Rankin–Selberg integrals, the descent method, and Langlands functoriality

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Abstract. In this article I survey the descent method of Ginzburg, Rallis and Soudry and its main applications to the Langlands functorial lift of automorphic, cuspidal, generic representations on a classical group to (appropriate) GL_n, and to establishing a local Langlands reciprocity law for (split) SO_{2n+1} (joint work with D. Jiang). The descent method arises when we consider certain residues of special cases of a family of global integrals, attached to pairs of automorphic, cuspidal representations, one on a classical group G and one on GL_n. The last part of this article focuses on the case $G = \mathrm{SO}_m$ (split), and the progress made in a joint work with S. Rallis, towards establishing, via the converse theorem, the functorial lift from any automorphic, cuspidal representation on G to $\mathrm{GL}_{2[\frac{m}{2}]}$.

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1. Introduction

Let F be a number field, and let $\mathbb{A}$ be its ring of Adeles. I will start with a general family of global integrals of Rankin–Selberg type or of Shimura type for $G \times \mathrm{GL}_n$, where G is an orthogonal group, a unitary group, a symplectic group, or a metaplectic group defined over F. This family contains, at one end, the integrals studied by Shahidi, giving rise to the Langlands–Shahidi method, and at another end, this family contains the global integrals, giving rise to the descent method of Ginzburg, Rallis and Soudry. In this section, I sketch the structure of these integrals, and in the next section I report on my joint work with D. Ginzburg and S. Rallis on the descent method and its many applications, and on my joint work with D. Jiang on the local Langlands reciprocity law for the split group SO_{2n+1}. In the third section I report on a joint work in progress with S. Rallis towards the (weak) functorial lift from SO_m to $\mathrm{GL}_{2[\frac{m}{2}]}$.

a. Let E be either F or a quadratic extension of F. Let V and V' be vector spaces over E of dimensions m and m' equipped with non-degenerate bilinear forms b and b', respectively, which are symmetric if $E = F$ and Hermitian (with respect to the Galois conjugation "-" of E over F) otherwise. Let $G = U(V)$ and $G' = U(V')$ be the isometry groups of (V, b) and (V', b'), respectively. Denote by X_i the (orthogonal) direct sum of i hyperbolic planes, where, in the second case, we mean two

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dimensional spaces over E with Hermitian form (according to some basis) given by $b((x_1, x_2), (y_1, y_2)) = x_1\bar{y}_2 + x_2\bar{y}_1$. Fix a decomposition $X_n = X_n^+ + X_n^-$, where $X_n^\pm$ are transversal n-dimensional totally isotropic subspaces. Assume that m and m' have different parities and that one of the following holds.

(1) There is an orthogonal decomposition of the form $V = X_j \oplus Y$ and there is a non-isotropic vector $y_0 \in Y$ such that $V' = Y \cap y_0^\perp$, and b' is the restriction of b to $V' \times V'$.

(2) The same as in (1), but reversing the roles of V and V', i.e. $V' = X_j \oplus Y'$ and there is a non-isotropic $y_0' \in Y'$ such that $V = Y' \cap {y_0'}^\perp$, and b is the restriction of b' to $V \times V$.

Let $W = X_n \oplus V'$ and $H = U(W)$. Let π, ρ, τ be irreducible, automorphic, cuspidal representations of $G_{\mathbb{A}}$, $G'_{\mathbb{A}}$, $\mathrm{GL}_n(\mathbb{A}_E)$, respectively. Let P be the parabolic subgroup of H which preserves X_n^+; its Levi part M is isomorphic to $\mathrm{Res}_{E/F}\,\mathrm{GL}_n \times G'$. Thus we may view $\tau|\det|^{s-\frac{1}{2}} \otimes \rho = \tau_s \otimes \rho$ as a representation of $M_{\mathbb{A}}$ and consider its parabolic induction $I(\tau_s, \rho)$ to $H_{\mathbb{A}}$. Let $E(h, f_{\tau_s,\rho})$ be an Eisenstein series corresponding to an analytic section $f_{\tau_s,\rho}$ in $I(\tau_s, \rho)$. Now we distinguish three cases.

(i) $m' < m < 2n + m'$.

(ii) $2n + m' < m$.

(iii) $m < m'$.

In cases (i) and (iii) we apply to $E(h, f_{\tau_s,\rho})$ a Fourier coefficient of Gelfand–Graev type stabilized by G; let us denote it by $E^{\psi,G}(h, f_{\tau_s,\rho})$, where ψ is a non-trivial character of $F\backslash\mathbb{A}$. We pair this coefficient with cusp forms φ_π in π,

$$\mathcal{L}(\varphi_\pi, f_{\tau_s,\rho}) = \int_{G_F\backslash G_{\mathbb{A}}} \varphi_\pi(g) E^{\psi,G}(g, f_{\tau_s,\rho})\, dg. \tag{1.1}$$

In case (ii) we apply to φ_π a Fourier coefficient of Gelfand–Graev type stabilized by H, and pair it with $E(h, f_{\tau_s,\rho})$,

$$\mathcal{L}(\varphi_\pi, f_{\tau_s,\rho}) = \int_{H_F\backslash H_{\mathbb{A}}} \varphi_\pi^{\psi,H}(h) E(h, f_{\tau_s,\rho})\, dh. \tag{1.2}$$

For the general notion of a Fourier coefficient of Gelfand–Graev type see [29]. The integrals above, in this generality, appear in [9] for orthogonal groups. See also [34]. These integrals are meromorphic in the whole plane, and their poles are included in the set of poles of the Eisenstein series involved. They can be unwinded for $\mathrm{Re}(s) \gg 0$ and can be shown to depend, through an inner integration, on an invariant bilinear pairing $b_{\pi,\rho}(\varphi_\pi, \varphi_\rho)$, where, in cases (i) and (ii),

$$b_{\pi,\rho}(\varphi_\pi, \varphi_\rho) = \int_{G'_F\backslash G'_{\mathbb{A}}} \varphi_\pi^{\psi,G'}(g)\varphi_\rho(g)\, dg, \tag{1.3}$$

and, in case (iii),

$$b_{\pi,\rho}(\varphi_\pi, \varphi_\rho) = \int_{H_F \backslash H_{\mathbb{A}}} \varphi_\pi(h)\varphi_\rho^{\psi,G}(h)\, dh. \tag{1.4}$$

In particular, if $b_{\pi,\rho}(\varphi_\pi, \varphi_\rho) = 0$ (identically), then the integrals (1.1), (1.2) are (identically) zero. The integrals above have easy variants when G, G' are special orthogonal groups.

b. Let now (V, b), (V', b') be symplectic spaces over F, and let X_j be a symplectic space of dimension $2j$ over F with two transversal Lagrangians $X_j^{\pm}$. Let $W = X_n \oplus V'$. Let G, G', H be the symplectic groups of V, V', W, respectively. Let P be the parabolic subgroup of H which preserves X_n^+. Denote by $\tilde{H}_{\mathbb{A}}$ the metaplectic cover of $H_{\mathbb{A}}$, and similarly for G, G'. Let π, ρ, τ be irreducible, automorphic, cuspidal representations of $G_{\mathbb{A}}$, $\tilde{G}'_{\mathbb{A}}$, $\mathrm{GL}_n(\mathbb{A})$, respectively. We assume that ρ is genuine. Consider the Eisenstein series $E(h, f_{\tau_s,\rho})$ on $\tilde{H}_{\mathbb{A}}$ corresponding to parabolic induction from $\tau_s \otimes \rho$ (we have to multiply τ_s by a Weil factor corresponding to ψ). Now we can either apply a Fourier–Jacobi coefficient, stabilized by G, to the Eisenstein series and pair it, as above, along $G_F \backslash G_{\mathbb{A}}$ with cusp forms φ_π, or apply such a coefficient, stabilized by H, to φ_π and pair it along $H_F \backslash H_{\mathbb{A}}$ with the Eisenstein series. In this way we get global integrals $\mathcal{L}(\varphi_\pi, \phi, f_{\tau_s,\rho})$; here ϕ is a Schwartz function in the space of the Weil representation, which occurs in the Fourier–Jacobi coefficients above. As before these integrals are meromorphic, can be unwinded, and depend through inner integrations on invariant bilinear pairings, which define a Fourier–Jacobi model of π with respect to ρ or vice-versa. Similar integrals can be written for π on $\tilde{G}_{\mathbb{A}}$, ρ on $G'_{\mathbb{A}}$, and τ as before (with $H_{\mathbb{A}}$ instead of $\tilde{H}_{\mathbb{A}}$). Finally, similar integrals can also be written when (V, b), (V', b') are Hermitian spaces over F. For more details see [29], [15], [34].

c. Two extreme cases

1. Assume that G is trivial. Also assume that when b' is symmetric, then G' is a special orthogonal group. Then, in all cases above, one can see that G' must be quasi-split, and the global integrals above are nothing but applications of a Whittaker coefficient to the Eisenstein series $E(h, f_{\tau_s,\rho})$, and using (1.4) and its analog in (b) above, we see that (for the global integrals to be non-trivial) ρ must be globally generic. This is the well-known Langlands–Shahidi method. It is worked out in a long series of papers (see [7] for a survey) by Shahidi and constitutes a beautiful chapter in mathematics. In particular, he established the complete theory of the standard L-functions $L(\rho \times \tau, s)$ (except the metaplectic case) and of $L(\tau, \wedge^2, s)$, $L(\tau, \mathrm{sym}^2, s)$, $L(\tau, \mathrm{Asai}, s)$. Together with the converse theorem of Cogdell and Piatetski–Shapiro [3] it yields the existence of the weak functorial lift from cuspidal generic representations of $G'_{\mathbb{A}}$ to automorphic representations of $\mathrm{GL}_N(\mathbb{A}_E)$ (appropriate N) [1], [2], [26].

2. Assume that G' is trivial. Also assume that when b is symmetric, then G is a special orthogonal group. Then the Levi part of the parabolic subgroup P is isomorphic to $\mathrm{Res}_{E/F}\,\mathrm{GL}_n$. The global integrals, in this case, were studied by many authors; see, for example, [8], [15], [6], [34], [35], [36]. As before, using (1.3) and its analog in (b) above, we see that π must be generic (or else, the global integrals are trivial). These integrals yield (up to a controllable factor) the quotient of the partial L-function $L^S(\pi \times \tau, s)$ by the following denominator. It is $L^S(\tau, r_G, 2s)$, by which we mean $L^S(\tau, \wedge^2, 2s)$ (G odd orthogonal), or $L^S(\tau, \mathrm{sym}^2, 2s)$ (G even orthogonal or symplectic), or $L^S(\tau, \mathrm{Asai}, 2s)$ (G odd unitary); it is $L^S\big(\tau, s + \frac{1}{2}\big)L^S(\tau, \wedge^2, 2s)$ if π is on a metaplectic group, and finally, if G is even unitary, the denominator is the partial Asai L-function of τ at $2s$, but in this case the numerator is $L^S(\pi \times \tau\gamma^{-1}, s)$, where we twist τ by a character γ^{-1}, which enters in the choice of the Weil representation defining the Fourier–Jacobi coefficient in this case. We remark that in the metaplectic case $L^S(\pi \times \tau, s)$ depends also on ψ. The descent method of Ginzburg, Rallis and Soudry is derived when we analyze the existence of a pole at $s = 1$ for the global integrals $\mathcal{L}(\varphi_\pi, f_{\tau_s})$, $\mathcal{L}(\varphi_\pi, \phi, f_{\tau_s})$. The descent method allows us to give a complete description of the image of the weak functorial lift from cuspidal generic representations of $G_{\mathbb{A}}$ to $\mathrm{GL}_N(\mathbb{A}_E)$, to prove the existence and give a description of endoscopic lifts to G (from generic cuspidal representations), to obtain a full local Langlands reciprocity law for generic representations of SO_{2n+1} (joint work with D. Jiang), and much more. I survey some of these applications in the next section.

The general case (G, G' non-trivial) was considered mainly for orthogonal groups in [9], where it is shown that the integrals $\mathcal{L}(\varphi_\pi, f_{\tau_s,\rho})$ yield the quotient

$$\frac{L^S(\pi \times \tau, s)}{L^S\big(\rho \times \tau, s + \frac{1}{2}\big)L^S(\tau, r, 2s)},$$

where $r = r_G = \wedge^2$ or sym^2, depending on G. See also [4] for local analogs in case $m < m'$.

In the third section, I report on the progress in a joint work with S. Rallis towards the existence of a weak functorial lift of cuspidal (not necessarily generic) representations of $\mathrm{SO}_m(\mathbb{A})$ to $\mathrm{GL}_{2[\frac{m}{2}]}(\mathbb{A})$.

2. The descent method and applications

We retain the notation from part c.2 of the previous section. So we have to assume that π is (globally) generic. We assume for simplicity that if G is even (special) orthogonal, then it is split. We also assume, as we may, that τ is unitary such that its central character ω_τ is trivial on the positive real numbers, embedded diagonally at all archimedean primes in the Idele group. The material in Section 2.1–2.3 is part of a long-term joint work with Ginzburg and Rallis [10]–[14], [34].

2.1.

Theorem 2.1. *$L^S(\pi \times \tau, s)$ is holomorphic for* $\mathrm{Re}(s) > 1$ *except when $n = 1$, τ is trivial, and π is on a metaplectic group; in this case the only such pole may occur at $s = \frac{3}{2}$. If $L^S(\pi \times \tau, s)$ has a pole at s_0 with* $\mathrm{Re}(s_0) = 1$*, then $s_0 = 1$. In this case $L^S(\tau, r_G, s)$ has a pole at $s = 1$, when G is orthogonal, symplectic, or odd unitary; if π is on a metaplectic group, then $L^S(\tau, \wedge^2, s)$ has a pole at $s = 1$ and $L\big(\tau, \frac{1}{2}\big) \neq 0$. Finally, if G is even unitary, then $L^S(\hat{\tau} \times \gamma$,* Asai, *$s)$ has a pole at $s = 1$.*

This follows from the fact that the analysis of poles of the integrals $\mathcal{L}(\varphi_\pi, f_{\tau_s})$, $\mathcal{L}(\varphi_\pi, \phi, f_{\tau_s})$ reduces to that of the Eisenstein series $E(h, f_{\tau_s})$ induced from τ.

In a similar way we get that if τ is the (standard) weak functorial lift of π, then τ is self-conjugate, and its central character is trivial on $\mathbb{A}^* = \mathbb{A}_F^*$, and the results of Theorem 2.1 hold for τ. The reason for this is that $L^S(\pi \times \hat{\tau}, s)$ has a pole at $s = 1$. Moreover, for the residue of the global integrals at $s = 1$ to be non-trivial, the L^2-pairing between π and $\mathrm{Span}\{\mathrm{Res}_{s=1} E^{\psi^{-1},G}(\,\cdot\,, f_{\tilde{\tau}_s\gamma})_{|G_{\mathbb{A}}}\} := \sigma_\psi(\tau)$ is non-trivial; in case π is on a metaplectic group, then, in the definition of $\sigma_\psi(\tau)$, we have to take restrictions to $\tilde{G}_{\mathbb{A}}$. Here γ is trivial except when G is even unitary. Note also that starting with τ, G (resp. $\tilde{G}$) is determined by n and the precise data about the pole at $s = 1$ related to τ. We keep all this implicit in the notation $\sigma_\psi(\tau)$. One of our main theorems is

Theorem 2.2. *Let τ be an irreducible, automorphic, cuspidal representation of* $\mathrm{GL}_n(\mathbb{A}_E)$ *such that ω_τ is trivial on $\mathbb{A}^*$. Assume that the results of Theorem* 2.1 *about the pole at $s = 1$ are satisfied for τ. Then $\sigma_\psi(\tau)$ is a non-trivial, automorphic, cuspidal, multiplicity free representation of $G_{\mathbb{A}}$* (*respectively of $\tilde{G}_{\mathbb{A}}$ if H is symplectic*)*. All irreducible summands of $\sigma_\psi(\tau)$ are* (ψ-)*generic and lift at almost all finite places to τ. Each such representation has a non-trivial L^2-pairing with $\sigma_\psi(\tau)$.*

2.2. We call $\sigma_\psi(\tau)$, for τ as in the last theorem, the descent of τ to G (resp. to $\tilde{G}$). Theorem 2.2 describes the cuspidal part of the weak functorial lift from generic cuspidal representations on G (resp. $\tilde{G}$) to GL_n, without knowing that such a lift exists in the sense that we know which cuspidal τ can occur in the image, and moreover, we do construct in a direct manner for such τ irreducible, cuspidal, (ψ-)generic representations which lift to τ; these are the summands of $\sigma_\psi(\tau)$.

When we analyze non-cuspidal τ on $\mathrm{GL}_n(\mathbb{A})$ which may be obtained as a weak lift from cuspidal generic representations of $G_{\mathbb{A}}$ (resp. a metaplectic group), we get that the central character of τ is trivial on $\mathbb{A}^*$, and, by successive applications of Theorem 2.1, we get that, except in the metaplectic case, τ must have the form $\tau = \tau_1 \times \cdots \times \tau_l$, where the τ_i are pairwise inequivalent, irreducible, unitary, self-conjugate automorphic representations of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$, cuspidal when $n_i > 1$; each one satisfying the results of Theorem 2.1 about the pole at $s = 1$. In the metaplectic case τ may also have the form above, with an added "tail" of the form $|\cdot|^{\frac{1}{2}} \times |\cdot|^{-\frac{1}{2}}$.

Thus, for such τ, except in the last case, we form $\sigma_\psi(\tau)$ as before by replacing the residue of the Eisenstein series at $s = 1$, with the multi-residue at $(1, \ldots, 1)$ of the Eisenstein series induced from $\tau_{1,s_1} \times \cdots \times \tau_{l,s_l}$. We prove

Theorem 2.3. *Let τ be an irreducible automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ as in the last paragraph* (*except the additional possibility in the metaplectic case*). *Then $\sigma_\psi(\tau)$ satisfies the conclusions of Theorem* 2.2.

Again, the descent $\sigma_\psi(\tau)$ of τ constructs cuspidal, generic representations of $G_\mathbb{A}$ (resp. $\tilde{G}_\mathbb{A}$) which lift to τ. Since all cuspidal, generic representations of $G_\mathbb{A}$ do lift to $\mathrm{GL}_n(\mathbb{A}_E)$, by [2] (recall again that n and G are related) we get the description of the image of this lift and that the descent is an explicit inverse map of this lift to the set of near equivalence classes of irreducible, cuspidal, generic representations of $G_\mathbb{A}$. Moreover, since each factor τ_i of τ, as above, satisfies the assumptions of Theorem 2.2, it determines a corresponding group G_i (or $\tilde{G}_i$) and a cuspidal, generic representation on it, π_i, which lifts to τ_i. Thus we establish the endoscopic lift from cuspidal, generic representations on $G_{1,\mathbb{A}} \times \cdots \times G_{l,\mathbb{A}}$ to automorphic representations on $G_\mathbb{A}$. In particular, if τ is non-cuspidal and in the image of the weak lift above, say, lifted from π (cuspidal, generic), then π is in the image of an endoscopic lift which can be described precisely.

Example. Let $G = \mathrm{Sp}_{2k}$ or SO_{2k} (split). In the first case $n = 2k + 1$, and in the second case $n = 2k$. Let π be an irreducible, automorphic, cuspidal, generic representation of $G_\mathbb{A}$. Assume that the lift τ of π to $\mathrm{GL}_n(\mathbb{A})$ is non-cuspidal. Then $\tau = \tau_1 \times \cdots \times \tau_l$, as above. Assume, for simplicity, that all $\omega_{\tau_i} = 1$. Then all partial L-functions $L^S(\tau_i, \mathrm{sym}^2, s)$ have a pole at $s = 1$. Let $G_i = \mathrm{Sp}_{2k_i}$ if $n_i = 2k_i + 1$ is odd, and $G_i = \mathrm{SO}_{2k_i}$ (split) if $n_i = 2k_i$ is even; $n = n_1 + \cdots + n_l$. Let π_i be an irreducible summand of the descent $\sigma_\psi(\tau_i)$ to G_i. Then π is a weak (generalized endoscopic) lift of $\pi_1 \otimes \cdots \otimes \pi_l$ from $G_1 \times \cdots \times G_l$.

2.3. The descent has a local analog, which was developed in [11], [12] for the target group $\tilde{G} = \tilde{\mathrm{Sp}}_{2k}$. We proved

Theorem 2.4. *Let K be a local non-archimedean field of characteristic zero. Let $\tau_1, \ldots, \tau_l$ be pairwise inequivalent, irreducible, supercuspidal representations of $\mathrm{GL}_{2k_1}(K), \ldots, \mathrm{GL}_{2k_l}(K)$, respectively, such that each local L-function $L(\tau_i, \wedge^2, s)$ has a pole at $s = 0$. Let $\tau = \tau_1 \times \cdots \times \tau_l$ be the corresponding parabolic induction to $\mathrm{GL}_{2k}(K)$, where $k = k_1 + \cdots + k_l$. Then there is a unique* (*up to isomorphism*) *irreducible, supercuspidal, ψ-generic representation π of $\tilde{\mathrm{Sp}}_{2k}(K)$ such that the local gamma factor $\gamma(\pi \times \tau, s, \psi)$ has a pole of order l at $s = 1$. The representation π is obtained by a local analogue, applied to τ, of the descent construction.*

For the local analogue of the descent construction, we induce, as in the global set-up, $\tau = \tau_{1,s_1} \times \cdots \times \tau_{l,s_l}$ at the point $s_1 = \cdots = s_l = 1$ from the Siegel parabolic subgroup P to $H = \mathrm{Sp}_{4k}(K)$, and we consider the corresponding Langlands quotient, call it e_τ; this is the analogue to the multi-residue at $(1, \dots 1)$ of the Eisenstein series in the global case. Then we apply to e_τ a Jacquet module, analogous to the Fourier–Jacobi coefficient, that we apply in the global case to the multi-residue of the Eisenstein series.

2.4. The local descent from $\mathrm{GL}_{2k}(K)$ to $\mathrm{SO}_{2k+1}(K)$ yields powerful results, among which are the local converse theorem for generic representations of $\mathrm{SO}_{2k+1}(K)$, a full local Langlands reciprocity law for generic representations of $\mathrm{SO}_{2k+1}(K)$, a rigidity property (strong multiplicity one, up to isomorphism) of irreducible, cuspidal generic representations of $\mathrm{SO}_{2k+1}(\mathbb{A})$, and more…. These are results of my joint work with D. Jiang and can all be found in [23], [24]. I survey this work in this subsection.

Consider the representations τ and π as in Theorem 2.4. Using the local Howe duality from $\widetilde{\mathrm{Sp}}_{2k}(K)$ to $\mathrm{SO}_{2k+1}(K)$, we lift π to an irreducible, supercuspidal, generic representation σ of $\mathrm{SO}_{2k+1}(K)$. It is unique, up to isomorphism, with the property that the local gamma factor $\gamma(\sigma \times \tau, s, \psi)$ has a pole of order l at $s = 1$. Using the existence of the weak lift of [1] we prove

Theorem 2.5 (The local converse theorem). *Let σ and σ' be two irreducible generic representations of $\mathrm{SO}_{2k+1}(K)$ such that, for all $j < 2k$ and all irreducible generic representations ρ of $\mathrm{GL}_j(K)$,*

$$\gamma(\sigma \times \rho, s, \psi) = \gamma(\sigma' \times \rho, s, \psi).$$

Then σ and σ' are isomorphic.

The idea is to reduce the proof to supercuspidal representations σ and σ', lift them locally to $\mathrm{GL}_{2k}(K)$, and use Henniart's local converse theorem for $\mathrm{GL}_n(K)$ [18] to conclude that both representations lift to the same representation τ of $\mathrm{GL}_{2k}(K)$. We prove that τ has the form as in Theorem 2.4, and then we conclude that both gamma factors of σ, twisted by τ, and of σ', twisted by τ, have a pole of order l at $s = 1$. Using the uniqueness mentioned in the last paragraph, we conclude that σ and σ' are isomorphic. As a result from this we prove

Theorem 2.6. *There is a one-to-one correspondence t between the isomorphism classes of irreducible, supercuspidal, generic representations of $\mathrm{SO}_{2k+1}(K)$ and the isomorphism classes of irreducible, generic representations of $\mathrm{GL}_{2k}(K)$, as in Theorem 2.4, such that if $\tau = t(\sigma)$, then for all irreducible generic representations ρ of $\mathrm{GL}_j(K)$, $j > 0$,*

$$\begin{aligned} \gamma(\sigma \times \rho, s, \psi) &= \gamma(\tau \times \rho, s, \psi), \\ L(\sigma \times \rho, s) &= L(\tau \times \rho, s). \end{aligned} \tag{2.1}$$

See [2] for partial results for the other groups G. Let us outline some applications.

Theorem 2.7 (Rigidity theorem). *Let σ and σ' be two irreducible, automorphic, cuspidal, generic representations of $\mathrm{SO}_{2k+1}(\mathbb{A})$ which are isomorphic, at almost all* *unramified places. Then σ and σ' are isomorphic. In particular, the weak lift from cuspidal, generic representations of* $\mathrm{SO}_{2k+1}(\mathbb{A})$ *to* $\mathrm{GL}_{2k}(\mathbb{A})$ *is injective.*

The point is that, by the strong multiplicity one theorem of Jacquet and Shalika [21] for GL_n, we know that both σ and σ' lift, following [1], to the same automorphic representation τ on $\mathrm{GL}_{2k}(\mathbb{A})$. Now it follows from [1], p. 26, that all twisted local gamma factors of our two representations are the same at all finite places and hence, by Theorem 2.6, they are isomorphic at all finite places. Since the local lift is already prescribed at the archimedean places, σ and σ' are isomorphic at all places. From Theorem 2.6 we derive

Theorem 2.8. *Let τ be in the domain of the descent map σ_ψ from* GL_{2k} *to* SO_{2k+1}*. Then $\sigma_\psi(\tau)$ is irreducible.*

This follows from the fact that $\sigma_\psi(\tau)$ is multiplicity free and all its summands are cuspidal, generic and nearly equivalent, and hence by the rigidity theorem they are all isomorphic and we conclude that $\sigma_\psi(\tau)$ is irreducible.

Let us return to Theorem 2.5. Using the local Langlands reciprocity law for $\mathrm{GL}_n(K)$ proved by Harris–Taylor [17] and Henniart [19], and a theorem of Henniart on the compatibility of the exterior square local L and ε-factors for $\mathrm{GL}_n(K)$ with their corresponding local exterior square Artin factors [20], we obtain

Theorem 2.9. *There exists a unique bijection, preserving twisted ε-factors, between the conjugacy classes of* $2k$*-dimensional admissible, absolutely irreducible, multiplicity free symplectic representations of the Weil group W_K of K and the isomorphism classes of irreducible, supercuspidal, generic representations of* $\mathrm{SO}_{2k+1}(K)$.

In [24] we extend Theorem 2.8 to a full local Langlands reciprocity law, using Muić's description of all generic representations of $\mathrm{SO}_{2k+1}(K)$ [30].

Theorem 2.10. *For each local Langlands parameter φ of* $\mathrm{SO}_{2k+1}(K)$ (*i.e. a conjugacy class of admissible homomorphisms from* $W_K \times \mathrm{SL}_2(\mathbb{C})$ *to* $\mathrm{Sp}_{2k}(\mathbb{C})$)*, there is a unique, up to isomorphism, irreducible representation $\sigma(\varphi)$ of* $\mathrm{SO}_{2k+1}(K)$ *which is the Langlands subquotient of a parabolic induction of the form* $\Sigma(\varphi) = \delta(\Sigma_1) \times \cdots \times \delta(\Sigma_f) \rtimes \sigma^{(t)}$*, where $\sigma^{(t)}$ is tempered generic* (*on appropriate* $\mathrm{SO}_{2k'+1}(K)$) *and $\delta(\Sigma_i)$ is an essentially square integrable representation of* $\mathrm{GL}_{n_i}(K)$ *associated to an imbalanced segment Σ_i* (*in the sense of* [37]). *The map $\varphi \mapsto \sigma(\varphi)$ preserves local twisted L and ε-factors. Moreover, $\sigma(\varphi)$ is generic if and only if $\Sigma(\varphi)$ is irreducible.*

As a corollary, we get that, for each tempered local Langlands parameter φ for $\mathrm{SO}_{2k+1}(K)$, the representation $\sigma(\varphi)$ is generic; it will eventually be "the generic member of the tempered local L-packet $\prod(\varphi)$". This is the case of SO_{2k+1} of a conjecture of Shahidi [33]. We also get, as another corollary, a conjecture of Gross–Prasad [16] and of Rallis [27] for $\mathrm{SO}_{2k+1}(K)$.

Theorem 2.11. *With notation as above, $\sigma(\varphi)$ is generic if and only if the local adjoint L-function $L(\mathrm{Ad}_{\mathrm{Sp}_{2k}} \circ \varphi, s)$ is regular at $s = 1$.*

Finally, we get the following applications to automorphic representations.

Theorem 2.12. 1. *Let* τ *be an irreducible, automorphic, cuspidal representation of* $\mathrm{GL}_{2k}(\mathbb{A})$ *such that* $L(\tau, \wedge^2, s)$ *has a pole at* $s = 1$*. Then the local components* τ_v *are symplectic at all places* v*.*

2. *The weak lift from irreducible, automorphic, cuspidal, generic representations of* $\mathrm{SO}_{2k+1}(\mathbb{A})$ *to automorphic representations of* $\mathrm{GL}_{2k}(\mathbb{A})$ *is compatible with the local Langlands functorial lift at all places.*

3. L-functions for orthogonal groups; non-generic representations

In this section I report on a joint work in progress with S. Rallis towards establishing the existence of a weak functorial lift from irreducible, automorphic, cuspidal representations on a split special orthogonal group, $G_m = \mathrm{SO}_m$ (regarded over F) in m variables, to $\mathrm{GL}_{2[\frac{m}{2}]}$.

Let π be an irreducible, automorphic, cuspidal representation of $G_m(\mathbb{A})$. Consider the Fourier coefficients of Gelfand–Graev type of the form φ_π^{ψ,G'_i} as in Section 1.a, where $G'_i = G_{m-2i-1}$ and φ_π varies in the space of π. These generate, upon restriction to $G'_i(\mathbb{A})$, a space of automorphic functions on $G'_i(\mathbb{A})$, which is invariant to right translations by $G'_i(\mathbb{A})$; denote this space by π^{ψ,G'_i}. Note that the last space is non-trivial for $i = 0$, and if it is non-trivial for $i = \left[\frac{m-1}{2}\right]$, then π is generic. Otherwise let i_1 be the largest index such that $\pi^{\psi,G'_{i_1}}$ is non-trivial. We prove that $\pi^{\psi,G'_{i_1}}$ is cuspidal on $G'_{i_1}(\mathbb{A})$. Choose an irreducible, cuspidal summand of $\pi^{\psi,G'_{i_1}}$ and let ρ_0 be its conjugate representation. Then the $G'_{i_1}(\mathbb{A})$-invariant bilinear pairing $b_{\pi,\rho_0}(\varphi_\pi, \varphi_{\rho_0})$ in (1.3) is non-trivial. We may consider the integrals of the form (1.2),

$$\int_{G_{m-1}(F)\backslash G_{m-1}(\mathbb{A})} \varphi_\pi(g) E(g, f_{s_1,\dots,s_{i_1};\rho_0})\, dg, \tag{3.1}$$

where $E(g, f_{s_1,\dots,s_{i_1};\rho_0})$ is an Eisenstein series on $G_{m-1}(\mathbb{A})$ corresponding to the parabolic induction from $|\cdot|^{s_1} \otimes \cdots \otimes |\cdot|^{s_{i_1}} \otimes \rho_0$ on the parabolic subgroup P_1 whose Levi part is isomorphic to $\mathrm{GL}_1^{i_1} \times G_{m-2i_1-1}$. (The difference from (1.2) is that the cuspidal representation τ on GL_{i_1} is replaced by the Eisenstein series induced from the Borel subgroup and its character $|\cdot|^{s_1} \otimes \cdots \otimes |\cdot|^{s_{i_1}}$.) The methods of [9] apply here, as well, and (3.1) is non-trivial and meromorphic. Let us choose and fix $s_1, \dots, s_{i_1}$ purely imaginary such that (3.1) is non-trivial and $\mathrm{Ind}_{P_1(\mathbb{A})}^{G_{m-1}(\mathbb{A})} |\cdot|^{s_1} \otimes \cdots \otimes |\cdot|^{s_{i_1}} \otimes \rho_0$ is in general position. Denote by ρ the automorphic representation of $G_{m-1}(\mathbb{A})$ generated by the Eisenstein series corresponding to this induced representation. Denote

$$b_{\pi,\rho}(\varphi_\pi, \xi_\rho) = \int_{G_{m-1}(F)\backslash G_{m-1}(\mathbb{A})} \varphi_\pi(g)\xi_\rho(g)\, dg, \tag{3.2}$$

where ξ_ρ is in the space of ρ. Let τ be an irreducible, automorphic, cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$. We consider the integrals (1.1) $\mathcal{L}(\varphi_\pi, f_{\tau_s,\rho})$, even when $i_1 > 0$, in which case ρ is not cuspidal. The methods of [9] still apply, and so we have, for $\mathrm{Re}(s) \gg 0$, the following "Eulerian" expression

$$\mathcal{L}(\varphi_\pi, f_{\tau_s,\rho}) = \int_{G_{m-1}(\mathbb{A})\backslash G_m(\mathbb{A})} \int_{U'_{\mathbb{A}}} b_{\pi,\rho}(\pi(g)\varphi_\pi, f_{\tau_s,\rho}(ug))\psi_{U'}(u)\, du dg, \quad (3.3)$$

where U' is a certain F-unipotent subgroup and $\psi_{U'}$ is a certain character of $U'(\mathbb{A})$, trivial on $U'(F)$. We can find data such that

$$\mathcal{L}(\varphi_\pi, f_{\tau_s,\rho}) = \mathcal{L}_{S_\infty}(\varphi_\pi, f_{\tau_s,\rho}) \frac{L^Z(\pi \times \tau, s)}{L^Z\big(\rho \times \tau, s + \frac{1}{2}\big)L^Z(\tau, r, 2s)}, \quad (3.4)$$

where Z is any finite set of places, including those at infinity, outside which π, ρ_0, τ are all unramified; $r = \wedge^2$ (resp. sym^2) if m is odd (resp. m is even). We denote by S_∞ the set of archimedean primes. $\mathcal{L}_{S_\infty}(\varphi_\pi, f_{\tau_s,\rho})$ is the integral, as in the right-hand side of (3.3), where we replace $\mathbb{A}$ by $\mathbb{A}_{S_\infty}$; it can be chosen to be holomorphic and non-zero at any given point s_0. Fix a finite set of finite places S such that π, ρ_0 are unramified outside $S_\infty \cup S$.

Let τ belong to the twisting set $\mathcal{T}(S, \chi)$ of the converse theorem [3], i.e. χ is a character of $F^*\backslash\mathbb{A}^*$ such that at the places ν of S, χ_ν is highly ramified, and τ_ν is χ_ν times an unramified representation.

Let us define now the (standard) local factors $L(\pi_\nu \times \tau_\nu, s)$ and $\gamma(\pi_\nu \times \tau_\nu, s, \psi_\nu)$ at all places. Let S_τ be a finite set of finite places disjoint from S such that τ is unramified outside $S' = S_\infty \cup S \cup S_\tau$. The definition of the local factors is clear outside S' via the unramified parameters. At the places of S_τ the definition is by multiplicativity of the local factors in the first variable π which is unramified, where for a character μ the local factors for $\mu \times \tau$ are the ones for GL_n. At the places of S we define $L(\pi_\nu \times \tau_\nu, s) = 1$, and we define the local gamma factor by multiplicativity in the second variable τ which is induced from the Borel subgroup, and for a character μ we define $\gamma(\pi_\nu \times \mu, s, \psi_\nu)$ by the doubling method [31], [28]. Finally, at S_∞ we define the local factors through the Langlands classification. We have similar definitions of local factors for $\rho \times \tau$. Denote by $\mathcal{T}_0(S, \chi)$ the subset of elements τ of $\mathcal{T}(S, \chi)$ which are unitary, with central character, which is trivial on the positive real numbers and diagonally embedded inside $\mathbb{A}^*_{S_\infty}$. We prove:

Theorem 3.1. *For $\tau \in \mathcal{T}_0(S, \chi)$, $L^{S_\infty}(\pi \times \tau, s)$ is holomorphic for* $\mathrm{Re}(s) \geq \frac{1}{2}$.

Indeed, taking the (highly ramified at S) character χ as in [1], p. 12, we get that τ is not self-dual, and we conclude as in loc. cit. that $\mathcal{L}(\varphi_\pi, f_{\tau_s,\rho})$ is holomorphic for $\mathrm{Re}(s) \geq \frac{1}{2}$. As in (3.4) we obtain that

$$\mathcal{L}_{S'}(\varphi_\pi, f_{\tau_s,\rho}) \frac{L^{S'}(\pi \times \tau, s)}{L^{S'}\big(\rho \times \tau, s + \frac{1}{2}\big)L^{S'}(\tau, r, 2s)}$$

is holomorphic for $\mathrm{Re}(s) \geq \frac{1}{2}$. Here $\mathcal{L}_{S'}$ is as in the right-hand side of (3.3), with $\mathbb{A}$ replaced by $\mathbb{A}_{S'}$. Next we can find data such that

$$\mathcal{L}_{S'}(\varphi_\pi, f_{\tau_s,\rho}) = \mathcal{L}_{S_\infty \cup S}(\varphi_\pi, f_{\tau_s,\rho}) \prod_{\nu \in S_\tau} \frac{L(\pi_\nu \times \tau_\nu, s)}{L\big(\rho_\nu \times \tau_\nu, s + \frac{1}{2}\big) L(\tau_\nu, r, 2s)}. \tag{3.5}$$

Finally we can find data such that $\mathcal{L}_{S_\infty \cup S}(\varphi_\pi, f_{\tau_s,\rho}) = \mathcal{L}_{S_\infty}(\varphi_\pi, f_{\tau_s,\rho})$. Thus, $\frac{L^{S_\infty}(\pi \times \tau, s)}{L^{S_\infty}(\rho \times \tau, s+\frac{1}{2}) L^{S_\infty}(\tau, r, 2s)}$ is holomorphic, for $\mathrm{Re}(s) \geq \frac{1}{2}$. Since $L^{S_\infty}(\tau, r, 2s)$ is holomorphic for $\mathrm{Re}(2s) \geq 1$, we conclude that $\frac{L^{S_\infty}(\pi \times \tau, s)}{L^{S_\infty}(\rho \times \tau, s+\frac{1}{2})}$ is holomorphic for $\mathrm{Re}(s) \geq \frac{1}{2}$. By induction on m, $L^{S_\infty}(\rho \times \tau, s + \frac{1}{2})$ is holomorphic for $\mathrm{Re}(s) \geq \frac{1}{2}$, and hence so is $L^{S_\infty}(\pi \times \tau, s)$. Note that S can be enlarged so that the process above can be repeated for a finite sequence $\pi, \rho_0, \rho_1, \ldots, \rho_l$ of irreducible cuspidal representations of $G_m(\mathbb{A})$, $G_{m-2i_1-1}(\mathbb{A})$, $G_{m-2(i_1+i_2)-2}(\mathbb{A}), \ldots$ such that they are all unramified outside $S_\infty \cup S$ and ρ_i appears in the space of Gelfand–Graev coefficients of ρ_{i-1}. The basic case of induction is when π is generic, and then the theorem is known by [1].

Let M be the intertwining operator corresponding to a Weyl element of $H = G_{2n+m-1}$, which flips X_n^+ and X_n^-. Then we have a functional equation

$$\mathcal{L}(\varphi_\pi, f_{\tau_s,\rho}) = \mathcal{L}(\varphi_\pi, M(f_{\tau_s,\rho})),$$

which unfolds to

$$\begin{aligned}\mathcal{L}_{S_\infty \cup S}(\varphi_\pi, f_{\tau_s,\rho}) &\prod_{\nu \in S_\tau} \mathcal{L}_\nu(\varphi_\pi, f_{\tau_s,\rho}) L^{S'}(\pi \times \tau, s) \\ &= \tilde{\mathcal{L}}_{S_\infty \cup S}(\varphi_\pi, M_{S_\infty \cup S}(f_{\tau_s,\rho})) \prod_{\nu \in S_\tau} \tilde{\mathcal{L}}_\nu(\varphi_\pi, M_\nu(f_{\tau_s,\rho})) \\ &\quad \cdot \frac{L^{S'}\big(\rho \times \tau, s - \frac{1}{2}\big) L^{S'}(\tau, r, 2s-1)}{L^{S'}\big(\rho \times \hat{\tau}, \frac{3}{2} - s\big) L^{S'}(\hat{\tau}, r, 2-2s)} L^{S'}(\pi \times \hat{\tau}, 1-s),\end{aligned} \tag{3.6}$$

where at the places $\nu \in S_\tau$, $\mathcal{L}_\nu$ are local analogs of the right-hand side of (3.3) and $\tilde{\mathcal{L}}_\nu$ are obtained from these after a slight modification. Here we use the uniqueness theorem of [25]; for unramified representations π_ν, ρ_ν the space $\mathrm{Hom}_{G_{m-1}(F_\nu)}(\pi_\nu, \hat{\rho}_\nu)$ is one-dimensional. Recall that π_ν and ρ_ν are unramified for $\nu \in S_\tau$. Using this result again, we also prove a local functional equation and compute the proportionality factor at the places of S_τ.

Theorem 3.2. *For $\nu \in S_\tau$ we have*

$$\frac{\gamma(\pi_\nu \times \tau_\nu, s, \psi_\nu)}{\gamma\big(\rho_\nu \times \tau_\nu, s - \frac{1}{2}, \psi_\nu\big) \gamma(\tau_\nu, r, 2s-1, \psi_\nu)} \mathcal{L}_\nu(\varphi_\pi, f_{\tau_s,\rho}) = \tilde{\mathcal{L}}_\nu(\varphi_\pi, M_\nu(f_{\tau_s,\rho})). \tag{3.7}$$

Note that all the local gamma factors in the left-hand side of (3.7) are well defined. We can also prove (3.7) for the places of S, but we prove it as an identity for each local pairing $b_\nu \in \mathrm{Hom}_{G_{m-1}(F_\nu)}(\pi_\nu \otimes \rho_\nu, 1)$, and local integrals $\mathcal{L}_\nu$, and $\tilde{\mathcal{L}}_\nu$, defined using this pairing b_ν. The point is that at S the representation τ is induced from the Borel subgroup, and we have a way to factorize the local integrals, "one GL_1-twist at a time", and then relate the local integrals above for $G_m \times \mathrm{GL}_1$ to the local integrals arising in the doubling method. Thus we get

$$\begin{aligned}\prod_{\nu\in S}\frac{\gamma(\pi_\nu\times\tau_\nu,s,\psi_\nu)}{\gamma\big(\rho_\nu\times\tau_\nu,s-\frac{1}{2},\psi_\nu\big)\gamma(\tau_\nu,r,2s-1,\psi_\nu)}&\mathcal{L}_{S_\infty\cup S}(\varphi_\pi,f_{\tau_s,\rho})\\ &= \mathcal{L}^{\sim}_{S_\infty\cup S}(\varphi_\pi,M_S(f_{\tau_s,\rho})).\end{aligned} \tag{3.8}$$

Here $\mathcal{L}^{\sim}$ refers to the modification $\sim$ taking place only at S. Note again that all local gamma factors which appear in the left-hand side of (3.8) are well defined. The formal steps of the proof of (3.8) carry on for S_∞ as well. They yield, in the left-hand side of (3.8), the product over all of $S_\infty \cup S$, where, in the case of S_∞, the local factors are the corresponding Artin local gamma factors, and in the right-hand side of (3.8) we have to replace M_S by $M_{S_\infty\cup S}$ and $\mathcal{L}^{\sim}_{S_\infty\cup S}$ by $\tilde{\mathcal{L}}_{S_\infty\cup S}$. However, there are fine details which need to be taken care of in order to justify the formal steps of the proof; this has to do with analytic continuation (in general) of the local integrals. Let us assume that (3.8) is valid, with $S_\infty \cup S$ replacing S, as we just explained, so that we can continue and point out what we have at present, and what is still missing towards a proof of existence of a weak functorial lift from SO_m (split) to $\mathrm{GL}_{2[\frac{m}{2}]}$. With this assumption, (3.6)–(3.8) imply that

$$\begin{aligned}L(\pi\times\tau,s)=\varepsilon(\pi\times\tau,s)L(\pi\times\hat{\tau},1-s)&\frac{L(\tau,r,2s-1)}{\varepsilon(\tau,r,2s-1)L(\hat{\tau},r,2-2s)}\\ \cdot\frac{L\big(\rho\times\tau,s-\frac{1}{2}\big)}{\varepsilon\big(\rho\times\tau,s-\frac{1}{2}\big)L\big(\rho\times\hat{\tau},\frac{3}{2}-s\big)}&.\end{aligned} \tag{3.9}$$

By the functional equation for $L(\tau, r, s)$, the first quotient in the right-hand side of (3.9) is 1. By induction on m, the second quotient is also 1, the basic case being that where ρ is generic (or just take the trivial cases $m = 0, 1$). This will prove

Theorem 3.3. *Let $\tau \in \mathcal{T}(S, \chi)$ (notation as above). Assume that (3.8) is valid, with $S_\infty \cup S$ replacing S (as explained above). Then*

$$L(\pi\times\tau,s)=\varepsilon(\pi\times\tau,s)L(\pi\times\hat{\tau},1-s). \tag{3.10}$$

As in [1], define an irreducible representation $\Pi = \otimes\Pi_\nu$ of $\mathrm{GL}_{2[\frac{m}{2}]}(\mathbb{A})$ as follows. For $\nu \notin S_\infty \cup S$, Π_ν is the unramified representation (of $\mathrm{GL}_{2[\frac{m}{2}]}(F_\nu)$) obtained from π_ν by the local unramified functorial lift. Similarly, for $\nu \in S_\infty$, Π_ν is obtained from π_ν via the Langlands classification. For $\nu \in S$ choose any irreducible, generic, self-dual representation Π_ν of $\mathrm{GL}_{2[\frac{m}{2}]}(F_\nu)$ which has a trivial central character. In particular, Π has a trivial central character.

Theorem 3.4. *Assume that χ is highly ramified at S (depending on π only). Then, for all places v,*

$$L(\Pi_v \times \tau_v, s) = L(\pi_v \times \tau_v, s), \tag{3.11}$$

$$\gamma(\Pi_v \times \tau_v, s, \psi_v) = \gamma(\pi_v \times \tau_v, s, \psi_v), \tag{3.12}$$

Similar identities apply to $\hat{\tau}$.

By construction, (3.11) and (3.12) are clear for all v outside S. For $v \in S$, since χ_v is highly ramified, both sides of (3.11) are 1. As for (3.12), let us write τ_v as the representation, induced from the Borel subgroup and a character $\mu_{1,v}\chi_v \otimes \cdots \otimes \mu_{n,v}\chi_v$, where $\mu_{1,v}, \dots, \mu_{n,v}$ are unramified. Then we need to prove

$$\prod_{i=1}^{n} \gamma(\Pi_v \times \mu_{i,v}\chi_v, s, \psi_v) = \prod_{i=1}^{n} \gamma(\pi_v \times \mu_{i,v}\chi_v, s, \psi_v).$$

For this it is enough to prove

$$\gamma(\Pi_v \times \chi_v, s, \psi_v) = \gamma(\pi_v \times \chi_v, s, \psi_v). \tag{3.13}$$

Recall again that the right-hand side of (3.13) is defined via the doubling method. Both sides of (3.13) are stable for sufficiently ramified χ_v. See [22] for the stability of the left-hand side, and [32] for the stability of the right-hand side. Thus we may replace π_v and Π_v with a pair of unramified representations, which correspond without changing their local gamma factors (twisted by χ_v). This proves (3.13).

The main property which is missing at this stage is the holomorphicity of the full L-function $L(\pi \times \tau, s)$ for $\mathrm{Re}(s) \geq \frac{1}{2}$. Recall from Theorem 3.1 that we know that $L^{S_\infty}(\pi \times \tau, s)$ is holomorphic for $\mathrm{Re}(s) \geq \frac{1}{2}$, when $\tau \in \mathcal{T}_0(S, \chi)$. Assume, for example, that π is tempered at S_∞. Then $\prod_{v \in S_\infty} L(\pi_v \times \tau_v, s)$ is holomorphic for $\mathrm{Re}(s) \geq \frac{1}{2}$, and hence so is $L(\pi \times \tau, s)$. In such a case Theorem 3.3 (where we assumed that (3.8) is valid for $S_\infty \cup S$ as well) will imply that $L(\pi \times \tau, s)$ is entire. Once we have this, we see that by Theorem 2 in [5], $L(\pi \times \tau, s)$ is an entire function of finite order, and we can conclude that it is bounded in vertical strips (and similarly for $\hat{\tau}$). Then we can apply the converse theorem and obtain an automorphic representation of $\mathrm{GL}_{2[\frac{m}{2}]}(\mathbb{A})$ which is isomorphic to Π at all places outside S. Finally, let us mention that the ideas of the descent method can be applied here as well, and moreover, cuspidal representations on any orthogonal group (split or otherwise) can be considered along similar lines. These topics are the subject of current work in progress, which we hope to report on in future times.

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