

Representation theory and the cohomology of arithmetic groups

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Abstract. Let G be a semisimple Lie group with finitely many connected components and Lie algebra \mathfrak{g} , K a maximal compact subgroup of G , and $X = G/K$ a symmetric space. A torsion free discrete subgroup Γ of G and a finite dimensional real or complex linear representation (ρ, E) of G define a locally symmetric space $X_\Gamma = \Gamma \backslash G/K$ with a local system \tilde{E} . Then $H^*(\Gamma, E) = H^*(\Gamma \backslash X, \tilde{E})$ is isomorphic to $H^*(\mathfrak{g}, K, C^\infty(\Gamma \backslash G) \otimes E)$. If Γ is an arithmetic group, then $H^*(\mathfrak{g}, K, C^\infty(\Gamma \backslash G) \otimes E)$ is isomorphic to the (\mathfrak{g}, K) -cohomology with coefficients in $\mathcal{A}(\Gamma \backslash G) \otimes E$ where $\mathcal{A}(\Gamma \backslash G)$ is the space of automorphic forms. Using representation theory and the theory of automorphic forms a large amount of information about $H^*(\Gamma, E)$ can be deduced.

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1. Introduction

Let G be a semi-simple simply connected algebraic group over \mathbb{Q} , K be a maximal compact subgroup of its real points $G = G(\mathbb{R})$ and $X = G/K$ the global symmetric space. Assume that $\Gamma \subset G(\mathbb{Q})$ is a torsion-free congruence subgroup. Then $S(\Gamma) = \Gamma \backslash X$ is a manifold with finite volume under the G -invariant metric and volume form on $X = G/K$.

Suppose that H is a \mathbb{Q} -rational reductive subgroup with real points H such that $K \cap H = K_H$ is maximal compact in H . Then the inclusion

$$X_H := H/K_H \rightarrow X$$

induces a map

$$j: \Gamma \cap H \backslash X_H \longrightarrow \Gamma \backslash X.$$

We assume from now on that $\Gamma \backslash X$ and $S_H(\Gamma) := \Gamma \cap H \backslash X_H$ are not compact.

Let $[\phi]$ be an element in the i -th cohomology with compact support $H_c^i(S(\Gamma), \mathbb{C})$ represented by a closed compactly supported i -form ϕ and that $i + k =$

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$\dim(X_H)$. Then $[\phi]$ determines a map

$$(\phi, S_H): H^k(S(\Gamma), \mathbb{C}) \rightarrow \mathbb{C},$$

$$[\omega] \mapsto \int_{\Gamma \cap H \backslash X_H} \phi \wedge j^* \omega.$$

We call this map the modular symbol attached to $([\phi], S_H)$. If we can find $[\omega] \in H^*(S(\Gamma), \mathbb{C})$ such that $([\phi], H)([\omega]) \neq 0$ then $([\phi], S_H)$ is a nontrivial modular symbol. If $S(\Gamma)$ is oriented we use Poincaré duality and identify $([\phi], S_H)$ with an element in $H_c^*(S(\Gamma), \mathbb{C})$.

On the other hand if $[\psi] \in H^i(\Gamma \cap H \backslash X_H, \mathbb{C})$ and $i + k = \dim(X_H)$ the map

$$([\psi], S_H): H_c^k(\Gamma \backslash X, \mathbb{C}) \rightarrow \mathbb{C},$$

$$[\omega] \mapsto \int_{\Gamma \cap H \backslash X_H} \psi \wedge j^* \omega$$

can be identified with an element in $H^*(\Gamma \backslash X, \mathbb{C})$ which we also call a modular symbol attached to $([\psi], S_H)$. If the degree of ψ is 0 then we say that the fundamental class $[S_H(\Gamma)]$ is a modular symbol in $H^{\dim X - \dim X_H}(S(\Gamma), \mathbb{C})$.

The Hecke algebra acts on $H^*(S(\Gamma), \mathbb{C})$ and on $H_c^*(S(\Gamma), \mathbb{C})$ and one may consider the span of Hecke translates of the modular symbols $([\psi], S_H)$ and in particular $[S_H(\Gamma)]$, where H ranges over all groups H . It is unknown, how much of the cohomology of $S(\Gamma)$ may be captured by nonzero modular symbols and their Hecke translates.

The cohomology $H^*(\Gamma \backslash X, \mathbb{C})$ of the locally symmetric space $\Gamma \backslash X$ can be identified with the (\mathfrak{g}, K) -cohomology of the G -module $\mathcal{A}(\Gamma \backslash G)$ of automorphic functions on $\Gamma \backslash G$. Thus it is determined by representation theoretic and arithmetic data. Understanding the relationship between nontrivial modular symbols attached to a subgroup H and the subrepresentations of $\mathcal{A}(\Gamma \backslash G)$ gives important arithmetic information, since many of the integrals $\int_{\Gamma \cap H \backslash X_H} \psi \wedge j^* \omega$ are period integrals or are related to special values of L -functions. This is discussed in [7], [11], [12], [4] and the recent work of S. Kudla [8].

For some compact locally symmetric spaces Kudla and Millson [9], by Clozel and Venkataranama [4] and by Tong and Wang [19], have obtained results relating some special $[S_H(\Gamma)]$ to cohomology classes defined by automorphic representations. I will discuss here some connections between cohomology classes defined using automorphic representations and nontrivial modular symbols for noncompact locally symmetric spaces.

2. The cohomology $H^*(S(\Gamma), \mathbb{C})$ and automorphic representations

We denote the Lie algebra by gothic letters and write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the Cartan decomposition of the Lie algebra \mathfrak{g} . The complex of the differential forms on $S(\Gamma)$ is

isomorphic to

$$\mathrm{Hom}_K(\wedge^* \mathfrak{g}/\mathfrak{k}, C^\infty(\Gamma \backslash G))$$

(see [3]). Here consider $C^\infty(\Gamma \backslash G)$ as a left \mathfrak{g} -module and

$$\begin{aligned} df(x_0, \dots, x_q) &= \sum_i x_i f(x_0, \dots, \hat{x}_i, \dots, x_q) \\ &\quad + \sum_{i < j} (-1)^{i+j} f([x_i, x_j], x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_q). \end{aligned}$$

We denote its cohomology by $H^*(S(\Gamma), \mathbb{C}) = H^*(\mathfrak{g}, K, C^\infty(\Gamma \backslash G))$. Similarly we define for an admissible representation π of G the (\mathfrak{g}, K) -cohomology $H^*(\mathfrak{g}, K, \pi)$.

We denote the space of automorphic functions on $\Gamma \backslash G$ by $\mathcal{A}(\Gamma \backslash G)$. An admissible (\mathfrak{g}, K) -module, which is isomorphic to a subrepresentation of $\mathcal{A}(\Gamma \backslash G)$, is called an automorphic representation. The space of square integrable automorphic functions is a direct sum of $\mathcal{A}_{\mathrm{res}}(\Gamma \backslash G)$, the space of residual automorphic functions and $\mathcal{A}_{\mathrm{cusp}}(\Gamma \backslash G)$, the space of the cuspidal automorphic functions [10]. An irreducible unitary (\mathfrak{g}, K) -submodule of $\mathcal{A}_{\mathrm{res}}(\Gamma \backslash G)$, respectively of $\mathcal{A}_{\mathrm{cusp}}(\Gamma \backslash G)$ is called a residual, respectively cuspidal (\mathfrak{g}, K) -module.

As a (\mathfrak{g}, K) -module

$$\mathcal{A}_{\mathrm{cusp}}(\Gamma \backslash G) = \bigoplus_{U \in \hat{G}_u} m(\Gamma, U)U.$$

Here \hat{G}_u denotes the set of all unitary irreducible (\mathfrak{g}, K) -modules and

$$m(\Gamma, U) = \dim \mathrm{Hom}_{\mathfrak{g}, K}(U, \mathcal{A}_{\mathrm{cusp}}(\Gamma \backslash G)).$$

A result by A. Borel [2] states that if π is a cuspidal (\mathfrak{g}, K) -module then

$$H^*(\mathfrak{g}, K, \pi) \hookrightarrow H^*(\mathfrak{g}, K, C^\infty(\Gamma \backslash G)) = H^*(S(\Gamma), \mathbb{C}). \quad (1)$$

In particular, if $S(\Gamma)$ is compact, then

$$\mathcal{A}_{\mathrm{cusp}}(\Gamma \backslash G) = \mathcal{A}(\Gamma \backslash G)$$

and hence

$$H^*(S(\Gamma), \mathbb{C}) = H_c^*(S(\Gamma), \mathbb{C}) = \bigoplus_{U \in \hat{G}_u} m(\Gamma, U)H^*(\mathfrak{g}, K, U).$$

See [3] for details.

If $\Gamma \backslash G$ and hence $S(\Gamma)$ are not compact a result of Franke [5] shows that

$$H^*(S(\Gamma), \mathbb{C}) = H^*(\mathfrak{g}, K, \mathcal{A}(\Gamma \backslash G)).$$

In contrast to (1)

$$H^*(\mathfrak{g}, K, \mathcal{A}_{\mathrm{res}}(\Gamma \backslash G)) \not\hookrightarrow H^*(\mathfrak{g}, K, C^\infty(\Gamma \backslash G))$$

as the following example illustrates.

The trivial representation I of G is an automorphic representation satisfying $H^*(\mathfrak{g}, K, I) \neq 0$. Let G_u be the compact subgroup of $G(\mathbb{C})$ with Lie algebra $\mathfrak{k} \oplus i\mathfrak{p}$ and write $\hat{X} = G_u/K$ for the compact dual of X . We identify $H^*(\hat{X})$ with $H^*(\mathfrak{g}, K, I)$. The Borel map j from the cohomology $H^*(\hat{X})$ into the cohomology $H^*(S(\Gamma), \mathbb{C})$ is defined by the inclusion of the constant functions \mathbb{C} in the space $\mathcal{A}(\Gamma \backslash G)$. The classes in the image of $H^*(\mathfrak{g}, K, I)$ in $H^*(S(\Gamma), \mathbb{C})$ are represented by invariant differential forms. If D is the dimension of $S(\Gamma)$ then $\dim H^D(\mathfrak{g}, K, I) = 1$. On the other hand $H^D(S(\Gamma), \mathbb{C}) = 0$ since $S(\Gamma)$ is not compact.

The kernel of the Borel map $j : H^*(\mathfrak{g}, K, I) \rightarrow H^*(S(\Gamma), \mathbb{C})$ is determined in [6].

According to Parthasaraty, Kumaresan and Vogan–Zuckerman the irreducible unitary representations π with $H^*(\mathfrak{g}, K, \pi) \neq 0$ are classified as follows: let $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{a}$ be a θ -stable fundamental Cartan subalgebra and $\mathfrak{q}_{\mathbb{C}} = \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}$ a θ -stable parabolic subgroup of $\mathfrak{g} \otimes \mathbb{C}$ containing $\mathfrak{h} \otimes \mathbb{C}$. Then $\mathfrak{l}_{\mathbb{C}}, \mathfrak{l}$ and L are the centralizers of an element $X \in i\mathfrak{k}$ in $\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}$ and G respectively. Associated to \mathfrak{q} is an irreducible unitary (\mathfrak{g}, K) -modules $A_{\mathfrak{q}}$ and an irreducible finite dimensional representation $V(\mathfrak{q}) \subset \wedge^{\dim(\mathfrak{p} \cap \mathfrak{n})} \mathfrak{p}$ of K so that

$$H^{\dim(\mathfrak{p} \cap \mathfrak{n})}(\mathfrak{g}, K A_{\mathfrak{q}}) \cong \text{Hom}(V(\mathfrak{q}), A_{\mathfrak{q}}) \cong \mathbb{C},$$

and we obtain forms representing all other nontrivial cohomology classes as the wedge product of $0 \neq \omega_{\mathfrak{q}} \in \text{Hom}(V(\mathfrak{q}), A_{\mathfrak{q}})$ with forms representing classes in $H^*(\mathfrak{l}, L \cap K, I)$. Every irreducible unitary representation with nontrivial (\mathfrak{g}, K) -cohomology is isomorphic to a representation $A_{\mathfrak{q}}$.

Suppose that \mathbf{P} is a rational parabolic subgroup and that $A_{\mathfrak{q}}$ is the Langlands quotient of $I(\mathbf{P}(\mathbb{R}), \pi, \nu)$. Assume furthermore that there is a map

$$E_{\text{res}} : I(\mathbf{P}(\mathbb{R}), \pi, \nu) \rightarrow \mathcal{A}_{\text{res}}(\Gamma \backslash G)$$

which defines a (\mathfrak{g}, K) -map $J : A_{\mathfrak{q}} \hookrightarrow \mathcal{A}_{\text{res}}(\Gamma \backslash G)$ into the residual spectrum. It extends to a map

$$J^* : \text{Hom}_K(\wedge^* \mathfrak{p}, A_{\mathfrak{q}}) \rightarrow \text{Hom}_K(\wedge^* \mathfrak{p}, \mathcal{A}_{\text{res}}(\Gamma \backslash G)).$$

One can show that $[J^*(\omega_{\mathfrak{q}})] \neq 0$.

I expect the following generalization of Franke’s result to hold.

Conjecture. Let $[\omega_L] \in H^*(\mathfrak{l}, K \cap L, I)$ be in the kernel of the Borel map for L . Then

$$[J^*(\omega_{\mathfrak{q}} \wedge \omega_L)] = 0.$$

The contribution to $H^*(S(\Gamma), \mathbb{C})$ by representations with nontrivial (\mathfrak{g}, K) -cohomology which are induced from parabolic subgroups and which are embedded via Eisenstein series into $\mathcal{A}(\Gamma \backslash G)$ has been considered by G. Harder and J. Schwermer [15].

3. Locally symmetric spaces in the adelic language

Let A be the adèles of \mathbb{Q} and A_f the finite adèles. For a linear algebraic semi-simple simply connected group G defined over \mathbb{Q} we define $G(A)$, $G(\mathbb{Q})$, $G(A_f)$ and a compact group

$$K_A = K_\infty \prod_p G(\mathcal{O}_p) = K_\infty K_{A_f},$$

where K_∞ is the maximal compact subgroup of $G = G(\mathbb{R})$. We have a locally symmetric space

$$S(K_{A_f}) = G(\mathbb{Q}) \backslash X \times G(A_f) / K_{A_f}.$$

For each subgroup $K'_{A_f} \subset K_{A_f}$ of finite index we also define a locally symmetric space $S(K'_A) = G(\mathbb{Q}) \backslash X \times G(A_f) / K'_{A_f}$. If K'_f is sufficiently small then

$$S(K'_f) = \bigcup_{i=1}^h \Gamma_i \backslash X$$

is a finite disjoint union of locally symmetric spaces $\Gamma_i \backslash X$, where the $\Gamma_i \subset G(\mathbb{Q})$ are congruence subgroups. Furthermore

$$H^*(S(K'_f), \mathbb{C}) \cong H^*(\mathfrak{g}, K_\infty, \mathcal{A}(G(\mathbb{Q}) \backslash G(A))^{K'_f}).$$

We give $G(A_f)$ the topology induced by the topology of A_f and define $X_A := X \times G(A_f)$. Then X_A with the product topology is the *adelic symmetric space* attached to G . The group $G(\mathbb{Q})$ of \mathbb{Q} -rational points of G acts by left translation freely and discontinuously on X_A . The space $G(\mathbb{Q}) \backslash X_A =: S_G$ with the quotient topology is called the *adelic locally symmetric space* attached to G .

Let $\tilde{\mathbb{C}}$ be a locally constant sheaf S_G . By $H^*(S_G, \tilde{\mathbb{C}})$ we denote the smooth sheaf cohomology of S_G with coefficients $\tilde{\mathbb{C}}$. The group $G(A_f)$ acts by right translation on S_G and $H^j(S_G, \tilde{\mathbb{C}})$ is a smooth $G(A_f)$ -module, i.e. if K_f runs in the set of compact open subgroups of $G(A_f)$ and if $H^j(S_G, \tilde{\mathbb{C}})^{K_f}$ denotes the K_f -invariants in $H^j(S_G, \tilde{\mathbb{C}})$ then

$$\bigcup_{K_f} H^j(S_G, \tilde{\mathbb{C}})^{K_f} = H^j(S_G, \tilde{\mathbb{C}}).$$

Moreover

$$H^j(S_G, \tilde{\mathbb{C}})^{K_f} = H^j(S_G/K_f, \tilde{\mathbb{C}})$$

where $S_G/K_f = S(K_f)$ is the topological quotient of S_G by the K_f -action, and $H^j(S_G/K_f, \tilde{\mathbb{C}})$ is the sheaf-cohomology of S_G/K_f with coefficients in the sheaf $\tilde{\mathbb{C}}$.

A finite dimensional representation $\rho: G \rightarrow \text{End}(V)$ of G defines a locally constant sheaf \tilde{V} on S_G and we have similar formulas for the sheaf \tilde{V} .

4. Modular symbols and automorphic representations

Consider first the trivial representation I which is in the residual spectrum of $\mathcal{A}(G(\mathbb{Q}) \backslash G(\mathbb{A}))$.

For a subgroup H define the compact dual \hat{X}_H of X_H analogously. One has an embedding of \hat{X}_H in \hat{X} , and the fundamental class of $[\hat{X}_H]$ is a cohomology class in $H^*(\hat{X})$.

Put $\Gamma_0 = G(\mathbb{Q}) \cap \prod_p G(\mathcal{O}_p)$ and let \mathcal{H}_0 to be the space of complex valued compactly supported $K_f = \prod_p G(\mathcal{O}_p)$ -bi-invariant functions on the finite adelic group $G(\mathbb{A}_f)$. This is the Hecke algebra corresponding to K_f under convolutions and acts on the cohomology group $H^*(S(\Gamma_0), \mathbb{C}) = H^*(S_G, \tilde{\mathbb{C}})^{K_f}$.

Theorem ([17]). *Suppose that G is a simply connected group which has no \mathbb{R} -anisotropic factors defined over \mathbb{Q} . Then, the class $j([\hat{X}_H])$ is a linear combination of Hecke translates of the generalized modular symbol $[S_H(\Gamma')] \in H^*(S(\Gamma'), \mathbb{C})$ for some congruence subgroup Γ' of Γ_0 .*

In particular, if $j([\hat{X}_H]) \neq 0$, then the modular symbol $[S_H(\Gamma')]$ does not vanish.

Since the kernel of the Borel map j has been determined in [6], this result is used to exhibit many modular symbols in [17] related to the trivial representation.

We now consider representations with nontrivial (\mathfrak{g}, K) -cohomology which are induced from parabolic subgroups and which are considered as subrepresentations of $\mathcal{A}(G(\mathbb{Q}) \backslash G(\mathbb{A}))$ via Eisenstein series. Let $P \supset B$ be a standard parabolic subgroup of G with standard Levi part L_P . Let $x_0 = K e \in X$. The orbit of x_0 under $L_P(\mathbb{A})$ in the globally symmetric space X_A is isomorphic to $L_P(\mathbb{R}) / (L_P(\mathbb{R}) \cap K) \times L_P(\mathbb{A}_f)$. We have a symmetric space $L_P(\mathbb{R}) / (L_P(\mathbb{R}) \cap K)$ and

$$S_{L_P}^\natural := L_P(\mathbb{Q}) \backslash L_P(\mathbb{R}) / (L_P(\mathbb{R}) \cap K) \times L_P(\mathbb{A}_f)$$

is a locally symmetric space and we have a continuous injection $S_{L_P}^\natural \hookrightarrow S_G$, which identifies it with a closed subspace of S_G . We call $S_{L_P}^\natural$ the modular manifold attached to P .

In [14] we consider the covering space $S_P := P(\mathbb{Q}) \backslash X_A$. One can see that

$$H^*(S_P, \mathbb{C}) = H^*(P(\mathbb{Q}), C^\infty(G(\mathbb{A}_f), \mathbb{C})).$$

Using the Steinberg representation we define the Poincaré dual $H_c^*(S_P, \mathbb{C})$. In [13] we attach to a class $[\phi] \in H_c^j(S_P, \mathbb{C})$ a class

$$\text{cor}_P([\phi]) \in H_c^j(S_G, \tilde{V}).$$

The classes are in a sense the Poincaré dual to Eisenstein classes defined by embeddings of representations induced from $P(\mathbb{A})$ into the space of automorphic functions [15]. In [14] we obtain formulas for a restriction map res_c of the classes $\text{cor}_P([\phi])$

to $H^*(S_{L\rho}^{\natural}, \mathbb{C})$, thus obtaining modular symbols. As a special case we prove that the fundamental class $[S_{L\rho}^{\natural}]$ defines a nontrivial modular symbol generalizing a result by Ash and Borel [1].

For $G = \mathrm{Gl}(2n, \mathbb{R})$ there is a unique unitary irreducible representation $J(n)$ with

$$H^{n(n+1)/2}(\mathfrak{g}, K, J(n)) \neq 0$$

and

$$H^j(\mathfrak{g}, K, J(n)) = 0 \quad \text{for } j < n(n+1)/2.$$

By Poincaré duality $H^{(n+1)(3n-2)/2}(\mathfrak{g}, K, J(n)) \neq 0$.

If Γ is a small congruence subgroup the residues of Eisenstein series define an embedding

$$E: J(n) \rightarrow \mathcal{A}_{\mathrm{res}}(\Gamma \backslash G)$$

in the residual spectrum and a nonzero class in $H^{n(n+1)/2}(S(\Gamma), \mathbb{C})$ [16]. By Poincaré duality we have a nonzero class in $H_c^{(n+1)(3n-2)/2}(S(\Gamma), \mathbb{C})$ which can be represented by a pseudo Eisenstein form ω_{sp} [13].

Now let $H = \mathrm{Sp}(n, \mathbb{R}) \subset \mathrm{Gl}(2n, \mathbb{R})$ be defined by the skew symmetric form $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. The work of Jacquet and Rallis implies that $J(n)$ admits an H -invariant linear functional and no other irreducible unitary infinite dimensional representation with nontrivial (\mathfrak{g}, K) -cohomology admits an H -invariant linear functional.

Proposition. *Suppose that $n = 2$. If Γ is a sufficiently small congruence subgroup, then $\int_{S_H(\Gamma)} \omega_{\mathrm{sp}} \neq 0$ and so the modular symbol $[S_H(\Gamma)]$ is not zero.*

Suppose that $n > 2$. Then the Oda restriction of $[\omega_{\mathrm{sp}}]$ to $S(\Gamma)$ is zero.

5. A conjecture

These examples support the conjecture that understanding the automorphic representations with nontrivial (\mathfrak{g}, K) -cohomology as H -modules and their (\mathfrak{h}, K_H) -quotients, respectively subrepresentations, is crucial to understanding their relationship to modular symbols. In general the restriction of π to \mathfrak{h} is not a direct sum of unitary irreducible (\mathfrak{h}, K_H) -modules and so we have to consider its irreducible quotients.

Suppose that $j: H \hookrightarrow G$ is a subgroup, π an irreducible unitary automorphic representation with $H^*(\mathfrak{g}, K, \pi) \neq 0$. Let $0 \neq [\omega] \in H^*(\mathfrak{g}, K, \pi)$. Assume also that there exists an irreducible (\mathfrak{h}, K_H) -module π_H and $Q \in \mathrm{Hom}_H(\pi, \pi_H)$ and $0 \neq [Q^*\omega] \in H^*(\mathfrak{h}, K \cap H, \pi_H)$.

Let

$$E: \pi \hookrightarrow \mathcal{A}_{\mathrm{res}}(\Gamma \backslash G) \oplus \mathcal{A}_{\mathrm{cusp}}(\Gamma \backslash G)$$

be an embedding.

Conjecture. Suppose that $[E^*(\omega)] \neq 0$. For sufficiently small congruence subgroup Γ' the Oda restriction of $[E^*\omega]$ to $H^*(S_H(\Gamma'), \mathbb{C})$ is non trivial.

By duality we have a map

$$E_c: H_c^*(S(\Gamma), \mathbb{C}) \rightarrow H^*(\mathfrak{g}, K, \pi).$$

Suppose again that $0 \neq [Q^*\omega] \in H^*(\mathfrak{h}, K \cap H, \pi_H)$.

Conjecture. Suppose that $[Q^*E_c^*\omega] \neq 0$. Then for sufficiently small Γ there exists a “restriction” map

$$\text{res}_c: H^*(S(\Gamma), \mathbb{C}) \rightarrow H_c^*(S_H(\Gamma'), \mathbb{C})$$

so that $\text{res}_c[\omega] \neq 0$.

If the restriction of π to \mathfrak{h} is a direct sum of unitary irreducible representations, the first conjecture is true.

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