Some results on compactifications of semisimple groups

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Abstract. This paper deals with recent results involving a compactification $X$ of a semisimple group $G$. The emphasis is on the case that $G$ is adjoint and $X$ is its wonderful compactification. Group theoretical constructions in $G$ have repercussions in $X$. The paper describes a number of them.

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1. Introduction

1.1. Notations. $G$ is a connected semisimple linear algebraic group over the algebraically closed field $k$ of characteristic $p \geq 0$. Fix a maximal torus $T$ of $G$ and a Borel subgroup $B = T.U$, where $U$ is the unipotent subgroup of $B$. The Weyl group $N_G(T)/T$ is denoted by $W$. It acts on $T$.

For $w \in W$ we denote by $\dot{w}$ a representative of $w \in N_G(T)$, not necessarily always the same.

$R$ is the root system of $(G, T)$ and $R^+ \subset R$ the system of positive roots defined by $B$. Its set of simple roots is denoted by $I$. We identify it with the set of simple reflections. $l$ is the corresponding length function on $W$.

For $J \subset I$, $W_J$ is the subgroup of $W$ generated by $J$ and $W^J$ the set of minimal length coset representatives for $W/W_J$. The longest element of $W_J$ is $w_{0,J}$.

1.2. Compactification. $G \times G$ acts on $G$ by $(x, y).z = xyz^{-1}$ ($x, y, z \in G$). By a compactification of $G$ we mean an irreducible normal projective $G \times G$-variety, containing $G$ as an open $G \times G$-stable subvariety. The theory of embeddings of spherical varieties (due to Luna and Vust, see [Kn]) can be applied to the $G \times G$-variety $G$ to analyze such compactifications.

We shall be concerned here mainly with the particular case that $G$ is adjoint and $X$ is the wonderful compactification of $G$. This was first constructed for $k = \mathbb{C}$ by De Concini and Procesi in [DP1]. The case of an arbitrary algebraically closed $k$ was dealt with in [St]. (See also [DS], where fields of definition are also taken into account).

In the construction given in these papers one uses a suitable finite-dimensional projective representation $\rho : G \to \text{PGL}(V)$ of $G$ and $X$ is defined to be the closure in
Then $X$ is a compactification of $G$. It is a smooth projective $G \times G$-variety. A key property is that $X$ contains a unique closed $G \times G$-orbit, isomorphic to $G/B^\times \times G/B$, where $B^-$ is the opposite of $B$ (see Lemma 1 below). The complement $X - G$ is a union of smooth divisors $D_i$ indexed by the simple roots $i \in I$, with normal crossings.

For $J \subset I$ let $X_J \subset \bigcap_{i \in I - J} D_i$ be the set of points not lying in a smaller intersection of the same kind. The $X_J$ ($J \subset I$) are the $G \times G$-orbits in $X$. Then $X_I = G$, $X_{I - \{i\}}$ is the orbit which is open in $D_i$ and $X_\emptyset$ is the closed orbit.

[DP2] analyzes general smooth compactifications of $G$. The wonderful one is shown to be “minimal”.

There are other constructions of $X$:
(a) as the closure of the $G \times G$-orbit of the diagonal in the variety of Lie subalgebras of $\text{Lie}(G \times G)$ (see [DP1, sect. 6]),
(b) as the closure of the $G \times G$-orbit of the diagonal of $G/B \times G/B$, viewed as a point of the Hilbert scheme of $G/B \times G/B$ (see [B2]).

If $G$ is arbitrary there does not seem to be a canonical smooth compactification. One can construct a not necessarily smooth one in the following manner.

Let $G_{ad}$ be the corresponding adjoint group and $X_{ad}$ its wonderful compactification. Let $X$ be the normalization of $X_{ad}$ in the function field $k(G)$ (a finite extension of $k(G_{ad})$).

Then $X$ is a compactification of $G$. The homomorphism $G \to G_{ad}$ extends to a $G \times G$-morphism $X \to X_{ad}$. It induces a bijection of the sets of $G \times G$-orbits.

We recall some facts, to be elaborated on later in the context of compactifications.

1.3. **Bruhat’s Lemma.** $G$ is the disjoint union of the locally closed subsets $G_w = B \dot{w} B$. In other words: $B \times B$ acts on $G$ with finitely many orbits, indexed by the elements of $W$.

There is an order $\leq$ on $W$ (the Bruhat–Chevalley order) such that for $x, w \in W$ the orbit $G_x$ lies in the closure $\overline{G_w}$ if and only if $x \leq w$.

The flag variety $G/B$ is a smooth projective $G$-variety. The closed subvarieties $S_w = \overline{G_w}/B$ ($w \in W$) are the Schubert varieties.

1.4. **Conjugation action.** Let $G_d \simeq G$ be the diagonal subgroup of $G \times G$. The restriction to $G_d$ of the $G \times G$-action on $G$ is the conjugation action of $G$ on itself.

We have a partition of $G$ into $G_d$-stable closed subsets, each of which consists of the elements whose semisimple part lies in a given conjugacy class. We call these subsets **Steinberg fibers**. Regular semisimple conjugacy classes are examples.

The Steinberg fibers are the fibers of a flat morphism $G \to W \setminus T$.

1.5. **Character sheaves.** There is a geometric character theory for $G$, embodied in Lusztig’s theory of character sheaves (see [L1]). Character sheaves are certain
conjugation-equivariant irreducible perverse sheaves on $G$. Ingredients in their construction are $B \times B$-equivariant perverse sheaves on $G$, supported by the closure of some $G_w$.

2. $B \times B$-action on a compactification

2.1. The $B \times B$-orbits. $G$ is assumed to be adjoint and $X$ is its wonderful compactification. The question of extending Bruhat’s Lemma to $X$, i.e. of describing the $B \times B$-action on $X$ arises naturally. It was studied in [B1], [Sp1] and more recently in [HT2]. The last paper also deals with non-adjoint groups.

We first describe in more detail the $G \times G$-orbits $X_J$ in $X$.

For $J \subset I$ denote by $P_J \supset B$ the corresponding standard parabolic subgroup and by $P_J^- \supset B^-$ its opposite. We denote by $L_J$ the Levi subgroup of $P_J$ and $P_J^-$ containing $T$ and by $G_J$ the adjoint group of $L_J$. The image of $T$ in $G_J$ is a maximal torus denoted by $T_J$.

Let $\lambda$ be a cocharacter of $T$ (a homomorphism $\lambda: \mathbb{G}_m \rightarrow T$). Then $\lambda(0) = \lim_{\xi \rightarrow 0} \lambda(\xi)$ is a well-defined point of $X$, as $X$ is complete. $\lambda$ also defines a linear function on the character group of $T$, denoted by the same symbol.

The $G \times G$-orbits $X_J$ ($J \subset I$) in $X$ can be described as follows.

Lemma 1. (i) There is a unique base point $h_J \in X_J$ such that for all cocharacters $\lambda$ with $\lambda(\alpha) = 0$ for $\alpha \in J$ and $\lambda(\alpha) > 0$ for $\alpha \in I - J$ we have $\lambda(0) = h_J$.

(ii) The orbit map $(x, y) \mapsto (x, y).h_J$ induces an isomorphism of the quotient variety $(G \times G) \times P_J^- P_J^- G_J$ onto $X_J$.

For (i) see [DS, sect. 3]. The quotient of (ii) is relative to the right action of $P_J^- \times P_J$ on $G \times G$ and the left action on $G_J$ given by $(x, y).z = \bar{x}z\bar{y}^{-1}$, the bars denoting projection on $G_J$ in $P_J$ and $P_J^-$. See [Sp1, p. 73].

By (ii) the closed orbit $X_\emptyset$ is isomorphic to $G/B^- \times G/B$ (as was already mentioned).

Proposition 1. (i) A $B \times B$-orbit in $X_J$ is of the form

$$[J, x, w] = (B \times B).(\hat{x}, \hat{w}).h_J,$$

with unique $w \in W$, $x \in W_J$.

(ii) $\dim [J, x, w] = l(w_0, J) - l(x) + l(w) + |J|$.

See [Sp1, p. 74]. The result is due to Brion [B1]. $[I, 1, w]$ is the set $G_w$ of 1.3.

The next result describes the closure relations between the $[I, x, w]$.

Proposition 2. Let $J, J' \subset I$, $x \in W_J$, $x' \in W_{J'}$, $w, w' \in W$. Then $[J', x', w'] \subset [J, x, w]$ if and only if $J' \subset J$ and there exists $u \in W_J$ such that $xu \leq x'$, $w' \leq wu$. 
See [HT2, Prop. 6.3]. [Sp1, Prop. 2.4] gives a somewhat more complicated description.

The closures $\overline{G_w}$ (in $X$) are the large Schubert varieties, first studied in [BP]. The large Schubert variety of minimal dimension is the closure $\overline{B}$. It follows from Proposition 2 that it is the union of the $[J, x, w]$ with $w \leq x$.

In [BP] it is shown how the geometry of $\overline{B}$ can be used to understand a result about the simply connected cover $G_{sc}$ of $G$, namely van der Kallen’s filtration (see [Kal]) of the coordinate algebra $k[B_{sc}]$ of the preimage $B_{sc}$ of $B$ in $G_{sc}$.

More generally, for each $J$ there is a unique $B \times B$-orbit of minimal dimension in $X_J$, viz. $[J, w_0, J]$. In the present case there is also a version of the Bott–Samelson–Demazure–Hansen variety. To formulate it succinctly write $B \times B = B$ and denote by $P_h$ minimal parabolic subgroups of $G \times G$ containing $B$ (so $P_h/B = \mathbb{P}^1$). For $h = (h_1, \ldots, h_s)$ put

$$Z_h = B^s \backslash (P_{h_1} \times \cdots \times P_{h_s} \times [J, w_0, J], 1),$$

the quotient for the $B^s$-action (with obvious notations)

$$(b_1, \ldots, b_s)(p_1, \ldots, p_s, x) = (p_1b_1^{-1}, b_1p_2b_2^{-1}, \ldots, b_{s-1}p_sb_s^{-1}, b_sx).$$

The $G \times G$-action on $X$ defines a morphism $\phi_h : Z_h \to X$.

**Proposition 3.** Given $x \in W_J$, $w \in W$ there exist $h$ such that $\phi_h$ is a proper birational morphism of $Z_h$ onto $[J, x, w]$.

The proof is along familiar lines, it uses reduced decompositions of $w$ and of $w_0,Jw_0,Jx$. However, one cannot claim that $\phi_h$ is a resolution, as the varieties $[J, w_0, J, 1]$ are usually not smooth.

For example, if $G$ is simple and of rank $> 1$, $\overline{B}$ is not smooth, see [Sp1, Cor. 4.8]. The following problem arises naturally.

**Problem 1.** Construct a $B \times B$-equivariant resolution of $[J, w_0, J, 1]$, in particular of $\overline{B}$.

**Proposition 4.** A large Schubert variety admits a cellular decomposition (paving by affine spaces).

See [Sp1, p. 81]. The cells can be described explicitly, which leads to a description of the cohomology groups of large Schubert varieties. In particular, their odd cohomology vanishes (see [loc. cit., 2.11]).

**Problem 2.** Do all $B \times B$-orbit closures have cellular decompositions?

**Proposition 5.** The odd (global) intersection cohomology of a $B \times B$-orbit closure vanishes.

See [loc. cit., Thm. 4.11]. The local intersection cohomology of orbit closures will appear in Section 4.
2.2. Algebro-geometric properties of orbit closures. In this subsection $G$ is an arbitrary semisimple group and $X$ is a compactification of $G$, as in 1.2. For Frobenius splittings and their various refinements we refer to [BK].

**Theorem 1.** Let $p > 0$. $X$ admits a $B \times B$-canonical Frobenius splitting which compatibly splits the closures of all $B \times B$-orbits.

This is [HT2, Prop. 7.1], where it is deduced from the slightly weaker result in [BK, Thm. 6.1.12]. The theorem has the following corollaries, in any characteristic. They are proved by familiar arguments. Let $Z \subset X$ be a $B \times B$-orbit closure.

**Corollary 1.** Let $\mathcal{L}$ be an ample line bundle on $Z$.

(i) $H^i(Z, \mathcal{L}) = 0$ for $i > 0$.

(ii) If $Z' \subset Z$ is another orbit closure then the restriction map

\[ H^i(Z, \mathcal{L}) \to H^i(Z', \mathcal{L}) \]

is surjective.

**Corollary 2.** $Z$ is normal and Cohen–Macaulay.

In fact, more is proved in [loc. cit.], namely that all $B \times B$-orbit closures are globally $F$-regular. This property also entails the two Corollaries (and more). We will not go into this.

Corollary 2 was first proved by Brion in [B3].

Let $\mathcal{L}$ be an ample line bundle on $X$. Chirivi and Maffei in [CM] constructed a “standard monomial basis” of the space of global sections $H^0(X, \mathcal{L})$. K. Appel recently showed (see [A]) that this basis is compatible with the $B \times B$-orbit closures.

3. The $G_d$-action

In this section $G$ is adjoint and $X$ is its wonderful compactification. This section discusses results about the $G_d$-action on $X$. Notations are as in Section 2.

If $\sigma$ is an automorphism of $G$ we have a $\sigma$-twisted $G \times G$-action on $X$: $(x, y) \cdot \sigma z = (x, \sigma y) \cdot z$. The induced $G_d$-action on $G$ is $\sigma$-twisted conjugacy: $(x, y) \mapsto xy(\sigma x)^{-1}$. Several of the results of this section extend to twisted actions.

3.1. A partition of $X$. The partition to be described is essentially due to Lusztig (see [L2, 12.3]). The present formulation was given by He (in [H1, sect. 2]). A similar result also occurs in [EL].

For $J \subset I$ and $W \in W^J$ put

\[ X_{J, w} = G_d, [J, w, 1]. \]
Theorem 2. (i) $X_J$ is the disjoint union of the $X_{J,w}$ ($w \in W_J$).

(ii) $X_{J,w}$ is locally closed and irreducible, of dimension $\dim G - l(w) - |I - J|$.

(iii) For $w \in W_J$ there exist a connected reductive group $G_w$ and an automorphism $\sigma_w$ of it such that there is a bijection of the set of $G^d$-orbits in $X_{J,w}$ onto the set of $\sigma_w$-twisted conjugacy classes of $G_w$.

The proofs of (i) in [L] and [H1] use a combinatorial machinery. (ii) and (iii) are also due to Lusztig (see [L2, sect. 8]).

Remark. For $J = \emptyset$ part (i) of the theorem is a familiar variant of Bruhat’s lemma.

We next describe the closure relations between the $X_{J,w}$, following [H2]. Let $I$ be the set of pairs $(J, x)$ with $J \subset I$, $x \in W_J$. Define a relation $\leq$ on $I$ by

$$(J, x) \leq (K, y) \text{ if and only if } J \subset K \text{ and } x \geq z^{-1}yz \text{ for some } z \in W_K.$$  

Theorem 3. (i) $\leq$ defines an order on $I$.

(ii) If $(J, x), (K, y) \in I$ then $X_{J,w} \subset X_{K,y}$ if and only if $(J, x) \leq (K, y)$.

See [H2, sect. 3, 4].

Proposition 6. If $\overline{X_{K,y}}$ contains only finitely many $G^d$-orbits then it has a cellular decomposition.

See [loc. cit., sect. 5].

3.2. The closure of Steinberg fibers. Let $F \subset G$ be a Steinberg fiber (see 1.4). Its closure $\overline{F}$ is an irreducible closed $G^d$-stable subset of $X$. An example is the unipotent variety of $X$, the closure of the unipotent variety $G_u$ of $G$.

Lemma 2. There is $t \in T$ such that $\overline{F} = G_d.t\overline{U}$

See [Sp2, Lemma 1.4].

This leads to the problem of describing $\overline{U}$. Some partial results are given in [loc. cit.]. They use the fact (a consequence of completeness) that a point of $\overline{U}$ can be obtained by “specializing $\xi$ to 0” from a point of $U(K)$ where $K = k((\xi))$, the field of formal Laurent series.

Let again $G_{sc}$ be the simply connected cover of $G$, with Borel group $B_{sc} = T_{sc}.U_{sc}$, $B_{sc}$ and $T_{sc}$ lying over $B$ and $T$. Then $U_{sc} \simeq U$.

Put

$$H = \{ g \in G_{sc}(k[[\xi]]) \mid g(0) \in B_{sc} \}.$$  

This is the Iwahori subgroup of $G(K)$ defined by $B_{sc}$. Let $W_a$ be the affine Weyl group associated to $T_{sc}$. Then we have the Bruhat decomposition $G(K) = HW_aH$.

Problem 3. Determine the image in $W_a$ of $U_{sc}(K)$. 
A solution of this problem will be useful for describing of \( \overline{U} \), see [loc. cit.].

The main fact about the closures \( \overline{F} \) is that they all intersect the boundary \( X - G \) of \( X \) in the same set. More precisely, we have the following result. For \( w \in W \) we denote by \( \text{supp}(w) \subset I \) the set of simple reflections occurring in a reduced decomposition of \( w \).

**Theorem 4.**

\[
\overline{F} - F = \bigsqcup_{J \neq I} \bigsqcup_{w \in W^J, \text{supp}(w) = I} X_{J,w}
\]

This was first proved by He in [H1, Thms. 4.3, 4.5], via a laborious case by case check. In [HT1] a shorter proof is given and the result is extended to the \( \sigma \)-twisted case.

3.3. We sketch a simplified version of the proof of the theorem. It uses the following steps. \( F \) is a Steinberg fiber.

(a) \( \overline{F} \cap X_I \neq \emptyset \).

By Lemma 2 it suffices to show that \( U \cap X_I \neq \emptyset \). This follows from the results of [Sp2, sect. 3], for example from [loc. cit., Cor. 3.8] with \( w = w_{0,I} \).

(b) If \( J \neq I \) and \( X_{J,w} \cap \overline{F} \neq \emptyset \) then \( \text{supp}(w) = I \).

This is established using an argument from the proof of [H1, Thm. 4.3]. Assume that \( i \notin \text{supp}(w) \). Let \( \varpi_i \) be the fundamental weight associated to \( i \) and let \((\rho, V)\) be an irreducible representation of \( G \) with lowest weight \(-n\varpi_i \) \((n > 0)\). Then \( \rho \) extends to a \( G_d \)-equivariant morphism \( X \to \mathbb{P}(\text{End}(V)) \) (see [DS, 3.15]). The image \( \rho(\overline{F} - F) \) consists of nilpotent lines in \( \text{End}(V) \). On the other hand a lowest weight vector is an eigenvector of \( \rho(\dot{w}) \) with a nonzero eigenvalue and \( \rho(h_J) \) is projection on the line of lowest weight vectors. Using (1) and Proposition 1 (i) one sees that this contradicts nilpotency.

(c) \( \overline{F} \) and \( X_J \) intersect properly if \( J \neq I \).

It suffices to prove this for \( J = \emptyset \). From (a) it follows that \( \dim \overline{F} \cap X_\emptyset \geq \dim F - |I| \). (b) implies, on the other hand, that the intersection has dimension \( \leq \dim F - |I| \).

(d) Let \( i \in I \). By (c) \( \dim(\overline{F} \cap X_{I-\{i\}}) = \dim F - 1 \). By Theorem 2 (i) there must be a set \( X_{J,w} \) whose intersection with \( \overline{F} \cap X_{I-\{i\}} \) is dense in \( X_{I-\{i\}} \). Then \( J \subset I - \{i\} \) and

\[
\dim G - l(w) - |I - J| = \dim X_{J,w} \geq \dim F - 1 = \dim G - |I| - 1,
\]

and \( l(w) \leq |J| + 1 \). But by (b) \( l(w) \geq |I| \). We conclude that \( |J| = |I| - 1 \) and \( l(w) = |I| \). We then must have \( J = I - \{i\} \). Moreover, \( w \) is a Coxeter element i.e., \( \text{supp}(w) = I \) and \( l(w) = |I| \).

(e) \( W^{I-\{i\}} \) contains a unique Coxeter element \( c_i \).

This is proved by induction on \( |I| \).
We conclude from (d) that $\bar{F} \cap X_{I-[i]} = \overline{X_{I-[i],c_i}}$ for all $i \in I$. Since for $J \neq I$

$$X_J = \bigcap_{i \notin J} X_{I-[i]}$$

this implies that the intersection $\bar{F} \cap (X - G)$ is independent of $F$. With a little more work the theorem follows.

### 3.4. Algebro-geometric properties.

Let $X_{sc}$ be a compactification of the simply connected cover $G_{sc}$. Let $X_i$ ($1 \leq i \leq n$) be the irreducible components of $X_{sc} - G_{sc}$. They all have codimension 1.

**Proposition 7.** Let $p > 0$. Let $F$ be a Steinberg fiber in $G_{sc}$.

(i) $X_{sc}$ admits a Frobenius splitting which compatibly splits $\bar{F}$ and the $X_i$ ($1 \leq i \leq n$).

(ii) $\bar{F}$ is normal and Cohen–Macaulay.

For (a somewhat stronger version of) (i) see [T, Thm. 8.2] and for (ii) [loc. cit., Thm. 10.2].

Notice that this result covers the wonderful compactification $X$ of $G$ if $G$ has trivial center. For arbitrary adjoint $G$ a partial result is proved in [LT]. For $i \in I$ let $\chi_i$ be the fundamental character of $G_{sc}$ associated to $i$. Put

$$\bar{F}_0 = \{g \in G_{sc} | \chi_i(g) = 0 \text{ for all } i\},$$

this is a Steinberg fiber in $G_{sc}$. Its image in $G$ is a Steinberg fiber $F_0$ in $G$, the zero fiber.

**Proposition 8.** Let $p > 0$. $X$ admits a Frobenius splitting which compatibly splits $F_0$ and the components of $X - G$.

See [loc. cit., Thm. 8.1]. It is also pointed out that the result cannot be true for arbitrary Steinberg fibers in $G$.

**Remark.** The appearance of the zero fiber is somewhat curious. Over $\mathbb{C}$ it appears in [Ka] in another context. I learned from J.-P. Serre (private communication) that for a quasi-simple group over $\mathbb{C}$, $F_0$ has been determined (case by case). Its elements are regular and have finite order. The characteristic $p$ case does not seem to have been analyzed.

**Problem 4.** ($p > 0$) Does $X$ admit a Frobenius splitting which compatibly splits an arbitrary $\bar{F}$?

**Problem 5.** Is $\bar{F}$ normal and Cohen–Macaulay?

**Problem 6.** Does $\bar{F}$ admit a cellular decomposition?

An example given in [Sp2, 4.3] with $G = \text{PGL}_3$ shows that $\bar{F}$ need not be smooth.

**Problem 7.** Determine the intersection cohomology (local and global) of $\bar{F}$. 

4. Character sheaves on $X$

In this section $G$ is an adjoint group and $X$ is its wonderful compactification.

4.1. $B \times B$-equivariant perverse sheaves. The definition of character sheaves on $X$ uses certain $B \times B$-equivariant perverse sheaves on $X$, which we first have to introduce.

If $S$ is a torus let $C(S)$ be its character group and put

$$\hat{C}(S) = C(S) \otimes (\mathbb{Z}(p)/\mathbb{Z}),$$

where $\mathbb{Z}(p)$ is the localization of $\mathbb{Z}$ at the prime ideal $(p)$. The elements of $\hat{C}(S)$ parametrize tame rank one local systems on $S$ (also called Kummer local systems).

We work in $l$-adic cohomology, with a coefficient field $E$ (e.g. $\overline{\mathbb{Q}}_l$).

Let $v = [J, x, w]$ be a $B \times B$-orbit in $X$, as in Proposition 1. Using that it is a homogeneous space for $B \times B$ one constructs a morphism $\phi: v \to T_J$, where $T_J$ is the maximal torus of $G_J$ of 2.1 (see [H3, 3.1]). For $\xi \in \hat{C}(T_J)$ we have a local system $\mathcal{L}_{\xi, v} = \phi^* \xi$ on $v$. Let $I_{\xi, v}$ be its perverse extension, a perverse sheaf on $X$ (for $l$-adic cohomology) supported by $v$, whose restriction to $v$ is $\mathcal{L}_{\xi, v}[\dim v]$.

[Sp1, sect. 5] deals with these perverse sheaves. It is shown that they are even, i.e. that their cohomology sheaves are zero in dimensions $\not\equiv \dim v (\mod 2)$.

In the next lemma one uses that $\hat{C}(T_J)$ can be viewed as a subset of $\hat{C}(T)$ (see [loc. cit., 1.7]) and that $W$ acts on $\hat{C}(T)$.

**Lemma 3.** If $x.\xi = w.\xi$ then $I_{\xi, v}$ is a $B_{d}$-equivariant irreducible perverse sheaf on $X$.

See [H3, 3.1].

4.2. Character sheaves. Character sheaves on a reductive group were introduced by G. Lusztig in the 1980s, in a long series of papers. [L1] gives a brief exposition of the results of these papers. [MS] is a report on part of the results. The definition of character sheaves used there is slightly different from Lusztig’s.

In [L2] Lusztig defines character sheaves on the compactification $X$. I noticed (unpublished) that the approach of [MS] could also be followed to do this. But it is not obvious that the two definitions of character sheaves on $X$ are equivalent.

Independently, Xuhua He also came to the definition based on [MS]. He proved in [H3] the equivalence with Lusztig’s definition. I shall not go into Lusztig’s definition. I will only report on the other one.

$B$ acts on $G \times X$ by $b.(g, x) = (gb^{-1}, (b, b).x)$. Let $G \times_B X$ be the quotient and $\alpha: G \times X \to G \times_B X$ the quotient map. The $G_d$-action on $X$ induces a proper morphism $\mu: G \times_B X \to X$.

Let $I_{\xi, v}$ be as in Lemma 3. Then $A = E[\dim G] \boxtimes I_{\xi, v}$ is an irreducible perverse sheaf on $G \times X$ and there is an irreducible perverse sheaf $\tilde{A}$ on $G \times_B X$ with $A = \alpha^* \tilde{A}$. 

Put $C_{\xi,v} = \mu_{\lambda}\tilde{A}$. By the decomposition theorem this is a semisimple complex on $X$, i.e. a direct sum of shifted irreducible perverse sheaves on $X$. The perverse sheaves occurring in the $C_{\xi,v}$ (if $\xi$ and $v$ vary) are the character sheaves on $X$. They are $G_d$-equivariant.

The nonzero restrictions of these character sheaves to the open subvariety $G$ of $X$ are Lusztig’s original character sheaves. More generally, for $J \subset D$ we call character sheaf on $X_J$ the restriction to $X_J$ of a character sheaf on $X$ which is obtained as above from an orbit $v \subset X_J$.

The character sheaves on $X$ deserve a further study. Here are a few problems.

**Problem 8.** Analyze the restriction of a character sheaf on $X$ to a $G \times G$-orbit $X_J$. Can such restrictions be described in terms of character sheaves on $X_J$?

**Problem 9.** Are character sheaves on $X$ even?

### 4.3. Finite ground fields.

Now let $k$ be an algebraic closure of the finite field $\mathbb{F}_q$ and let $F: a \mapsto a^q$ be the Frobenius automorphism of $k$. Assume that $G$ is defined over $k$. Then so is $X$, by [DS, Prop. 3.11].

Let $A$ be a character sheaf on $G$ whose support is not contained in $X - G$. The restriction of $A$ to $G$ is a character sheaf on $G$. Assume that $A$ “comes from $\mathbb{F}_q$”, meaning that $F^*A \simeq A$. Fix an isomorphism $\phi: F^*A \simeq A$. It can be normalized such as to be unique up to a root unity (see [L1, p. 178]).

$x \in X^F = X(\mathbb{F}_q)$ being an $\mathbb{F}_q$-rational point of $X$, $\phi$ defines linear maps $\phi^i_x$ of the cohomology stalks $H^i(A)_x$. Define a function $\chi_{\phi}$ on $X^F$ by

$$\chi_{\phi}(x) = \Sigma(-1)^i \text{Tr}(\phi^i_x, H^i(A)_x).$$

For $x \in G$ one obtains a class function on the finite Lie group $G^F$, which can be viewed as a generalized character of $G^F$. This function on $G^F$ has boundary values, viz. the values of $\chi_{\phi}$ on points of $X^F - G^F$.

**Problem 10.** Can one define and compute boundary values of irreducible characters of $G^F$?

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**References**


Some results on compactifications of semisimple groups


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