study and use of reduced cohomology in relation to property (T) was first pursued in [95], motivated by establishing rigidity results for lattices in products of groups. We shall need it in Subsection III for the proof of Theorem 1.1, and in the next subsection some previous applications of it are mentioned. Recall first the basic correspondence between (affine) isometric actions of a group Γ on a Hilbert space V, and first cohomology. Any such action is of the form $\rho(\gamma)v = \pi(\gamma)v + b(\gamma)$, where π is a unitary Γ-representation on V, and the affine part $b\colon \Gamma \to V$ satisfies the 1-cocycle identity corresponding to π (the identity sufficient and necessary to make ρ an *action*). Fixing π, the set of all such b is a vector space, denoted $Z^1(\Gamma, \pi)$, and those "trivial" elements of the form $b_v(\gamma) = v - \pi(\gamma)v$, the coboundaries, form a subspace denoted $B^1(\Gamma, \pi)$. It is immediate that the Γ-action on V has a fixed point (v_0) iff the corresponding 1-cocycle b is a coboundary (b_{v_0}). We define the quotient space $H^1(\Gamma, \pi) = Z^1(\Gamma, \pi)/B^1(\Gamma, \pi)$, and can now consider also the topological version of it. Namely, fixing π, introducing the topology of pointwise convergence on the space $Z^1(\Gamma, \pi)$ makes it a Frechet space, in which B^1 is not always closed. Forming its closure, the first *reduced cohomology* $\overline{H}^1(\Gamma, \pi)$ can now be defined as $Z^1/\overline{B}^1$. The following analogous relation between fixed points of affine actions and coboundaries, can be verified easily:

Lemma 4.1. *Given $b \in Z^1(\Gamma, \pi)$, we have $b \in \overline{B}^1(\Gamma, \pi)$ iff the corresponding affine action $\rho(\gamma)v = \pi(\gamma)v + b(\gamma)$ has almost fixed points in the metric sense, namely, for every finite $S \subseteq \Gamma$, and $\varepsilon > 0$, there is $v \in V$ with $\|\rho(\gamma)v - v\| < \varepsilon$ for all $\gamma \in S$.*

All the notions and results here extend naturally to the class of second countable locally compact groups. The discussion above shows that the characterization of property (T) in terms of vanishing of usual cohomology is a tautology if one takes the fixed point property as a definition. However, the following characterization in terms of the (generally smaller) reduced cohomology is less transparent, and holds only under the assumption of finite generation (or compact generation, in the locally compact setting):

Theorem 4.2 ([95]). *Let* Γ *be a finitely generated group without property* (T). *Then there exists some unitary* Γ*-representation* π, *with* $\overline{H}^1(\Gamma, \pi) \neq 0$. *Moreover, one can find such* π *which is irreducible.*

II. Some applications of the reduced cohomology

(1) Theorem 4.2 implies the existence of an irreducible π with $H^1(\Gamma, \pi) \neq 0$, as conjectured by Vershik–Karpushev [106]. Together with [106, Theorem 2] (see Louvet's [68] for a detailed exposition) it implies: *A discrete group* Γ *has property* (T) *iff it is finitely generated, has finite abelianization, and it does not admit any non-trivial irreducible unitary representation not separated* (*in the Fell topology*) *from the trivial representation*.

(2) As an application of the proof of Theorem 4.2, it is shown in [95] that every finitely generated Kazhdan group is a quotient of a *finitely presented* Kazhdan group (answering questions of Grigorchuk and of Żuk; the result is generalized by Fisher–Margulis [39] to locally compact groups). However, the existence seems entirely non-constructive, and there are concrete interesting groups which would be of interest to understand in this regards – see II in Section 5 below.

(3) Although in general, the existence of the *irreducible* cohomological π in Theorem 4.2 is non-constructive, somewhat surprisingly, in many cases one can actually classify all such π, and show that there are only *finitely many* of them. It may seem particularly unexpected that this finiteness phenomenon appears among *amenable* groups. For example, this is the case for all polycyclic (or lamplighter) groups, a result from [98] shown there to have applications in geometric group theory (e.g., any group quasi-isometric to a polycyclic group has a finite index subgroup with infinite abelianization). Martin [77] showed that all connected locally compact groups also have only finitely many such representations π.

(4) Inspired by Margulis' remarkable strategy in proving the normal subgroup theorem for higher rank lattices, as well as by more recent beautiful work of Burger–Mozes (cf. [22]), the following result was completed very recently: over sufficiently large finite fields, *any irreducible Kac–Moody group of non-affine* (*and non-spherical*) *type, has a finite index commutator subgroup, which is a* (*finitely generated*) *simple group, modulo its finite center*. Building on fundamental work of Rémy, the proof consists of three entirely independent results on any quotient of the simple group by a non-trivial normal subgroup: it is *Kazhdan* [95], *amenable* (Bader–Shalom [7]), and *infinite* (Caprace–Rémy [23]), classes of groups which do not intersect. The proof of the first relies crucially on the reduced cohomology of (infinite dimensional) unitary representations. Incidentally, we remark that while in virtually all of the applications of property (T), an appropriate Kazhdan group comes to the rescue, the ones made through normal subgroup theorems are of quite unique nature, as throughout the proof

no (non-trivial) Kazhdan group appears.

III. Sketch of proof of Theorem 1.1. Fix $n \geq 3$ (note that if $\dim R = 0$, by finite generation R must be finite). We proceed via the following steps:

1. Setting and notation. Set $\Gamma = \mathrm{EL}_n(R)$, and define the following subgroups: $\Lambda =$ matrices whose first row and first column begin with 1 and have 0s elsewhere, $N_1, N_2 \cong R^{n-1}$ the subgroups sitting in the upper row and left column with a "common" 1 at the upper left corner. Notice that Λ normalizes each one of the N_i's. The standard Steinberg commutator relations show that N_1 and N_2 together generate Γ. In fact, letting $r_1 = 1, r_2, \dots, r_k$ be generators of R as a ring, the set S of all elementary matrices belonging to one of the two N_i's, having one of the r_j as the only non-zero element off the diagonal, forms a finite generating set for Γ.

Finally, for an isometric Γ-action ρ on a Hilbert space V_ρ and $v \in V_\rho$, denote

$$\delta_S(v) = \max\{\|\rho(s)v - v\| \mid s \in S\}, \quad \delta_S(\rho) = \inf\{\delta_S(v) \mid v \in V_\rho\}.$$

2. Reduced cohomology. Assume that Γ does not have property (T). We argue to get a contradiction. By Theorem 4.2 and Lemma 4.1 above, there exists some isometric Γ-action ρ on a Hilbert space V_ρ, with $\delta_S(\rho) > 0$. By rescaling we may assume that $\delta_S(\rho) \geq 1$. Denote by $\mathcal{A}$ the set of all isometric Γ-actions ρ with $\delta_S(\rho) \geq 1$.

3. Relative property (T) – the spectral ingredient. By (an obvious extension of) Theorem 2.4, for each i the pair $(\Lambda \ltimes N_i, N_i)$ has the relative property (T). Consequently, by the equivalence in Definition 2.3, the following infimum is not taken over the empty set:

$$d = \inf\{\|v^1 - v^2\| \mid v^i \in V_\rho \text{ with } \rho \in \mathcal{A}, \text{ and } \rho(N_i)v^i = v^i \text{ for } i = 1, 2\}.$$

4. Attaining *d* through a limiting process. Let $\rho_n \in \mathcal{A}$ and $v_n^i \in V_{\rho_n}^{N_i}$ with $\|v_n^1 - v_n^2\| = d_n \to d$. We may assume that $d_n < d + 1$, and this gives for all n $\delta_S(v_n^1) < 2(d+1)$, since v_n^1 is fixed by the N_1-generators of S, and a vector of distance at most $d + 1$ from it is fixed by the N_2-generators of S. This uniform bound implies (using an ultra-product argument as in [39], or a negative definite kernel argument as in [95]), that a subsequence of the actions (ρ_n, V_{ρ_n}), pointed at v_n^1, converges to an isometric Γ-action on a Hilbert space (ρ_∞, V_∞), with two points $v^i \in V_\infty^{N_i}$ satisfying $\|v^1 - v^2\| = d$. One shows that indeed $\rho_\infty \in \mathcal{A}$, hence d defined in step 3 is attained. Notice that $d \neq 0$, for otherwise v^1 is fixed by Γ, contradicting $\rho \in \mathcal{A}$.

5. Can assume π_∞ has no invariant vectors. Write $\rho_\infty(\gamma)v = \pi_\infty(\gamma)v + b_\infty(\gamma)$, where π_∞ is the (unitary) linear part (see the discussion at the beginning of the previous subsection). Decompose orthogonally $\pi_\infty = \pi_0 \oplus \pi_1$ where $\pi_0 = \pi^\Gamma$, and correspondingly, $b_\infty = b_0 + b_1$. Being a 1-cocycle for a trivial Γ-action, b_0 is an additive character, and since Γ is perfect (as it is generated by commutators), $b_0 = 0$. Replacing ρ_∞ by $\rho'_\infty(\gamma)v := \pi_1(\gamma)v + b_1(\gamma)$ yields the required reduction.

6. A fixed point for Λ via a geometric argument. Assume that for some $\lambda_0 \in \Lambda$, $w^1 := \rho_\infty(\lambda_0)v^1 \neq v^1$, and denote $w^2 = \rho_\infty(\lambda_0)v^2$. We will show this to be impossible. As Λ normalizes N_i, we have $v^i, w^i, u^i := \frac{1}{2}(v_i + w_i) \in V_\infty^{N_i}$, for $i = 1, 2$. Because λ_0 is an isometry, $\|v^1 - v^2\| = \|w^1 - w^2\| = d$, while by the definition of d as infimum, $\|u^1 - u^2\| \geq d$. By a standard convexity argument this is possible only if $v^1 - v^2 = w^1 - w^2$, and hence $v^1 - w^1 = v^2 - w^2 \neq 0$. But since v^1, w^1 are N_1-fixed, the left hand side is a non-zero *vector* invariant under the *linear* action of N_1, and similarly for $v^2 - w^2$ w.r.t. N_2. Since $\langle N_1, N_2 \rangle = \Gamma$, it follows that this common non-zero vector is Γ-invariant, contradicting the reduction made in step 5. Thus, $\rho(\Lambda)v^1 = v^1$.

7. Finishing with bounded generation and the stable range. A fundamental result of Bass (cf. Theorem 4.1.14 in [49]), asserts that if R is a commutative Noetherian (in particular, by Hilbert's basis theorem, if it is a finitely generated) ring, then for any $n \geq 2 + \text{Krull}\dim R$, the ring R satisfies the following property: For every $a_1, \dots, a_n \in R$ such that $a_1R + \cdots + a_nR = R$ (such an n-tuple is called *unimodular*), there exist $\alpha_2, \dots, \alpha_n \in R$, such that $(a_2 + \alpha_2 a_1)R + \cdots + (a_n + \alpha_n a_1)R = R$. The minimal n satisfying this property is called the *stable range* of R, denoted $\text{sr}(R)$ (so $\text{sr}(R) \leq \dim R + 2$ – strict inequality can hold. Note also that there is "inconsistency up to ± 1" in the literature regarding the definition of the stable range). This property enables one to reduce any $\gamma \in \Gamma$ to $\lambda \in \Lambda$, using a bounded number of elementary operations. Indeed, notice that since all matrices in Γ are invertible, the first row of γ is unimodular. Then, by performing $n - 1$ elementary operations we may create a unimodular $(n-1)$-tuple in the last entries of the first row, and since $1 \in R$, proceed to place 1 in the upper left corner, and use it to annihilate all of the rest of the first row and column. In group theoretic terms, since all the elementary subgroups are conjugate, and any elementary operation is obtained as multiplication by an elementary matrix, this means that Γ is *boundedly generated* by finitely many conjugates of Λ. Like Λ, all of these conjugate fix some point in V_∞, and the Bounded Generation Lemma 2.2 yields a fixed point for Γ, a contradiction which finishes the proof.

It is clear that all that was really relevant to the proof was the stable range of R. Moreover, this notion is similarly defined for *any* ring, not necessarily commutative, only that here one has to distinguish between left and right ideals (although the left and right stable ranges were shown by Vaserstein to be equal [105]). After stating the condition on n in Theorem 1.1 in terms of the stable range in place of $\dim R$, the above proof goes through in this general setting. See [99] for the complete details.

5. Some concluding remarks, questions, and speculations

I. More on Theorem 1.1. The following arises immediately from Theorem 1.1: *Given a finitely generated commutative ring R, when does $\text{EL}_n(R)$ begin to have property* (T)? By that theorem the answer lies between $n = 3$ and $n = 2 + \dim R$.

In fact, the proof gives the generally better upper bound $\mathrm{sr}(R)$ (which we suspect is not always optimal). It seems that to address this issue, one should understand better the relation between bounded generation and property (T). We next speculate about a possible strategy.

The proof of Theorem 1.1 actually establishes the following result for any finitely generated ring R and every $n \geq 3$: if $\mathrm{EL}_n(R)$ is boundedly generated by conjugates of $\mathrm{EL}_n(R) \cap \mathrm{GL}_{n-1}(R)$ ($= \Lambda$ in our previous notation), then $\mathrm{EL}_n(R)$ has property (T). The purely algebraic assumption involved here is satisfied when $n \geq \dim R + 2$, and it seems that a full understanding of when it happens (in terms of n as a function of R), takes one beyond the "property (T) territory". However, an attempt at understanding its inverse relation to property (T) should be made: can one show that its failure reflects back on the failure of property (T)? The only device which currently seems available towards such a result, is that of "spaces with walls" defined in [48] (whose more general measurable counterpart is known to capture the lack of property (T) [87]; see also [28]). This setting enables one to construct a negative definite kernel on a group, out of its action on ("half spaces" of) a discrete set, satisfying simple axioms. For our purposes, a natural strategy would thus be to "encode" the algebraic framework into such a set, where the value of the negative definite kernel at $\gamma \in \mathrm{EL}_n(R)$ corresponds to the number of multiplications needed to generate it.

II. Quantifying the robustness of property (T). As mentioned is Section 4.II (2), one of the consequences of the reduced cohomology approach is that any finitely generated Kazhdan group Γ is a quotient of a *finitely presented* Kazhdan group. An intriguing example to which this applies is the group $\Gamma = \mathrm{SL}_3(\mathbb{F}_p[t])$ (which has property (T) because $3 > 2$, and is not finitely presented because $3 < 4$ – see the first mention of the group in this context on [76, p. 134]). Thus, finitely many among the well understood infinite sequence of relations in this group, already suffice to define a Kazhdan group. However, even in this particular case (where explicit Kazhdan constants are known), it is an open problem to *make the existence proof effective*.

The question can in fact be seen as merely one instance of trying to *quantify the robust behavior of property* (T), a phenomenon which was applied in Fisher–Margulis local rigidity results [39]. If Γ has property (T), then for some $\varepsilon > 0$ there is a fixed point for any ε-isometric action of it on a Hilbert space. Moreover, it is actually enough to impose this condition on a generating set only, and for spaces which are Hilbert only locally (on some $1/\varepsilon$ ball). Even further, the action may be well defined only on elements inside a $1/\varepsilon$-ball of Γ, and a moment reflection shows that the latter yields the previously mentioned result about the existence of a finitely presented Kazhdan cover. In fact, one may go further, in assuming the latter action to be only a *near* (ε-)action. In short, by appropriately using a (rather standard by now) limiting argument as needed in the proof of Theorem 1.1, essentially everything in the characterizations of property (T) can be perturbed, yet there is no single non-trivial case when it is known how to do this effectively. To put matters in perspective, we remark that a similar robustness phenomenon *does not* hold for the "opposite" fixed point property – amenability. Notice also that the proof of Theorem 1.1 *does not yield any explicit Kazhdan constants*. While we believe that some new ideas are needed for the previous questions, it may be that a variation on the proof of Theorem 1.1 will make this an easier task.

III. Property (τ) for irreducible lattices. Although Clozel's recent property (τ) paper [29] largely closes a chapter on the study of property (τ) for arithmetic groups, the story is not quite over as might be perceived. What Clozel actually establishes is the *Selberg property*, and the precise semantics here is important exactly because the congruence subgroup property for many arithmetic groups (e.g., co-compact irreducible lattices in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$) is only conjectured. While Selberg's property, just like his original $\lambda_1 \geq 3/16$ result for $\mathrm{SL}_2(\mathbb{Z})$, yields a spectral gap over the *congruence subgroups*, property (τ) seeks a spectral gap over *all* finite index subgroups. We remark that the incompleteness of property (τ) becomes worse for arithmetic groups over local fields of *positive characteristic*, which were not treated by Clozel.

For higher rank lattices in *simple* algebraic groups, property (T) automatically takes care of property (τ), hence all the unsettled cases lie in a general setting which was extensively and successfully studied in recent years; that of an irreducible lattice Γ in a product $G = G_1 \times G_2$. We believe that under fairly general conditions, certainly ones satisfied when each G_i is a simple algebraic group over a local field, one should be able to get a "softer" proof (certainly avoiding number theory, or a subtle identification of the finite index subgroups), that Γ has property (τ). While as explained, this would have some advantages compared to Clozel's theorem, we note that such a general approach cannot compete with Clozel's: his theorem gives *explicit* spectral bounds on the spectrum of $L_0^2(G/\Gamma_n)$, when restricting to *each simple factor* G_i. Property (τ) is equivalent to the existence of some bound on the joint spectrum of the G_i's.

It may be that there is actually a deeper, more interesting representation theoretic phenomenon underlying the conjectural property (τ) for irreducible lattices: *Is it true that whenever π is a unitary Γ-representation, possibly infinite dimensional, which admits almost invariant vectors, then π has some subrepresentation which extends to G*? Obviously, this would immediately imply property (τ) for Γ (using only that G_i have no non-trivial finite dimensional unitary representations). Such a result would be extremely useful. Some support may be provided by the superrigidity for reduced cohomology in [95], as the failure of property (T) for Γ is always detected by representations with reduced cohomology (Theorem 4.2 above), which in return, *must come from the ambient group G* [95, Theorem 3.1].

IV. Burnside groups and Zelmanov's theorem. The existence of infinite Burnside groups, i.e., finitely generated groups of bounded torsion (first established by Adyan and Novikov, see also [83] and the references therein), is still one of the impressive achievements of infinite group theory. Even harder non-finiteness results are non-amenability of such groups [2]. It is an extremely intriguing question, to which we believe the answer is positive, *whether all Burnside groups should have property* (T). One of the puzzling features of the problem is that it is unclear if it should be attacked within a *geometric*, or an *algebraic* approach. While serious attempts in the negative have been made within the former direction, we believe that it is actually the latter which should be used, and that reduced cohomology may again become useful. Of course, one of the rewards of establishing property (T) for Burnside groups would be an immediate "size dichotomy" of independent interest: either they are *finite*, or *non-amenable*. There is no known counterexample to this plausible statement.

With the hope of establishing property (T) for Burnside groups in mind, one can wildly speculate further, about a possible approach to reproving Zelmanov's celebrated positive solution to the restricted Burnside problem (cf. [110]): A *residually finite Burnside group is finite*. The idea is to try to adapt Margulis' "(T) $\cap$ (amenable) = (finite)" strategy in proofs of normal subgroup theorems, to this setting. More precisely, assume that Γ is a Burnside group and $N_i < \Gamma$ is a decreasing sequence of finite index normal subgroups. If Γ was shown to have (T), we would know that Γ/N_i are expanders. On the other hand, a reduction (due to Hall and Higman, already used by Zelmanov) enables one to assume that Γ has prime power torsion, in which case all the finite quotients would have a similar property (and are hence nilpotent). While nilpotency may hint (although in itself is not enough to show) that the quotients should not be expanders, the additional uniform torsion may be sufficient to establish this opposite behavior. Its contrast with the first, property (T) part, would then imply Zelmanov's theorem.

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