Some results on compactifications of semisimple groups

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Abstract. This paper deals with recent results involving a compactification X of a semisimple group G. The emphasis is on the case that G is adjoint and X is its wonderful compactification. Group theoretical constructions in G have repercussions in X. The paper describes a number of them.

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1. Introduction

1.1. Notations. G is a connected semisimple linear algebraic group over the algebraically closed field k of characteristic $p \geq 0$. Fix a maximal torus T of G and a Borel subgroup $B = T.U$, where U is the unipotent subgroup of B. The Weyl group $N_G(T)/T$ is denoted by W. It acts on T.

For $w \in W$ we denote by $\dot{w}$ a representative of $w \in N_G(T)$, not necessarily always the same.

R is the root system of (G, T) and $R^+ \subset R$ the system of positive roots defined by B. Its set of simple roots is denoted by I. We identify it with the set of simple reflections. l is the corresponding length function on W.

For $J \subset I$, W_J is the subgroup of W generated by J and W^J the set of minimal length coset representatives for W/W_J. The longest element of W_J is $w_{0,J}$.

1.2. Compactification. $G \times G$ acts on G by $(x, y).z = xyz^{-1}$ $(x, y, z \in G)$. By a *compactification* of G we mean an irreducible normal projective $G \times G$-variety, containing G as an open $G \times G$-stable subvariety. The theory of embeddings of spherical varieties (due to Luna and Vust, see [Kn]) can be applied to the $G \times G$-variety G to analyze such compactifications.

We shall be concerned here mainly with the particular case that G is adjoint and X is the wonderful compactification of G. This was first constructed for $k = \mathbb{C}$ by De Concini and Procesi in [DP1]. The case of an arbitrary algebraically closed k was dealt with in [St]. (See also [DS], where fields of definition are also taken into account).

In the construction given in these papers one uses a suitable finite-dimensional projective representation $\rho\colon G \to \mathrm{PGL}(V)$ of G and X is defined to be the closure in

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$\mathbb{P}(\mathrm{End}(V))$ of the image $\rho(G)$ (this turns out to be independent of the choice of ρ).

Then X is a compactification of G. It is a smooth projective $G \times G$-variety. A key property is that X contains a unique closed $G \times G$-orbit, isomorphic to $G/B^- \times G/B$, where B^- is the opposite of B (see Lemma 1 below). The complement $X - G$ is a union of smooth divisors D_i indexed by the simple roots $i \in I$, with normal crossings.

For $J \subset I$ let $X_J \subset \bigcap_{i \in I-J} D_i$ be the set of points not lying in a smaller intersection of the same kind. The X_J $(J \subset I)$ are the $G \times G$-orbits in X. Then $X_I = G$, $X_{I-\{i\}}$ is the orbit which is open in D_i and $X_\emptyset$ is the closed orbit.

[DP2] analyzes general smooth compactifications of G. The wonderful one is shown to be "minimal".

There are other constructions of X:

(a) as the closure of the $G \times G$-orbit of the diagonal in the variety of Lie subalgebras of Lie$(G \times G)$ (see [DP1, sect. 6]),

(b) as the closure of the $G \times G$-orbit of the diagonal of $G/B \times G/B$, viewed as a point of the Hilbert scheme of $G/B \times G/B$ (see [B2]).

If G is arbitrary there does not seem to be a canonical smooth compactification. One can construct a not necessarily smooth one in the following manner.

Let G_{ad} be the corresponding adjoint group and X_{ad} its wonderful compactification. Let X be the normalization of X_{ad} in the function field $k(G)$ (a finite extension of $k(G_{\mathrm{ad}})$).

Then X is a compactification of G. The homomorphism $G \to G_{\mathrm{ad}}$ extends to a $G \times G$-morphism $X \to X_{\mathrm{ad}}$. It induces a bijection of the sets of $G \times G$-orbits.

We recall some facts, to be elaborated on later in the context of compactifications.

1.3. Bruhat's Lemma. G is the disjoint union of the locally closed subsets $G_w = B\dot{w}B$. In other words: $B \times B$ acts on G with finitely many orbits, indexed by the elements of W.

There is an order $\leq$ on W (the Bruhat–Chevalley order) such that for $x, w \in W$ the orbit G_x lies in the closure $\overline{G_w}$ if and only if $x \leq w$.

The flag variety G/B is a smooth projective G-variety. The closed subvarieties $S_w = \overline{G_w}/B$ $(w \in W)$ are the Schubert varieties.

1.4. Conjugation action. Let $G_{\mathrm{d}} \simeq G$ be the diagonal subgroup of $G \times G$. The restriction to G_{d} of the $G \times G$-action on G is the conjugation action of G on itself.

We have a partition of G into G_{d}-stable closed subsets, each of which consists of the elements whose semisimple part lies in a given conjugacy class. We call these subsets *Steinberg fibers*. Regular semisimple conjugacy classes are examples.

The Steinberg fibers are the fibers of a flat morphism $G \to W\backslash T$.

1.5. Character sheaves. There is a geometric character theory for G, embodied in Lusztig's theory of character sheaves (see [L1]). Character sheaves are certain conjugation-equivariant irreducible perverse sheaves on G. Ingredients in their construction are $B \times B$-equivariant perverse sheaves on G, supported by the closure of some G_w.

2. $B \times B$-action on a compactification

2.1. The $B \times B$-orbits. G is assumed to be adjoint and X is its wonderful compactification. The question of extending Bruhat's Lemma to X, i.e. of describing the $B \times B$-action on X arises naturally. It was studied in [B1], [Sp1] and more recently in [HT2]. The last paper also deals with non-adjoint groups.

We first describe in more detail the $G \times G$-orbits X_J in X.

For $J \subset I$ denote by $P_J \supset B$ the corresponding standard parabolic subgroup and by $P_J^- \supset B^-$ its opposite. We denote by L_J the Levi subgroup of P_J and P_J^- containing T and by G_J the adjoint group of L_J. The image of T in G_J is a maximal torus denoted by T_J.

Let λ be a cocharacter of T (a homomorphism $\lambda\colon \mathbb{G}_m \to T$). Then $\lambda(0) = \lim_{\xi\to 0}\lambda(\xi)$ is a well-defined point of X, as X is complete. λ also defines a linear function on the character group of T, denoted by the same symbol.

The $G \times G$-orbits X_J $(J \subset I)$ in X can be described as follows.

Lemma 1. (i) *There is a unique base point $h_J \in X_J$ such that for all cocharacters λ with $\lambda(\alpha) = 0$ for $\alpha \in J$ and $\lambda(\alpha) > 0$ for $\alpha \in I - J$ we have $\lambda(0) = h_J$.*

(ii) *The orbit map $(x, y) \mapsto (x, y).h_J$ induces an isomorphism of the quotient variety $(G \times G) \times_{P_J^- \times P_J} G_J$ onto X_J.*

For (i) see [DS, sect. 3]. The quotient of (ii) is relative to the right action of $P_J^- \times P_J$ on $G \times G$ and the left action on G_J given by $(x, y).z = \bar{x}z\bar{y}^{-1}$, the bars denoting projection on G_J in P_J, respectively P_J^-. See [Sp1, p. 73].

By (ii) the closed orbit $X_\emptyset$ is isomorphic to $G/B^- \times G/B$ (as was already mentioned).

Proposition 1. (i) *A $B \times B$-orbit in X_J is of the form*

$$[J, x, w] = (B \times B).(\dot{x}, \dot{w}).h_J,$$

with unique $w \in W$, $x \in W^J$.

(ii) $\dim[J, x, w] = l(w_{0,J}) - l(x) + l(w) + |J|$.

See [Sp1, p. 74]. The result is due to Brion [B1]. $[I, 1, w]$ is the set G_w of 1.3.

The next result describes the closure relations between the $[I, x, w]$.

Proposition 2. *Let $J, J' \subset I$, $x \in W^J$, $x' \in W^{J'}$, $w, w' \in W$. Then $[J', x', w'] \subset \overline{[J, x, w]}$ if and only if $J' \subset J$ and there exists $u \in W_J$ such that $xu \leq x'$, $w' \leq wu$.*

See [HT2, Prop. 6.3]. [Sp1, Prop. 2.4] gives a somewhat more complicated description.

The closures $\overline{G_w}$ (in X) are the *large Schubert varieties*, first studied in [BP]. The large Schubert variety of minimal dimension is the closure $\bar{B}$. It follows from Proposition 2 that it is the union of the $[J, x, w]$ with $w \leq x$.

In [BP] it is shown how the geometry of $\bar{B}$ can be used to understand a result about the simply connected cover G_{sc} of G, namely van der Kallen's filtration (see [Kal]) of the coordinate algebra $k[B_{\mathrm{sc}}]$ of the preimage B_{sc} of B in G_{sc}.

More generally, for each J there is a unique $B \times B$-orbit of minimal dimension in X_J, viz. $[J, w_{0,I}w_{0,J}, 1]$.

In the present case there is also a version of the Bott–Samelson–Demazure–Hansen variety. To formulate it succinctly write $B \times B = \boldsymbol{B}$ and denote by $\boldsymbol{P}_h$ minimal parabolic subgroups of $G \times G$ containing $\boldsymbol{B}$ (so $\boldsymbol{P}_h/\boldsymbol{B} = \mathbb{P}^1$). For $\boldsymbol{h} = (h_1, \ldots, h_s)$ put

$$Z_{\boldsymbol{h}} = \boldsymbol{B}^s \backslash (\boldsymbol{P}_{h_1} \times \cdots \times \boldsymbol{P}_{h_s} \times \overline{[J, w_{0,I}w_{0,J}, 1]}),$$

the quotient for the $\boldsymbol{B}^s$-action (with obvious notations)

$$(b_1, \ldots, b_s).(p_1, \ldots, p_s, x) = (p_1 b_1^{-1}, b_1 p_2 b_2^{-1}, \ldots, b_{s-1} p_s b_s^{-1}, b_s.x).$$

The $G \times G$-action on X defines a morphism $\phi_{\boldsymbol{h}} \colon Z_{\boldsymbol{h}} \to X$.

Proposition 3. *Given $x \in W^J$, $w \in W$ there exist $\boldsymbol{h}$ such that $\phi_{\boldsymbol{h}}$ is a proper birational morphism of $Z_{\boldsymbol{h}}$ onto $\overline{[J, x, w]}$.*

The proof is along familiar lines, it uses reduced decompositions of w and of $w_{0,J}w_{0,I}x$. However, one cannot claim that $\phi_{\boldsymbol{h}}$ is a resolution, as the varieties $\overline{[J, w_{0,I}w_{0,J}, 1]}$ are usually not smooth.

For example, if G is simple and of rank > 1, $\bar{B}$ is not smooth, see [Sp1, Cor. 4.8]. The following problem arises naturally.

Problem 1. Construct a $B \times B$-equivariant resolution of $\overline{[J, w_{0,I}w_{0,J}, 1]}$, in particular of $\bar{B}$.

Proposition 4. *A large Schubert variety admits a cellular decomposition* (*paving by affine spaces*).

See [Sp1, p. 81]. The cells can be described explicitly, which leads to a description of the cohomology groups of large Schubert varieties. In particular, their odd cohomology vanishes (see [loc. cit., 2.11]).

Problem 2. Do all $B \times B$-orbit closures have cellular decompositions?

Proposition 5. *The odd* (*global*) *intersection cohomology of a $B \times B$-orbit closure vanishes.*

See [loc. cit., Thm. 4.11]. The local intersection cohomology of orbit closures will appear in Section 4.

2.2. Algebro-geometric properties of orbit closures. In this subsection G is an arbitrary semisimple group and X is a compactification of G, as in 1.2. For Frobenius splittings and their various refinements we refer to [BK].

Theorem 1. *Let $p > 0$. X admits a $B \times B$-canonical Frobenius splitting which compatibly splits the closures of all $B \times B$-orbits.*

This is [HT2, Prop. 7.1], where it is deduced from the slightly weaker result in [BK, Thm. 6.1.12].
The theorem has the following corollaries, in any characteristic. They are proved by familiar arguments. Let $Z \subset X$ be a $B \times B$-orbit closure.

Corollary 1. *Let $\mathcal{L}$ be an ample line bundle on Z.*

(i) $H^i(Z, \mathcal{L}) = 0$ *for* $i > 0$.

(ii) *If $Z' \subset Z$ is another orbit closure then the restriction map*

$$H^i(Z, \mathcal{L}) \to H^i(Z', \mathcal{L})$$

is surjective.

Corollary 2. *Z is normal and Cohen–Macaulay.*

In fact, more is proved in [loc. cit.], namely that all $B \times B$-orbit closures are globally F-regular. This property also entails the two Corollaries (and more). We will not go into this.

Corollary 2 was first proved by Brion in [B3].

Let $\mathcal{L}$ be an ample line bundle on X. Chirivì and Maffei in [CM] constructed a "standard monomial basis" of the space of global sections $H^0(X, \mathcal{L})$. K. Appel recently showed (see [A]) that this basis is compatible with the $B \times B$-orbit closures.

3. The G_d-action

In this section G is adjoint and X is its wonderful compactification. This section discusses results about the G_{d}-action on X. Notations are as in Section 2.

If σ is an automorphism of G we have a σ-twisted $G \times G$-action on X: $(x, y)._\sigma z = (x, \sigma y).z$. The induced G_{d}-action on G is σ-twisted conjugacy: $(x, y) \mapsto xy(\sigma x)^{-1}$. Several of the results of this section extend to twisted actions.

3.1. A partition of X. The partition to be described is essentially due to Lusztig (see [L2, 12.3]). The present formulation was given by He (in [H1, sect. 2]). A similar result also occurs in [EL].

For $J \subset I$ and $W \in W^J$ put

$$X_{J,w} = G_{\mathrm{d}}.[J, w, 1]. \tag{1}$$

Theorem 2. (i) *X_J is the disjoint union of the $X_{J,w}$ ($w \in W^J$).*

(ii) *$X_{J,w}$ is locally closed and irreducible, of dimension* $\dim G - l(w) - |I - J|$.

(iii) *For $w \in W^J$ there exist a connected reductive group G_w and an automorphism σ_w of it such that there is a bijection of the set of G_{d}-orbits in $X_{J,w}$ onto the set of σ_w-twisted conjugacy classes of G_w.*

The proofs of (i) in [L] and [H1] use a combinatorial machinery. (ii) and (iii) are also due to Lusztig (see [L2, sect. 8]).

Remark. For $J = \emptyset$ part (i) of the theorem is a familiar variant of Bruhat's lemma.

We next describe the closure relations between the $X_{J,w}$, following [H2]. Let $\mathcal{I}$ be the set of pairs (J, x) with $J \subset I,\ x \in W^J$. Define a relation $\leq$ on $\mathcal{I}$ by

$$(J, x) \leq (K, y) \text{ if and only if } J \subset K \text{ and } x \geq z^{-1}yz \text{ for some } z \in W_K.$$

Theorem 3. (i) *$\leq$ defines an order on $\mathcal{I}$.*

(ii) *If $(J, x),\ (K, y) \in \mathcal{I}$ then $X_{J,w} \subset \overline{X_{K,y}}$ if and only if $(J, x) \leq (K, y)$.*

See [H2, sect. 3, 4].

Proposition 6. *If $\overline{X_{K,y}}$ contains only finitely many G_{d}-orbits then it has a cellular decomposition.*

See [loc. cit., sect. 5].

3.2. The closure of Steinberg fibers. Let $F \subset G$ be a Steinberg fiber (see 1.4). Its closure $\overline{F}$ is an irreducible closed G_{d}-stable subset of X. An example is the unipotent variety of X, the closure of the unipotent variety G_u of G.

Lemma 2. *There is $t \in T$ such that $\overline{F} = G_{\mathrm{d}}.t\overline{U}$*

See [Sp2, Lemma 1.4].

This leads to the problem of describing $\overline{U}$. Some partial results are given in [loc. cit.]. They use the fact (a consequence of completeness) that a point of $\overline{U}$ can be obtained by "specializing ξ to 0" from a point of $U(K)$ where $K = k((\xi))$, the field of formal Laurent series.

Let again G_{sc} be the simply connected cover of G, with Borel group $B_{\mathrm{sc}} = T_{\mathrm{sc}}.U_{\mathrm{sc}}$, B_{sc} and T_{sc} lying over B and T. Then $U_{\mathrm{sc}} \simeq U$.

Put

$$H = \{g \in G_{\mathrm{sc}}(k[[\xi]]) \mid g(0) \in B_{\mathrm{sc}}\}.$$

This is the Iwahori subgroup of $G(K)$ defined by B_{sc}. Let W_a be the affine Weyl group associated to T_{sc}. Then we have the Bruhat decomposition $G(K) = HW_aH$. whence a map $G(K) \to W_a$.

Problem 3. Determine the image in W_a of $U_{\mathrm{sc}}(K)$.

A solution of this problem will be useful for describing of $\overline{U}$, see [loc. cit.].

The main fact about the closures $\overline{F}$ is that they all intersect the boundary $X - G$ of X in the same set. More precisely, we have the following result. For $w \in W$ we denote by $\mathrm{supp}(w) \subset I$ the set of simple reflections occurring in a reduced decomposition of w.

Theorem 4.

$$\overline{F} - F = \coprod_{J \neq I} \coprod_{\substack{w \in W^J \\ \mathrm{supp}(w)=I}} X_{J,w}$$

This was first proved by He in [H1, Thms. 4.3, 4.5], via a laborious case by case check. In [HT1] a shorter proof is given and the result is extended to the σ-twisted case.

3.3. We sketch a simplified version of the proof of the theorem. It uses the following steps. F is a Steinberg fiber.

(a) $\overline{F} \cap X_\emptyset \neq \emptyset$.

By Lemma 2 it suffices to show that $\overline{U} \cap X_\emptyset \neq \emptyset$. This follows from the results of [Sp2, sect. 3], for example from [loc. cit., Cor. 3.8] with $w = w_{0,I}$.

(b) If $J \neq I$ and $X_{J,w} \cap \overline{F} \neq \emptyset$ then $\mathrm{supp}(w) = I$.

This is established using an argument from the proof of [H1, Thm. 4.3]. Assume that $i \notin \mathrm{supp}(w)$. Let ϖ_i be the fundamental weight associated to i and let (ρ, V) be an irreducible representation of G with lowest weight $-n\varpi_i$ $(n > 0)$. Then ρ extends to a G_{d}-equivariant morphism $X \to \mathbb{P}(\mathrm{End}(V))$ (see [DS, 3.15]). The image $\rho(\overline{F} - F)$ consists of nilpotent lines in $\mathrm{End}(V)$. On the other hand a lowest weight vector is an eigenvector of $\rho(\dot{w})$ with a nonzero eigenvalue and $\rho(h_J)$ is projection on the line of lowest weight vectors. Using (1) and Proposition 1 (i) one sees that this contradicts nilpotency.

(c) $\overline{F}$ and X_J intersect properly if $J \neq I$.

It suffices to prove this for $J = \emptyset$. From (a) it follows that $\dim \overline{F} \cap X_\emptyset \geq \dim F - |I|$. (b) implies, on the other hand, that the intersection has dimension $\leq \dim F - |I|$.

(d) Let $i \in I$. By (c) $\dim(\overline{F} \cap X_{I-\{i\}}) = \dim F - 1$. By Theorem 2 (i) there must be a set $X_{J,w}$ whose intersection with $\overline{F} \cap X_{I-\{i\}}$ is dense in $X_{I-\{i\}}$. Then $J \subset I - \{i\}$ and

$$\dim G - l(w) - |I - J| = \dim X_{J,w} \geq \dim F - 1 = \dim G - |I| - 1,$$

and $l(w) \leq |J| + 1$. But by (b) $l(w) \geq |I|$. We conclude that $|J| = |I| - 1$ and $l(w) = |I|$. We then must have $J = I - \{i\}$. Moreover, w is a Coxeter element i.e., $\mathrm{supp}(w) = I$ and $l(w) = |I|$.

(e) $W^{I-\{i\}}$ contains a unique Coxeter element c_i.

This is proved by induction on $|I|$.

(f) We conclude from (d) that $\overline{F} \cap X_{I-\{i\}} = \overline{X_{I-\{i\},c_i}}$ for all $i \in I$. Since for $J \neq I$

$$X_J = \bigcap_{i \notin J} X_{I-\{i\}}$$

this implies that the intersection $\overline{F} \cap (X - G)$ is independent of F. With a little more work the theorem follows.

3.4. Algebro-geometric properties. Let X_{sc} be a compactification of the simply connected cover G_{sc}. Let X_i $(1 \leq i \leq n)$ be the irreducible components of $X_{\text{sc}} - G_{\text{sc}}$. They all have codimension 1.

Proposition 7. *Let $p > 0$. Let F be a Steinberg fiber in G_{sc}.*

(i) *X_{sc} admits a Frobenius splitting which compatibly splits $\overline{F}$ and the X_i $(1 \leq i \leq n)$.*

(ii) *$\overline{F}$ is normal and Cohen–Macaulay.*

For (a somewhat stronger version of) (i) see [T, Thm. 8.2] and for (ii) [loc. cit., Thm. 10.2].

Notice that this result covers the wonderful compactification X of G if G has trivial center. For arbitrary adjoint G a partial result is proved in [LT]. For $i \in I$ let χ_i be the fundamental character of G_{sc} associated to i. Put

$$\tilde{F}_0 = \{g \in G_{\text{sc}} \mid \chi_i(g) = 0 \text{ for all } i\},$$

this is a Steinberg fiber in G_{sc}. Its image in G is a Steinberg fiber F_0 in G, the zero fiber.

Proposition 8. *Let $p > 0$. X admits a Frobenius splitting which compatibly splits F_0 and the components of $X - G$.*

See [loc. cit., Thm. 8.1]. It is also pointed out that the result cannot be true for arbitrary Steinberg fibers in G.

Remark. The appearance of the zero fiber is somewhat curious. Over $\mathbb{C}$ it appears in [Ka] in another context. I learned from J.-P. Serre (private communication) that for a quasi-simple group over $\mathbb{C}$, F_0 has been determined (case by case). Its elements are regular and have finite order. The characteristic p case does not seem to have been analyzed.

Problem 4. $(p > 0)$ Does X admit a Frobenius splitting which compatibly splits an arbitrary $\overline{F}$?

Problem 5. Is $\overline{F}$ normal and Cohen–Macaulay?

Problem 6. Does $\overline{F}$ admit a cellular decomposition?

An example given in [Sp2, 4.3] with $G = \mathrm{PGL}_3$ shows that $\overline{F}$ need not be smooth.

Problem 7. Determine the intersection cohomology (local and global) of $\overline{F}$.

4. Character sheaves on X

In this section G is an adjoint group and X is its wonderful compactification.

4.1. $B \times B$-equivariant perverse sheaves. The definition of character sheaves on X uses certain $B \times B$-equivariant perverse sheaves on X, which we first have to introduce.

If S is a torus let $C(S)$ be its character group and put

$$\hat{C}(S) = C(S) \otimes (\mathbb{Z}_{(p)}/\mathbb{Z}),$$

where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at the prime ideal (p). The elements of $\hat{C}(S)$ parametrize tame rank one local systems on S (also called Kummer local systems). We work in l-adic cohomology, with a coefficient field E (e.g. $\overline{\mathbb{Q}_l}$).

Let $v = [J, x, w]$ be a $B \times B$-orbit in X, as in Proposition 1. Using that it is a homogeneous space for $B \times B$ one constructs a morphism $\phi\colon v \to T_J$, where T_J is the maximal torus of G_J of 2.1 (see [H3, 3.1]). For $\xi \in \hat{C}(T_J)$ we have a local system $\mathcal{L}_{\xi,v} = \phi^*\xi$ on v. Let $\mathcal{I}_{\xi,v}$ be its perverse extension, a perverse sheaf on X (for l-adic cohomology) supported by $\bar{v}$, whose restriction to v is $\mathcal{L}_{\xi,v}[\dim v]$.

[Sp1, sect. 5] deals with these perverse sheaves. It is shown that they are even, i.e. that their cohomology sheaves are zero in dimensions $\not\equiv \dim v \pmod 2$.

In the next lemma one uses that $\hat{C}(T_J)$ can be viewed as a subset of $\hat{C}(T)$ (see [loc. cit., 1.7]) and that W acts on $\hat{C}(T)$.

Lemma 3. *If $x.\xi = w.\xi$ then $\mathcal{I}_{\xi,v}$ is a B_{d}-equivariant irreducible perverse sheaf on X.*

See [H3, 3.1].

4.2. Character sheaves. Character sheaves on a reductive group were introduced by G. Lusztig in the 1980s, in a long series of papers. [L1] gives a brief exposition of the results of these papers. [MS] is a report on part of the results. The definition of character sheaves used there is slightly different from Lusztig's.

In [L2] Lusztig defines character sheaves on the compactification X. I noticed (unpublished) that the approach of [MS] could also be followed to do this. But it is not obvious that the two definitions of character sheaves on X are equivalent.

Independently, Xuhua He also came to the definition based on [MS]. He proved in [H3] the equivalence with Lusztig's definition. I shall not go into Lusztig's definition. I will only report on the other one.

B acts on $G \times X$ by $b.(g, x) = (gb^{-1}, (b, b).x)$. Let $G \times_B X$ be the quotient and $\alpha\colon G \times X \to G \times_B X$ the quotient map. The G_{d}-action on X induces a proper morphism $\mu\colon G \times_B X \to X$.

Let $\mathcal{I}_{\xi,v}$ be as in Lemma 3. Then $A = E[\dim G] \boxtimes \mathcal{I}_{\xi,v}$ is an irreducible perverse sheaf on $G \times X$ and there is an irreducible perverse sheaf $\tilde{A}$ on $G \times_B X$ with $A = \alpha^*\tilde{A}$.

Put $C_{\xi,v} = \mu_! \tilde{A}$. By the decomposition theorem this is a semisimple complex on X, i.e. a direct sum of shifted irreducible perverse sheaves on X. The perverse sheaves occurring in the $C_{\xi,v}$ (if ξ and v vary) are the character sheaves on X. They are G_{d}-equivariant.

The nonzero restrictions of these character sheaves to the open subvariety G of X are Lusztig's original character sheaves. More generally, for $J \subset D$ we call character sheaf on X_J the restriction to X_J of a character sheaf on X which is obtained as above from an orbit $v \subset X_J$.

The character sheaves on X deserve a further study. Here are a few problems.

Problem 8. Analyze the restriction of a character sheaf on X to a $G \times G$-orbit X_J. Can such restrictions be described in terms of character sheaves on X_J?

Problem 9. Are character sheaves on X even?

4.3. Finite ground fields. Now let k be an algebraic closure of the finite field $\mathbb{F}_q$ and let $F: a \mapsto a^q$ be the Frobenius automorphism of k. Assume that G is defined over k. Then so is X, by [DS, Prop. 3.11].

Let A be a character sheaf on G whose support is not contained in $X - G$. The restriction of A to G is a character sheaf on G. Assume that A "comes from $\mathbb{F}_q$", meaning that $F^*A \simeq A$. Fix an isomorphism $\phi\colon F^*A \simeq A$. It can be normalized such as to be unique up to a root unity (see [L1, p. 178]).

$x \in X^F = X(\mathbb{F}_q)$ being an $\mathbb{F}_q$-rational point of X, ϕ defines linear maps ϕ_x^i of the cohomology stalks $H^i(A)_x$. Define a function χ_ϕ on X^F by

$$\chi_\phi(x) = \Sigma_i (-1)^i \operatorname{Tr}(\phi_x^i, H^i(A)_x).$$

For $x \in G$ one obtains a class function on the finite Lie group G^F, which can be viewed as a generalized character of G^F. This function on G^F has boundary values, viz. the values of χ_ϕ on points of $X^F - G^F$.

Problem 10. Can one define and compute boundary values of irreducible characters of G^F?

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