Quasiconformal geometry of fractals

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Abstract. Many questions in analysis and geometry lead to problems of quasiconformal geometry on non-smooth or fractal spaces. For example, there is a close relation of this subject to the problem of characterizing fundamental groups of hyperbolic 3-orbifolds or to Thurston's characterization of rational functions with finite post-critical set.

In recent years, the classical theory of quasiconformal maps between Euclidean spaces has been successfully extended to more general settings and powerful tools have become available. Fractal 2-spheres or Sierpiński carpets are typical spaces for which this deeper understanding of their quasiconformal geometry is particularly relevant and interesting.

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1. Introduction

A homeomorphism on $\mathbb{R}^n$ is called *quasiconformal* if it maps infinitesimal balls to infinitesimal ellipsoids with uniformly controlled eccentricity. While in its substance this notion (for $n = 2$) was introduced by Grötzsch in the late 1920s (see [Kü] for an account of Grötzsch's work), the term "quasiconformal" was first used by Ahlfors in 1935 [Ah, p. 213 and p. 242].

The importance of planar quasiconformal mappings was only fully realized after Teichmüller had published his groundbreaking work on the classical moduli problem for Riemann surfaces around 1940. It took another two decades after the foundational issues of the theory of quasiconformal mappings had been clarified. A great subtlety here is what a priori smoothness assumption to impose on a quasiconformal map in order to get a theory with the desired compactness properties under limits. It turned out that some requirement of Sobolev regularity is appropriate.

Nowadays planar quasiconformal maps are recognized as a standard tool in various areas of complex analysis such as Teichmüller theory, Kleinian groups, and complex dynamics. One of the main reasons for this is that in the plane a flexible existence theorem for quasiconformal maps is available in the Measurable Riemann Mapping Theorem.

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In higher dimensions there is no such existence theorem putting severe limitations to the theory. Accordingly, quasiconformal maps for $n \geq 3$ initially had a less profound impact than their planar relatives. This situation changed when Mostow used the theory of higher-dimensional quasiconformal maps in the proof of his celebrated rigidity theorems for rank-one symmetric spaces [Mo]. In this context it also became desirable to extend the classical theory of quasiconformal mappings on $\mathbb{R}^n$ to other settings such as Heisenberg groups which arise as boundaries of complex hyperbolic spaces [KR1], [KR2], [Pa1]. A continuation of this trend was a theory of quasiconformal maps in a general metric space context and recently led to Heinonen and Koskela's theory of quasiconformal maps on Loewner spaces and spaces satisfying a Poincaré inequality [HK].

In this survey we will focus on questions of quasiconformal geometry of low-dimensional fractals such as 2-spheres and Sierpiński carpets. In particular, we will discuss uniformization and rigidity theorems that are motivated by questions in geometric group theory and complex dynamics. For some additional topics not discussed here, we refer to B. Kleiner's article in these conference proceedings.

2. Quasiconformal and quasisymmetric maps

We first give a precise definition of various classes of maps that are related to the concept of quasiconformality (see [BI], [He], [Vä1] for more details).

A homeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 2$, is called *quasiconformal*, if f belongs to the Sobolev space $W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ and if there exists a constant $K \geq 1$ such that the inequality

$$\|Df(x)\|^n \leq K|J_f(x)| \tag{1}$$

is valid for almost every $x \in \mathbb{R}^n$. Here $\|Df(x)\|$ is the norm of the formal differential Df and $J_f = \det(Df)$ is the Jacobian determinant of f. If we want to emphasize K, then we say that f is *K-quasiconformal*. We will use similar language below for concepts that depend on parameters. We exclude $n = 1$ in the following, because in this case the theory of quasiconformal maps has somewhat different features than for $n \geq 2$. An important fact about quasiconformal maps is that they are differentiable almost everywhere. Hence we can replace the formal quantity Df in (1) by the classical derivative.

The "analytic" definition of quasiconformality above applies to more general smooth settings, for example, if f is a map between Riemannian n-manifolds. If one drops the requirement that f is a homeomorphism and allows branching, then one is led to the concept of a *quasiregular* map [Re], [Ri].

A general definition of a quasiconformal map in a metric space context (the "metric" definition) can be given as follows. Suppose $f\colon X \to Y$ is a homeomorphism between metric spaces (X, d_X) and (Y, d_Y). Then f is *quasiconformal* if there exists a constant $H \geq 1$ such that

$$H_f(x) := \limsup_{r \to 0^+} \frac{\sup\{d_Y(f(x'), f(x)) : d_X(x, x') \leq r\}}{\inf\{d_Y(f(x'), f(x)) : d_X(x, x') \geq r\}} \leq H$$

for all $x \in X$. In $\mathbb{R}^n$, $n \geq 2$, this is equivalent to the analytic definition. In general, this concept is rather weak and does not lead to a useful theory, if one does not impose further restrictions on the spaces X and Y.

A related, but stronger concept is the notion of a quasisymmetric map [TV]. By definition a homeomorphism $f: X \to Y$ is called *quasisymmetric*, if there exists a homeomorphism $\eta: [0, \infty) \to [0, \infty)$ (which plays the role of a distortion function) such that

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta\left(\frac{d_X(x, y)}{d_X(x, z)}\right),$$

whenever $x, y, z \in X$, $x \neq z$.

The geometric meaning of this condition is that balls are mapped to "round" sets with quantitative control for their "eccentricity". This is a global version of the geometric property of a quasiconformal map. In $\mathbb{R}^n$, $n \geq 2$, a map is quasiconformal if and only if it is quasisymmetric. Inverse maps or compositions of quasisymmetric maps are again quasisymmetric. So the quasisymmetric maps $f: X \to X$ on a metric space X form a group that we denote by $\mathrm{QS}(X)$.

In particular when a group action is present, the concept of a quasi-Möbius map is often more suitable than that of a quasisymmetry. Here a homeomorphism $f: X \to Y$ is called a *quasi-Möbius map* [Vä2] if there exists a homeomorphism $\eta: [0, \infty) \to [0, \infty)$ such that for every 4-tuple (x_1, x_2, x_3, x_4) of distinct points in X, we have the inequality

$$[f(x_1), f(x_2), f(x_3), f(x_4)] \leq \eta([x_1, x_2, x_3, x_4])$$

for the *metric cross-ratio*

$$[z_1, z_2, z_3, z_4] = \frac{d_X(z_1, z_3) d_X(z_2, z_4)}{d_X(z_1, z_4) d_X(z_2, z_3)}.$$

Every quasisymmetric map $f: X \to Y$ between metric spaces is also quasi-Möbius. This statement is "quantitative" in the following sense: If f is η-quasisymmetric, then f is $\tilde{\eta}$-quasi-Möbius with $\tilde{\eta}$ only depending on η. Conversely, every quasi-Möbius map between bounded metric spaces is quasisymmetric, but in contrast to the other implication this statement is not quantitative in general. One can also get a quantitative statement here if one assumes that the spaces are bounded and adds a normalization condition for the map.

There is a third way to characterize quasiconformal maps on $\mathbb{R}^n$ based on the concept of the modulus of a path family. We first review some definitions related to this.

Suppose (X, d, μ) is a metric measure space, i.e., (X, d) is a metric space and μ a Borel measure on X. Moreover, we assume that (X, d) is locally compact and that μ is locally finite and has dense support.

The space (X, d, μ) is called (*Ahlfors*) *Q-regular*, $Q > 0$, if the measure μ satisfies

$$C^{-1}R^Q \le \mu(\bar{B}(a, R)) \le CR^Q$$

for each closed ball $\bar{B}(a, R)$ of radius $0 < R \le \operatorname{diam}(X)$ and for some constant $C \ge 1$ independent of the ball. If the measure is not specified, then it is understood that μ is Q-dimensional Hausdorff measure. Note that a complete Ahlfors regular space X is *proper*, i.e., closed balls in X are compact.

A *density* on X is a Borel function $\rho\colon X \to [0, \infty]$. A density ρ is called *admissible* for a path family Γ in X, if

$$\int_\gamma \rho\, ds \ge 1$$

for each locally rectifiable path $\gamma \in \Gamma$. Here integration is with respect to arclength on γ.

If $Q \ge 1$, the *Q-modulus* of a family Γ of paths in X is the number

$$\operatorname{Mod}_Q(\Gamma) = \inf \int \rho^Q\, d\mu,$$

where the infimum is taken over all densities $\rho\colon X \to [0, \infty]$ that are admissible for Γ. If E and F are subsets of X with positive diameter, we denote by

$$\Delta(E, F) = \frac{\operatorname{dist}(E, F)}{\min\{\operatorname{diam}(E), \operatorname{diam}(F)\}}$$

the *relative distance* of E and F, and by $\Gamma(E, F)$ the family of all paths in X connecting E and F.

Suppose (X, d, μ) is a connected metric measure space. Then X is called a *Q-Loewner space*, $Q \ge 1$, if there exists a positive decreasing function $\Psi\colon (0, \infty) \to (0, \infty)$ such that

$$\operatorname{Mod}_Q(\Gamma(E, F)) \ge \Psi(\Delta(E, F)), \tag{2}$$

whenever E and F are disjoint continua in X.

Condition (2) axiomatizes a property of n-modulus in $\mathbb{R}^n$ relevant for quasiconformal geometry to a general metric space setting. Examples for Loewner spaces are $\mathbb{R}^n$, all compact Riemannian manifolds, Carnot groups (such as the Heisenberg group), and boundaries of some Fuchsian buildings. Equivalent to the Loewner property of a space is that it satisfies a Poincaré inequality (see [HK], [He] for more discussion on these topics).

The following theorem is a combination of results by Heinonen and Koskela [HK] and by Tyson [Ty], and characterizes quasiconformal maps on Loewner spaces by a distortion property for modulus.

Theorem 2.1. *Let X and Y be Q-regular Q-Loewner spaces, $Q > 1$, and $f: X \to Y$ a homeomorphism. Then f is quasiconformal if and only if there exists a constant $K \geq 1$ such that*

$$\frac{1}{K}\operatorname{Mod}_Q(\Gamma) \leq \operatorname{Mod}_Q(f(\Gamma)) \leq K \operatorname{Mod}_Q(\Gamma) \tag{3}$$

for every family Γ of paths in X, where $f(\Gamma) = \{f \circ \gamma : \gamma \in \Gamma\}$.

In $\mathbb{R}^n$ this characterization of a quasiconformal map is known as the "geometric" definition. The reason for this terminology is that (3) immediately leads to strong geometric consequences. For example, using this one can show that f is quasisymmetric.

The fact that the analytic, metric, and geometric definitions of a quasiconformal map on $\mathbb{R}^n$ are (quantitatively) equivalent is a rather deep fact. This has recently been generalized to the general Loewner space setting [HKST].

3. The quasisymmetric uniformization problem

The classical uniformization theorem implies that every Riemann surface is conformally equivalent to a "standard" surface carrying a metric of constant curvature. One can ask whether a general metric space version of this fact is true where the class of conformal maps is replaced by the more flexible class of quasisymmetric homeomorphisms.

Quasisymmetric uniformization problem. Suppose X is a metric space homeomorphic to some "standard" metric space Y. When is X quasisymmetrically equivalent to Y?

Here we call two metric spaces X and Y *quasisymmetrically equivalent* if there exists a quasisymmetric homeomorphism from X onto Y. Equivalently, one may ask for a quasisymmetric characterization of Y. Of course, it depends on the context how the term "standard" is precisely interpreted. This general question is motivated by problems in geometric group theory, for example (see Sections 5 and 7). One can also pose a similar uniformization problem for other classes of maps, for example bi-Lipschitz maps (recall that a homeomorphism between metric spaces is called *bi-Lipschitz* if it distort distances by at most a fixed multiplicative amount).

The prime instance for a quasisymmetric uniformization result is the following theorem due to Tukia and Väisälä that characterizes *quasicircles*, i.e., quasisymmetric images of the unit circle $\mathbb{S}^1$ [TV].

Theorem 3.1. *Let X be a metric space homeomorphic to $\mathbb{S}^1$. Then X is quasisymmetrically equivalent to $\mathbb{S}^1$ if and only if X is doubling and linearly locally connected.*

Here a metric space X is called *doubling* if there exists a number N such that every ball in X of radius R can be covered by N balls of radius $R/2$. A metric space X

is called *linearly locally connected* if there exists a constant $\lambda \geq 1$ satisfying the following conditions: If $B(a, r)$ is an open ball in X and $x, y \in B(a, r)$, then there exists a continuum in $B(a, \lambda r)$ connecting x and y. Moreover, if $x, y \in X \setminus B(a, r)$, then there exists a continuum in $X \setminus B(a, r/\lambda)$ connecting x and y. The geometric significance of this condition is that it rules out "cusps" of the space.

A quasisymmetric characterization of the standard $1/3$-Cantor set can be found in [DS]. Work by Semmes [Se1], [Se2] shows that the quasisymmetric characterization of $\mathbb{R}^n$ or the standard sphere $\mathbb{S}^n$ for $n \geq 3$ is a problem that seems to be beyond reach at the moment. The intermediate case $n = 2$ is particularly interesting.

For a metric 2-sphere X to be quasisymmetrically equivalent to the standard 2-sphere $\mathbb{S}^2$ it is necessary that X is linearly locally connected. This alone is not sufficient, but will be if a mass bound assumption is added [BK1].

Theorem 3.2. *Suppose X is a metric space homeomorphic to $\mathbb{S}^2$. If X is linearly locally connected and Ahlfors 2-regular, then X is quasisymmetrically equivalent to $\mathbb{S}^2$.*

This answers a question by Heinonen and Semmes [HS]. A similar result for other simply connected surfaces has been obtained by K. Wildrick [Wi].

The proof of Theorem 3.2 uses approximations of X by graphs that are combinatorially equivalent to triangulations of $\mathbb{S}^2$. Realizing such a triangulation as an incidence graph of a circle packing on $\mathbb{S}^2$, one finds a mapping of X to $\mathbb{S}^2$ on a coarse scale. The main difficulty is to show the subconvergence of this procedure. For this one controls the quasisymmetric distortion by modulus estimates. Incidentally, a very similar algorithm has recently been used to obtain mappings of the surface of the human brain [H-R].

The assumption of Ahlfors regularity for some exponent $Q \geq 2$ is quite natural, because it is satisfied in many interesting cases (for boundaries of Gromov hyperbolic groups for example; see Section 4). There are metric 2-spheres X though that are linearly locally connected and Q-regular with $Q > 2$, but are not quasisymmetrically equivalent to $\mathbb{S}^2$ [Vä3].

The following result can be proved similarly as Theorem 3.2 [BK1].

Theorem 3.3. *Let $Q \geq 2$ and Z be an Ahlfors Q-regular metric space homeomorphic to $\mathbb{S}^2$. If Z is Q-Loewner, then $Q = 2$ and Z is quasisymmetrically equivalent to $\mathbb{S}^2$.*

Note that an analog of this is false in higher dimensions. For example, one can equip $\mathbb{S}^3$ with a Carnot–Carathéodory metric d modeled on the geometry of the Heisenberg group. Then $(\mathbb{S}^3, d)$ is 4-regular and 4-Loewner, but not quasisymmetrically equivalent to standard $\mathbb{S}^3$ [Se1].

In [BK1] a necessary and sufficient condition was established for a metric 2-sphere to be quasisymmetrically equivalent to $\mathbb{S}^2$. This condition is in terms of the behavior of some combinatorially defined moduli of ring domains and too technical to be stated here. The usefulness of any such characterization depends on whether one can verify its hypotheses in concrete situations such as for fractal 2-spheres coming from dynamical systems as considered in the following Sections 5 and 6.

4. Gromov hyperbolic spaces and quasisymmetric maps

As a preparation for the next sections, we quickly review some standard material on Gromov hyperbolic spaces and groups [GH], [Gr].

Let (X, d) be a metric space, and $\delta \geq 0$. Then X is called *δ-hyperbolic*, if the inequality

$$(x \cdot z)_p \geq \min\{(x \cdot y)_p, (y \cdot z)_p\} - \delta$$

holds for all $x, y, z, p \in X$, where

$$(u \cdot v)_p = \frac{1}{2}\big(d(u, p) + d(v, p) - d(u, v)\big)$$

for $u, v \in X$. If the space X is *geodesic* (this means that any two point in X can be joined by a path whose length is equal to the distance of the points), then the δ-hyperbolicity of X is equivalent to a thinness condition for geodesic triangles. We say that X is *Gromov hyperbolic* if X is δ-hyperbolic for some $\delta \geq 0$. Roughly speaking, this requires that the space is "negatively curved" on large scales. Examples for such spaces are simplicial trees, or Cartan–Hadamard manifolds with a negative upper curvature bound such as the real hyperbolic spaces $\mathbb{H}^n$, $n \geq 2$.

To each Gromov hyperbolic space one can associate a boundary at infinity $\partial_\infty X$ as follows. Fix a basepoint $p \in X$, and consider sequences of points $\{x_i\}$ in X *converging to infinity* in the sense that

$$\lim_{i,j\to\infty} (x_i \cdot x_j)_p = \infty.$$

We declare two such sequences $\{x_i\}$ and $\{y_i\}$ in X as *equivalent* if

$$\lim_{i\to\infty} (x_i \cdot y_i)_p = \infty.$$

Now the *boundary at infinity* $\partial_\infty X$ is defined as the set of equivalence classes of sequences converging to infinity. It is easy to see that the choice of the basepoint p does not matter here. If X is in addition proper and geodesic, then there is an equivalent definition of $\partial_\infty X$ as the set of equivalence classes of geodesic rays emanating from the basepoint p. One declares two such rays as equivalent if they stay within bounded Hausdorff distance. Intuitively, a ray represents its "endpoint" on $\partial_\infty X$. If $\mathbb{H}^n$ is given by the unit ball model, then from this point of view it is clear that $\partial_\infty \mathbb{H}^n = \mathbb{S}^{n-1}$.

The boundary $\partial_\infty X$ comes naturally equipped with a class of "visual" metrics. By definition a metric ρ on $\partial_\infty X$ is called *visual* if there exist $p \in X$, $C \geq 1$, and $\varepsilon > 0$ such that

$$\frac{1}{C}\exp(-\varepsilon(a \cdot b)_p) \leq \rho(a, b) \leq C\exp(-\varepsilon(a \cdot b)_p) \tag{4}$$

for all $a, b \in \partial_\infty X$. In this inequality we used the fact that a "product" $(a\cdot b)_p \in [0, \infty]$ can also be defined for points $a, b \in \partial_\infty X$ in a natural way. Here we have $(a\cdot b)_p = \infty$ if and only if $a = b \in \partial_\infty X$. If X is δ-hyperbolic, then there exists a visual metric ρ with parameter ε if $\varepsilon > 0$ is small enough depending on δ.

In the following we always think of $\partial_\infty X$ as a metric space by equipping it with a fixed visual metric. If ρ_1 and ρ_2 are two visual metrics on $\partial_\infty X$, then the identity map is a quasisymmetric map between (X, ρ_1) and (X, ρ_2) (the visual metrics form a *conformal gauge* – see [He, Ch. 15] for this terminology and further discussion). So the ambiguity of the visual metric is irrelevant if one wants to speak of quasisymmetric maps on $\partial_\infty X$. One should consider the space $\partial_\infty X$ equipped with such a visual metric ρ as very "fractal". For example, if the parameter ε in (4) is very small, then $(\partial_\infty X, \rho)$ will not contain any non-constant rectifiable curves.

Suppose (X, d_X) and (Y, d_Y) are metric spaces. A map $f\colon X \to Y$ is called a *quasi-isometry* if there exist constants $\lambda \geq 1$ and $k \geq 0$ such that

$$\frac{1}{\lambda} d_X(u, v) - k \leq d_Y(f(u), f(v)) \leq \lambda d_X(u, v) + k$$

for all $u, v \in X$ and if

$$\inf_{x \in X} d_Y(f(x), y) \leq k$$

for all $y \in Y$. The spaces X and Y are called *quasi-isometric* if there exists a quasi-isometry $f\colon X \to Y$.

Quasi-isometries form a natural class of maps in the theory of Gromov hyperbolic spaces. For example, Gromov hyperbolicity of geodesic metric spaces is invariant under quasi-isometries. The following fact links these concepts to quasisymmetric maps (see [BS] for more on this subject).

Proposition 4.1. *Let X and Y be proper and geodesic Gromov hyperbolic spaces. Then every quasi-isometry $f\colon X \to Y$ induces a natural quasisymmetric boundary map $\tilde{f}\colon \partial_\infty X \to \partial_\infty Y$.*

The boundary map $\tilde{f}$ is defined by assigning to a point $a \in \partial_\infty X$ represented by the sequence $\{x_i\}$ the point $b \in \partial_\infty Y$ represented by the sequence $\{f(x_i)\}$.

This proposition constitutes a core element in Mostow's proof of rigidity of rank-one symmetric spaces. The point is that a quasi-isometry may locally exhibit very irregular behavior, but gives rise to a quasisymmetric boundary map that can be analyzed by analytic tools.

Suppose G is a finitely generated group, and let S be a set of generators. We will always assume that S is finite and *symmetric* in the sense that if $s \in S$, then $s^{-1} \in S$. The *Cayley graph* $C(G, S)$ of G with respect to S is a graph that has G as its set of vertices. Moreover, we connect two distinct vertices $x, y \in G$ by an edge if there exists $s \in S$ such that $y = xs$. The graph $C(G, S)$ carries a natural path metric that assigns length 1 to each edge. By considering its Cayley graph, one can study properties of the group from a geometric point of view.

The group G is called *Gromov hyperbolic* if $C(G, S)$ is Gromov hyperbolic for some set S of generators of G. If this is the case, then $C(G, S')$ is Gromov hyperbolic for all generating sets S'. This essentially follows from the fact that $C(G, S)$ and $C(G, S')$ are bi-Lipschitz equivalent (and in particular quasi-isometric). This also implies that if we define $\partial_\infty G := \partial_\infty C(G, S)$, then by Proposition 4.1 the boundary at infinity of G is well-defined up to quasisymmetric equivalence. Examples of Gromov hyperbolic groups are free groups and fundamental groups of negatively curved manifolds.

Letting a group element $g \in G$ act on the vertices of $C(G, S)$ by left-translation, we get a natural action $G \curvearrowright C(G, S)$ by isometries. According to Proposition 4.1 this induces a group action $G \curvearrowright \partial_\infty G$, where each group element acts as a quasisymmetric map. In general the distortion function η will be different for different group elements. This changes if one uses the concept of quasi-Möbius maps. In this case the action $G \curvearrowright \partial_\infty G$ is *uniformly quasi-Möbius*, i.e., there exists a distortion function η such that every element $g \in G$ acts as an η-quasi-Möbius map on $\partial_\infty G$ [Pau].

Another important property of the action $G \curvearrowright \partial_\infty G$ is its "cocompactness on triples". Denote by $\mathrm{Tri}(X)$ the space of triples of distinct points in a space X. The action $G \curvearrowright \partial_\infty G$ induces an action $G \curvearrowright \mathrm{Tri}(\partial_\infty G)$. This action is discrete and cocompact, and Gromov hyperbolic groups are characterized by this property according to a theorem by Bowditch [Bow].

5. Cannon's conjecture and fractal 2-spheres

It is a natural question to what extent the structure of a Gromov hyperbolic group G is reflected in its boundary $\partial_\infty G$. For example, $\partial_\infty G$ is totally disconnected iff G is virtually free, i.e., it contains a free group of finite index. Similarly, $\partial_\infty G$ is a topological circle iff G is virtually Fuchsian (see [KB] for these and related results). The case where $\partial_\infty G$ is a topological 2-sphere is covered by the following conjecture [Ca].

Conjecture 5.1 (*Cannon's conjecture, Version* I). Suppose G is a Gromov hyperbolic group whose boundary at infinity $\partial_\infty G$ is homeomorphic to $\mathbb{S}^2$. Then there exists an action of G on hyperbolic 3-space $\mathbb{H}^3$ that is isometric, properly discontinuous, and cocompact.

If true, this conjecture would essentially give a characterization of fundamental groups of closed hyperbolic 3-orbifolds from the point of view of geometric group theory. The conjecture is equivalent to a quasisymmetric uniformization problem.

Conjecture 5.2 (*Cannon's conjecture, Version* II). Suppose G is a Gromov hyperbolic group whose boundary at infinity $\partial_\infty G$ is homeomorphic to $\mathbb{S}^2$. Then $\partial_\infty G$ is quasisymmetrically equivalent to $\mathbb{S}^2$.

Indeed, if Conjecture 5.2 holds, then we can conjugate the natural action of G on $\partial_\infty G$ to a uniformly quasiconformal action of G on $\mathbb{S}^2$. By a well-known theorem due to Sullivan [Su1] and to Tukia [Tu] such an action is conjugate to an action of G on $\mathbb{S}^2$ by Möbius transformations. Considering $\mathbb{S}^2$ as the boundary at infinity of $\mathbb{H}^3$, we can extend this action to an isometric action of G on $\mathbb{H}^3$ with the desired properties.

Conversely, if G acts on $\mathbb{H}^3$ isometrically, properly discontinuously, and cocompactly, then the Cayley graph of G with respect to any (finite and symmetric) set of generators is quasi-isometric to $\mathbb{H}^3$. This quasi-isometry induces the desired quasisymmetric equivalence between $\partial_\infty G$ and $\partial_\infty \mathbb{H}^3 = \mathbb{S}^2$.

Cannon, Floyd, and Parry [CFP] have attempted to settle Conjecture 5.1 by using subdivision rules and Cannon's Combinatorial Riemann Mapping Theorem [Ca]. A different approach is due to B. Kleiner and the author. In [BK4] recently developed techniques from analysis on metric spaces were used and led to the following theorem.

Theorem 5.3. *Suppose G is a Gromov hyperbolic group whose boundary at infinity $\partial_\infty G$ is homeomorphic to $\mathbb{S}^2$. If the Ahlfors regular conformal dimension of $\partial_\infty G$ is attained as a minimum, then ∂G_∞ is quasisymmetrically equivalent to $\mathbb{S}^2$.*

Here the (*Ahlfors regular*) *conformal dimension* of a metric space X is defined as

$$\dim_{\mathrm{c}} X = \inf\{Q : \text{there exists an Ahlfors } Q\text{-regular metric space } Y \\ \text{that is quasisymmetrically equivalent to } X\}. \tag{5}$$

This concept was implicitly introduced by Bourdon and Pajot [BP]. Pansu [Pa2] has defined a related, but different concept of conformal dimension of a space. If $X = \partial_\infty G$, where G is a Gromov hyperbolic group, then the set over which the infimum in (5) is taken is nonempty, because the boundary of a Gromov hyperbolic group equipped with a visual metric is Ahlfors regular [Co].

In more intuitive terms, the above result can be formulated as follows: Let G be a Gromov hyperbolic group as in Cannon's conjecture. If a certain infimum for $\partial_\infty G$ is attained as a minimum (related to how much we can "squeeze" the space by a quasisymmetric map while retaining Ahlfors regularity for some exponent), then the desired conclusion holds.

The proof of this theorem depends on a recent result by Keith and Laakso [KL] which essentially says that if $Q > 1$ is the conformal dimension of a Q-regular space X, then X has a *weak tangent* (see [BBI, Ch. 8] for the definition and [BK2] for related discussions), carrying a family of non-constant paths with positive Q-modulus.

In our situation $X = \partial_\infty G$ is the boundary of a Gromov hyperbolic group G, and is equipped with a metric that comes from the minimizer in (5). Every weak tangent of X is quasisymmetrically equivalent to $\partial_\infty G$ minus a point [BK2]. Therefore, $\partial_\infty G$ itself carries a family of positive Q-modulus, where Q is the conformal dimension of $\partial_\infty G$. The main work now consists in showing that the natural group action $G \curvearrowright \partial_\infty G$ allows one to promote this to the stronger conclusion that $\partial_\infty G$ has families of non-constant paths with uniformly positive Q-modulus on all locations and scales; more precisely, that $X = \partial_\infty G$ is a Q-regular Q-Loewner space. Up to this point, the assumption that $\partial_\infty G$ is homeomorphic to $\mathbb{S}^2$ was not used. The proof of the above statement is now finished by invoking Theorem 3.3 which shows that $\partial_\infty G$ is quasisymmetrically equivalent to $\mathbb{S}^2$.

Related to these questions is the following problem due to P. Papasoglu: Suppose that G is a Gromov hyperbolic group whose boundary $\partial_\infty G$ is homeomorphic to $\mathbb{S}^2$. Cannon's conjecture predicts that in this situation $\partial_\infty G$ is quasisymmetrically equivalent to $\mathbb{S}^2$; in particular, $\partial_\infty G$ should contain many quasicircles. While Cannon's conjecture is still open, can one at least prove that $\partial_\infty G$ contains a *single* quasicircle? The following result proved in [BK5] settles this in the affirmative.

Theorem 5.4. *The boundary of a Gromov hyperbolic group contains a quasicircle if and only if the group is not virtually free.*

In order to get a better understanding of the relevant issues in Cannon's conjecture, it seems natural to study uniformly quasi-Möbius actions on compact metric spaces X such that the induced action on the space $\mathrm{Tri}(X)$ of triples is discrete and cocompact. In addition to these assumptions it is reasonable to require that X is Ahlfors regular.

If a metric space X is Q-regular, then the exponent Q is at least as big as the topological dimension of X. The borderline case where Q equals the topological dimension of X is of particular interest. In [BK2] (for $n \geq 2$) and [BK3] (for $n = 1$) the following rigidity theorem was proved in all dimensions (see [Su2] and [Yu] for related results).

Theorem 5.5. *Let X be a compact, Ahlfors n-regular metric space of topological dimension $n \in \mathbb{N}$. Suppose that a group G acts on X by uniformly quasi-Möbius maps and that the induced action on $\mathrm{Tri}(X)$ is discrete and cocompact. Then the action $G \curvearrowright X$ is quasisymmetrically conjugate to a Möbius group action on the standard sphere $\mathbb{S}^n$.*

Note that we do not assume that X is homeomorphic to $\mathbb{S}^n$. We get the quasisymmetric equivalence of X and $\mathbb{S}^n$ as part of the conclusion.

6. Post-critically finite rational maps

Apart from Gromov hyperbolic groups, there are other dynamical systems where quasisymmetric uniformization problems arise. Interesting examples are provided by post-critically finite rational maps R on the Riemann sphere $\overline{\mathbb{C}}$ [DH].

Suppose $R\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a holomorphic map of $\overline{\mathbb{C}}$ onto itself, i.e., a rational function. Let Ω_R denote the set of critical points of R, and $P_R = \bigcup_{n\in\mathbb{N}} R^n(\Omega_R)$ be the set of post-critical points of R (here R^n denotes the n-th iterate of R). We make the following assumptions on R:

(i) R is post-critically finite, i.e., P_R is a finite set,

(ii) R has no periodic critical points; this implies that $\mathcal{J}(R) = \overline{\mathbb{C}}$ for the Julia set of R,

(iii) the orbifold $\mathcal{O}_R$ associated with R is hyperbolic (see [DH] for the definition of $\mathcal{O}_R$); this implies that the dynamics of R on the Julia set $\mathcal{J}(R) = \overline{\mathbb{C}}$ is expanding.

A characterization of post-critically finite rational maps is due to Thurston. The right framework is the theory of topologically holomorphic self-maps $f : \mathbb{S}^2 \to \mathbb{S}^2$ of the sphere. By definition these maps have the local form $z \mapsto z^n$ with $n \in \mathbb{N}$ in appropriate local coordinates, and one defines the critical set, the post-critical set, and the associated orbifold similarly as for rational maps. In our context, Thurston's theorem can be stated as follows [DH].

Theorem 6.1. *Let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be a post-critically finite topologically holomorphic map with hyperbolic orbifold. Then f is equivalent to a rational map R if and only if f has no "Thurston obstructions".*

Equivalence has to be understood in an appropriate sense. If f and R are both expanding, this just means conjugacy of the maps.

A Thurston obstruction is defined as follows. A *multicurve* $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ is a system of Jordan curves in $\mathbb{S}^2 \setminus P_f$ with the following properties: the curves have pairwise empty intersection, are pairwise non-homotopic in $\mathbb{S}^2 \setminus P_f$, and non-peripheral (this means that each of the complementary components of a curve contains at least two points in P_f). A multicurve Γ is called *f-stable* if for all j every component of $f^{-1}(\gamma_j)$ is either peripheral or homotopic in $\mathbb{S}^2 \setminus P_f$ to one of the curves γ_i.

If Γ is an f-stable multicurve, fix i and j and label by α the components $\gamma_{i,j,\alpha}$ of $f^{-1}(\gamma_j)$ homotopic to γ_i in $\mathbb{S}^2 \setminus P_f$. Then f restricted to $\gamma_{i,j,\alpha}$ has a mapping degree $d_{i,j,\alpha} \in \mathbb{N}$. Let

$$m_{ij} = \sum_\alpha \frac{1}{d_{i,j,\alpha}}$$

and define the *Thurston matrix* $A(\Gamma)$ of the f-stable multicurve Γ by $A(\Gamma) = (m_{ij})$. This is a matrix with nonnegative coefficients; therefore, it has a largest eigenvalue $\lambda(f, \Gamma) \geq 0$. Then Γ is a *Thurston obstruction* if $\lambda(f, \Gamma) \geq 1$.

In Figure 1 we see topological 2-spheres obtained by gluing together 16 squares (colored black and white in an alternating fashion) for the surface on the left, and two large squares for the surface on the right. The map f is constructed by scaling a white square so that it corresponds to the top square on the right and extending the partially defined map to the whole surface by "Schwarz reflection". Then f is post-critically finite with a set of 4 post-critical points (the corners of the large squares). An f-stable multicurve Γ consisting of one Jordan curve γ is indicated on the right. It has 4 preimages on the left. Two of them are peripheral, and the other ones are homotopic to γ in the complement of P_f. Since the degree of the map on these curves is 2, the Thurston matrix is a (1×1)-matrix with the entry $1/2 + 1/2 = 1$. Hence Γ is a Thurston obstruction and f is not equivalent to a rational map.

Post-critically finite rational maps are related to *subdivision rules* [CFKP], [CFP]. For example, if R is a real rational map, i.e., $R(\overline{\mathbb{R}}) \subseteq \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, satisfying

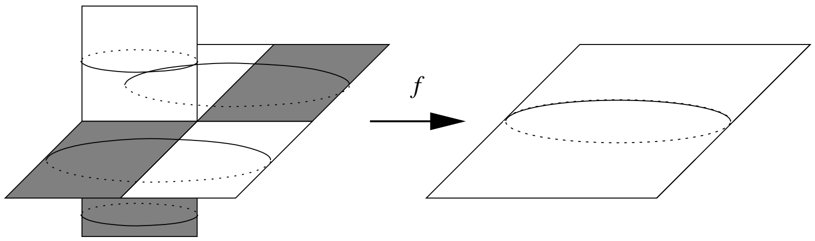

Figure 1. A post-critically finite topologically holomorphic map with a Thurston obstruction.

the above conditions (i)–(iii), and $P_R \subseteq \overline{\mathbb{R}}$, then $R^{-1}(\overline{\mathbb{R}})$ is a graph providing a subdivision of the upper and lower half-planes. It will give rise to a *one-tile subdivision rule*, because the upper and the lower half-planes are subdivided in the same way. The combinatorics of the graphs $R^{-n}(\overline{\mathbb{R}})$ with its corresponding tiles of level n (the closures of the complementary components of the graph $R^{-n}(\overline{\mathbb{R}})$) is determined by iterating the subdivision rule n times. Note that once the subdivision rule is given, the map R admits a completely combinatorial description as an "expanding map" of the subdivision rule by specifying which tiles on level n are mapped to which tiles on level $n-1$. The map f in Figure 1 is also associated with a one-tile subdivision rule which describes how the squares on the right are subdivided into 8 squares each to obtain the combinatorics of the squares on the surface on the left.

For general, not necessarily real rational functions, one expects at least two tile types. More precisely, one can ask whether every rational map satisfying (i)–(iii), or at least a sufficiently high iterate of such a map, is associated with a two-tile subdivision rule. This is indeed the case [BMy], showing that the behavior of the rational maps as discussed admits a combinatorial description.

Theorem 6.2. *Let R be a rational function satisfying* (i)–(iii). *Then there exists an iterate R^n and a quasicircle $C \subseteq \overline{\mathbb{C}}$ such that $P_{R^n} \subseteq C$ and $R^n(C) \subseteq C$.*

A related result has been announced by Cannon, Floyd, and Parry (unpublished).

Conversely, one can start with a two-tile subdivision rule of $\mathbb{S}^2$ (satisfying additional technical assumptions encoding the properties (i)–(iii)). One can associate a natural metric d_λ on $\mathbb{S}^2$ with such a subdivision rule. Roughly speaking, one fixes a parameter $\lambda < 1$ and declares tiles on level n to have size λ^n. The distance $d_\lambda(x, y)$ between two points $x, y \in \mathbb{S}^2$ is then defined as the infimum of all sums of tile-sizes in chains of tiles connecting x and y. Here one has to allow tiles of different levels in a chain. If $\lambda < 1$ is sufficiently close to 1, this gives a metric d_λ on $\mathbb{S}^2$ such that the diameter of a tile on level n is comparable to λ^n. The ambiguity in the parameter λ is not very serious and leads to quasisymmetrically equivalent metrics. These metrics d_λ form an analog of the visual metrics on the boundary of a Gromov hyperbolic group.

If we denote by X the sphere $\mathbb{S}^2$ equipped with this metric, then X is Ahlfors regular and linearly locally connected, and the subdivision rule produces a topologically holomorphic expanding map $f: X \to X$ which is post-critically finite, and which is “uniformly” quasiregular with respect to a suitable notion of quasiregularity in this metric space context. The question when the dynamical system $f: X \to X$ comes from a rational map can be formulated as a quasisymmetric uniformization problem. The following result is essentially contained in [My].

Theorem 6.3. *The map $f: X \to X$ is conjugate to a rational map $R: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ if and only if X is quasisymmetrically equivalent to $\mathbb{S}^2$.*

If f is not equivalent to a rational function, then it is natural to ask whether the dynamical system $f: X \to X$ is conjugate to a corresponding dynamical system on a better and less “fractal” space. More precisely, we want to replace X by a quasisymmetrically equivalent Ahlfors regular space of lower Hausdorff dimension. As in Theorem 5.3 discussed above, this leads to the problem of finding the conformal dimension $\dim_c X$ of the self-similar space X. By Theorem 6.3 the conformal case is characterized by the fact that we can squeeze X to a 2-regular space (and hence to the standard sphere $\mathbb{S}^2$ according to Theorem 3.2) by a quasisymmetric map. So we have a situation that is very similar to Cannon’s conjecture.

In discussions with L. Geyer and K. Pilgrim the following conjecture for $\dim_c X$ in terms of dynamical data emerged. To state it, let $Q \geq 2$ and Γ be an f-stable multicurve, define the modified Thurston matrix $A(\Gamma, Q)$ as $A(\Gamma, Q) = (m_{ij}^Q)$, where

$$m_{ij}^Q = \sum_\alpha \frac{1}{d_{i,j,\alpha}^{Q-1}},$$

and let $\lambda(f, \Gamma, Q)$ be the largest nonnegative eigenvalue of $A(\Gamma, Q)$.

Conjecture 6.4. If X comes from a subdivision rule with associated expanding map f, then $\dim_c X$ is the infimum of all $Q \geq 2$ such that $\lambda(f, \Gamma, Q) < 1$ for all f-stable multicurves Γ.

As in the proof of Theorem 6.1 (related to the necessity of the condition), there is one part of Conjecture 6.4 that seems to be rather easy to establish: If there exists an f-stable multicurve Γ with $\lambda(f, \Gamma, Q) \geq 1$, then $\dim_c X \geq Q$. The idea for proving this is to find path families related to ring domains associated with Γ which have positive Q-modulus. Any such path family on an Ahlfors regular space provides an obstruction to lowering its dimension by a quasisymmetric map [He, Thm. 15.10].

For some nontrivial cases one can show that Conjecture 6.4 is true [BMy]. For example, if X is the fractal obtained from the subdivision rule suggested by Figure 1, then $\dim_c X = 2$. This corresponds to the prediction of Conjecture 6.4 in this case. Note that here the infimum defining $\dim_c X$ is not attained as a minimum; otherwise f would be equivalent to a rational map by Theorem 6.3. We have seen above that this is not the case.

7. Sierpiński carpets

Sierpiński carpets are fractal spaces with a very interesting quasiconformal geometry. Recall that the "standard" Sierpiński carpet is obtained as follows: Start with the closed unit square, and subdivide it into $9 = 3 \times 3$ equal subsquares. Remove the interior of the middle square, and repeat this procedure for each of the remaining 8 subsquares. The limiting object of this construction is the standard Sierpiński carpet.

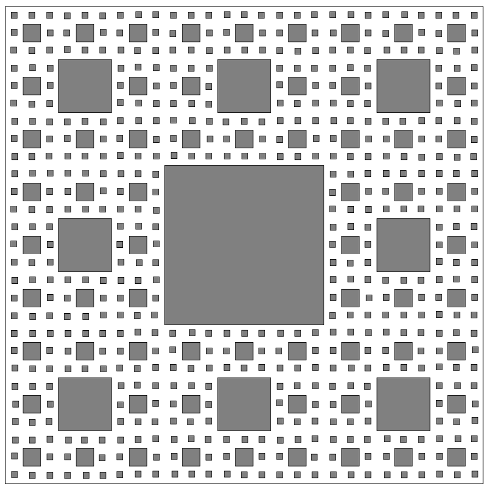

Figure 2. The standard Sierpiński carpet S_3.

One can run a similar construction, where one subdivides each square into $(p \times p)$-subsquares, p odd, and removes the middle square in each step. We denote the resulting space by S_p. So the standard Sierpiński carpet is S_3. We equip S_p with the restriction of the Euclidean metric on $\mathbb{R}^2$.

The spaces S_p are all homeomorphic to each other as follows from the following topological characterization theorems due to Whyburn [Wh].

Theorem 7.1. *Let X be a metric space. Then X is homeomorphic to the standard Sierpiński carpet if and only if X is a locally connected continuum, is topologically planar, has topological dimension* 1*, and has no local cut points.*

Here we call a set *topologically planar* if it is homeomorphic to a subset of the plane $\mathbb{R}^2$. A *local cut point* p of X is a point that has a connected neighborhood U such that $U \setminus \{p\}$ is not connected.

Theorem 7.2. *Let $X = \mathbb{S}^2 \setminus \bigcup_{i\in\mathbb{N}} D_i$ be the complement in $\mathbb{S}^2$ of countably many pairwise disjoint open Jordan regions D_i. Then X is homeomorphic to the standard Sierpiński carpet if and only if X has empty interior, $\partial D_i \cap \partial D_j = \emptyset$ for $i \neq j$, and* $\operatorname{diam}(D_i) \to 0$ *as $i \to \infty$.*

In the following we will call a metric space X a *carpet* if it is homeomorphic to S_3. A topological circle J in a carpet X is called a *peripheral circle* if J does not separate X, i.e., if $X \setminus J$ is connected. If X is a carpet as in Theorem 7.2, then the peripheral circles of X are precisely the Jordan curves ∂D_i, $i \in \mathbb{N}$. Note that every homeomorphism between two carpets X and Y has to map every peripheral circle of X to a peripheral circle of Y.

The deeper reason why all spaces as in Theorem 7.1 are homeomorphic to each other is that the homeomorphisms on a carpet form a rather large and flexible class. This is illustrated by the following transitivity result: If X is a carpet, $\{C_1, \dots, C_n\}$ and $\{C'_1, \dots, C'_n\}$ are two collections of peripheral circles of X with n elements, then there exists a homeomorphism $f: X \to X$ such that $f(C_i) = C'_i$. In other words, the homeomorphism group of X acts n-transitively on the set of peripheral circles of X for every $n \in \mathbb{N}$. This changes drastically if one restricts attention to quasisymmetric homeomorphisms and surprising rigidity phenomena emerge (cf. the Three-Circle Theorem 8.3 below). To discuss instances of this we first introduce some more terminology.

We say that a carpet $X \subseteq \mathbb{S}^2$ is *round* if its peripheral circles are round circles, i.e., if X is as in Theorem 7.2, where the Jordan regions D_i are round disks. If X is a round carpet and $f: \mathbb{S}^2 \to \mathbb{S}^2$ is a (possibly orientation reversing) Möbius transformation, then $f(X)$ is also a round carpet. We say that a round carpet X is *rigid* if this is the only way to obtain another round carpet as a quasisymmetric image of X, i.e., if every quasisymmetric map $g: X \to Y$ to another round carpet Y is the restriction of a Möbius transformation. Rigid round carpets admit a simple characterization [BKM].

Theorem 7.3. *A round carpet X is rigid if and only if it has measure zero.*

Actually, in [BKM] a related rigidity result is proved in all dimensions. Call a subset $X \subseteq \mathbb{S}^n$, $n \geq 2$, a *Schottky set* if it is the complement of pairwise disjoint open balls, and call a Schottky set $X \subseteq \mathbb{S}^n$ *rigid* if every quasisymmetric map $f: X \to Y$ to another Schottky set $Y \subseteq \mathbb{S}^n$ is the restriction of a Möbius transformation. Then one can show that every Schottky set of measure zero is rigid. This is a strengthening of a result due M. Kapovich, B. Kleiner, B. Leeb, and R. Schwartz (unpublished).

A corollary of Theorem 7.3 is that the set of quasisymmetric equivalence classes of round carpets has the cardinality of the continuum. So even though topologically there is only one Sierpiński carpet, from the point of view of quasiconformal geometry, there are many different ones.

An important source of round carpets is the theory of Kleinian groups. Let M be a compact hyperbolic 3-orbifold with nonempty totally geodesic boundary. Its universal cover $\tilde{M}$ is isometric to a convex subset K of $\mathbb{H}^3$ bounded by a nonempty collection of pairwise disjoint hyperplanes. Then the boundary at infinity $\partial_\infty K \subseteq \partial_\infty \mathbb{H}^3 = \mathbb{S}^2$ of K is a round carpet. The fundamental group $G = \pi_1(M)$ of M acts in a natural way on K by isometries. This induces an action $G \curvearrowright \partial_\infty K$ of G on the round carpet $S = \partial_\infty K$ by Möbius transformations. The group G is Gromov hyperbolic and its boundary $\partial_\infty G$ is quasisymmetrically equivalent to S. Hence the group $\mathrm{QS}(S) \supseteq G$

of quasisymmetric self-maps of S is rather large, because it acts cocompactly on triples of S and so there are only finitely many distinct orbits of peripheral circles. Accordingly, one should think of these round carpets as particularly "symmetric" ones.

It is tempting to try to characterize this situation from the point of view of Gromov hyperbolic groups. An analog of Cannon's conjecture is the following conjecture due to Kapovich and Kleiner who studied Gromov hyperbolic groups with carpet boundaries [KK]: *Suppose G is a Gromov hyperbolic group such that $\partial_\infty G$ is a carpet. Then G admits a properly discontinuous, cocompact and isometric action on a convex subset of $\mathbb{H}^3$ with nonempty totally geodesic boundary.*

This can be reformulated as a quasisymmetric uniformization problem.

Conjecture 7.4 (*Kapovich–Kleiner conjecture).* Suppose G is a Gromov hyperbolic group with $\partial_\infty G$ homeomorphic to the standard Sierpiński carpet. Then $\partial_\infty G$ is quasisymmetrically equivalent to a round carpet.

We call a carpet a *group carpet* if it arises as (i.e., is quasisymmetrically equivalent to) a boundary of a Gromov hyperbolic group. So the Kapovich–Kleiner conjecture asks whether every group carpet is quasisymmetrically equivalent to a round carpet.

Group carpets X should be thought of as very self-similar fractal spaces. As in the Kleinian case, the group $\mathrm{QS}(X)$ of quasisymmetric self-maps of X is rather large. It acts cocompactly on triples, and so there are only finitely many distinct orbits of peripheral circles. In addition, the collection of peripheral circles of a group carpet has the following geometric properties:

(i) The peripheral circles are *uniform quasicircles*, i.e., each one is quasisymmetrically equivalent to $\mathbb{S}^1$ by an η-quasisymmetric map with η independent of the peripheral circle.

(ii) The peripheral circles are *uniformly separated*, i.e., there is a uniform positive lower bound for the relative distance

$$\frac{\operatorname{dist}(C, C')}{\min\{\operatorname{diam}(C), \operatorname{diam}(C)\}}$$

of two distinct peripheral circles C and C'.

(iii) The peripheral circles *occur on all locations and scales*, i.e., if B is a ball in the carpet, then there exists a peripheral circle that intersects B and has a size comparable to B.

In view of the Kapovich–Kleiner conjecture one can ask whether these conditions are sufficient for X to be quasisymmetrically equivalent to a round carpet. It turns out that this is true for carpets that can be quasisymmetrically embedded into $\mathbb{S}^2$. This is a consequence of the following uniformization result [Bo].

Theorem 7.5. *Let $X \subseteq \mathbb{S}^2$ be a carpet, and suppose that the peripheral circles of X are uniform quasicircles and are uniformly separated. Then there exists a quasisymmetric map $f : X \to Y$ to a round carpet $Y \subseteq \mathbb{S}^2$.*

This theorem applies for example to the carpets S_p. So they are quasisymmetrically equivalent to round carpets. Note that if X as in the theorem has measure zero in addition (which is true if X is quasisymmetrically equivalent to a group carpet), then the uniformizing map f is uniquely determined up to a post-composition by a Möbius transformation (this essentially follows from Theorem 7.3). This shows that one can expect very little flexibility in constructing the uniformizing map f.

Theorem 7.5 is an analog of Koebe's well-known result on uniformization by circle domains. It says that every region in $\mathbb{S}^2$ with finitely many complementary components is conformally equivalent to a *circle domain*, i.e., a region whose complementary components are round (possibly degenerate) disks. This statement is actually used in the proof of Theorem 7.5. One considers regions Ω_n obtained by removing from $\mathbb{S}^2$ the closures of n complementary components of the given carpet X. By circle uniformization one can map the regions Ω_n to circle domains by (suitably normalized) conformal maps f_n. The uniformizing map f of X to a round carpet is then obtained as a sublimit of the sequence of maps f_n. The main difficulty is to show that such a sublimit exists. For this one proves that the maps f_n are uniformly quasisymmetric, i.e., η-quasisymmetric with η independent of n. It is a standard idea to use modulus estimates to get the required uniform distortion estimates for the maps f_n. If X has measure zero, then X does not support path families of positive modulus. Accordingly, one cannot expect any control for the distortion coming from such estimates involving classical modulus. This situation is remedied by a new quasisymmetric invariant, the modulus of a path family with respect to a carpet, which is the main technical ingredient in the proof of Theorem 7.5.

Let $X \subseteq \mathbb{S}^2$ be a carpet with peripheral circles C_i, $i \in \mathbb{N}$, and Γ a family of paths in $\mathbb{S}^2$. Then the *modulus of* Γ *with respect to* X is defined as

$$M_X(\Gamma) = \inf\left\{ \sum_{i \in \mathbb{N}} \rho_i^2 : \rho = \{\rho_i\} \text{ is admissible for } \Gamma \right\}.$$

Here a sequence $\rho = \{\rho_i\}$ of nonnegative weights ρ_i is called *admissible for* Γ if there exists an exceptional path family $\Gamma_0 \subseteq \Gamma$ with $\mathrm{Mod}_2(\Gamma_0) = 0$ such that

$$\sum_{\gamma \cap C_i \neq \emptyset} \rho_i \geq 1$$

for all paths $\gamma \in \Gamma \setminus \Gamma_0$.

So in contrast to classical modulus where ρ is a density, the test function is an assignment of discrete weights ρ_i to the peripheral circles C_i. This is similar to Schramm's notion of "transboundary extremal length" [Sc], where the test function consists both of a density and a discrete part. In the definition of $M_X(\Gamma)$ one wants to infimize the total mass $\sum_{i \in \mathbb{N}} \rho_i^2$ for all admissible sequences $\rho = \{\rho_i\}$. The admissibility requires that essentially every path picks up at least total weight 1 from all the peripheral circles that it meets. An important subtlety here is to allow the exceptional path family Γ_0. Otherwise, the quantity $M_X(\Gamma)$ would be infinite (and hence useless) for sufficiently large families Γ.

In contrast to classical modulus which is distorted by a multiplicative amount (cf. Theorem 2.1), the quantity $M_X(\Gamma)$ is invariant under quasisymmetric maps on $\mathbb{S}^2$.

Proposition 7.6. *Let $X \subseteq \mathbb{S}^2$ be a carpet, Γ a path family in $\mathbb{S}^2$, and $f : \mathbb{S}^2 \to \mathbb{S}^2$ a quasisymmetric map. Then*

$$M_X(\Gamma) = M_{f(X)}(f(\Gamma)).$$

The restriction to global maps f is not very serious here if one requires that the peripheral circles of X are uniform quasicircles. Then one can extend every quasisymmetric embedding of X into $\mathbb{S}^2$ to a quasisymmetric homeomorphism on $\mathbb{S}^2$.

As we remarked, Theorem 7.5 would settle the Kapovich–Kleiner conjecture if one could always quasisymmetrically embed a group carpet into $\mathbb{S}^2$. The conditions (i)–(iii) for the peripheral circles discussed above are not enough to guarantee this, because there are some carpets with these properties which do not admit such an embedding. In the positive direction one can show that if $\dim_{\mathrm{c}} X < 2$ for such a carpet X, then one gets the desired quasisymmetric embedding into $\mathbb{S}^2$. This was recently proved by B. Kleiner and the author [BK6]. The idea for the proof (due to J. Heinonen) is that each peripheral circle can be filled by a metric disk to obtain a sphere to which Theorem 3.2 can be applied. To get fillings of the right type one uses conformal densities as in [BHR].

8. Rigidity of square carpets

A carpet $X \subseteq \mathbb{R}^2$ is called a *square carpet* if its peripheral circles are boundaries of squares. Examples are the carpets $S_3, S_5, \dots$ introduced in the previous section. Obviously, these carpets are very symmetric and self-similar, so one may wonder whether they are group carpets. If so, the groups $\mathrm{QS}(S_p)$ should be rather large, and in particular infinite. It turns out that this is not the case [BMe].

Theorem 8.1. *Suppose $f : S_3 \to S_3$ is a quasisymmetric map. Then f is an isometry.*

The only isometries of S_3 are the obvious symmetries given by reflections and rotations; so $\mathrm{QS}(S_3)$ is a dihedral group containing 8 elements. It is very likely that an analog of Theorem 8.1 is true for all carpets S_p, p odd. At the moment it is only known that $\mathrm{QS}(S_p)$ is always a finite dihedral group. This implies that no carpet S_p is a group carpet.

The proof of Theorem 8.1 is surprisingly difficult. To explain some of the ingredients, suppose $f : S_3 \to S_3$ is a quasisymmetric map. For simplicity assume that f

is orientation preserving. Denote by C_1 the boundary of the unit square, and by C_2 the boundary of the middle square that was deleted in the first step of the construction of S_3. So C_1 and C_2 are peripheral circles of S_3.

If C and C' are two distinct peripheral circles of S_3, let $\Gamma(C, C')$ be the family of all open paths $\gamma|_{(0,1)}$, where $\gamma\colon [0, 1] \to \mathbb{S}^2$ is a path connecting C and C' such that $\gamma(0) \in C$, $\gamma(1) \in C'$, and $\gamma(0, 1) \cap (C \cup C') = \emptyset$. Then the (unordered) pair $\{C_1, C_2\}$ is distinguished from all other pairs $\{C, C'\}$ due to the following fact.

Lemma 8.2. *If C and C' are two distinct peripheral circles of S_3, then*

$$M_{S_3}(\Gamma(C, C')) \le M_{S_3}(\Gamma(C_1, C_2))$$

with equality if and only if $\{C, C'\} = \{C_1, C_2\}$.

The proof crucially uses the self-similarity of S_3 combined with monotonicity properties of the modulus invariant $M_X(\Gamma)$ defined in the previous section.

An immediate consequence of Lemma 8.2 and Proposition 7.6 is that

$$\{f(C_1), f(C_2)\} = \{C_1, C_2\}.$$

In other words, f preserves the peripheral circles C_1 and C_2 setwise or exchanges them.

Let us again make a simplifying assumption, namely that $f(C_1) = C_1$ and $f(C_2) = C_2$. Now one analyzes the possibilities for the images of the eight peripheral circles of S_3 that constitute the boundaries of the squares deleted in the second step of the construction of S_3. These eight peripheral circles come in two groups: "corner" circles and "side" circles. Using ideas as in the proof of Lemma 8.2, one can show that at least one of these eight second-generation peripheral circles is mapped to another second-generation peripheral circle; say one of the corner circles C_3 is mapped to a corner circle or a side circle C'_3.

In the first case where C'_3 is also a corner circle there exists a rotation R of S_3 such that $R(C_3) = C'_3$. Since R also preserves C_1 and C_2 setwise, we conclude that $f = R$ by the following theorem (applied to $g = R^{-1} \circ f$).

Theorem 8.3 (Three-circle theorem). *Let $X \subseteq \mathbb{S}^2$ be a carpet of measure zero whose peripheral circles are uniform quasicircles and are uniformly separated. Suppose C_1, C_2, C_3 are three distinct peripheral circles of X and $g\colon X \to X$ is an orientation preserving quasisymmetric map such that $g(C_i) = C_i$, $i = 1, 2, 3$. Then g is the identity on X.*

In other words, if an orientation preserving quasisymmetric map of the carpet fixes three peripheral circles setwise, then it is the identity. The same proof will show that if g fixes three points (instead of three peripheral circles), then the same conclusion holds, i.e., g is the identity.

Proof. By Theorem 7.5 there exists a quasisymmetric uniformization map $h\colon X \to Y$ to a round carpet $Y \subseteq \mathbb{S}^2$. One can show that the map h can be extended to a global quasiconformal map $H\colon \mathbb{S}^2 \to \mathbb{S}^2$. Since quasiconformal maps preserve sets of measure zero, the round carpet Y has measure zero. Hence Y is rigid by Theorem 7.3. Therefore, the quasisymmetric map $\tilde{g} = h \circ g \circ h^{-1}\colon Y \to Y$ is the restriction of an orientation-preserving Möbius transformation. Since it fixes the three round circles $h(C_i)$ setwise, it is the identity on Y. Hence g is the identity on X. □

The second case where the corner circle C_3 is mapped to a side circle C_3' does not occur. To rule out the existence of such a "ghost" map, one argues by contradiction. Suppose the situation is as represented in Figure 3. If R_D and R_M denote the reflections in the indicated symmetry lines D and M of S_3, respectively, one can show that

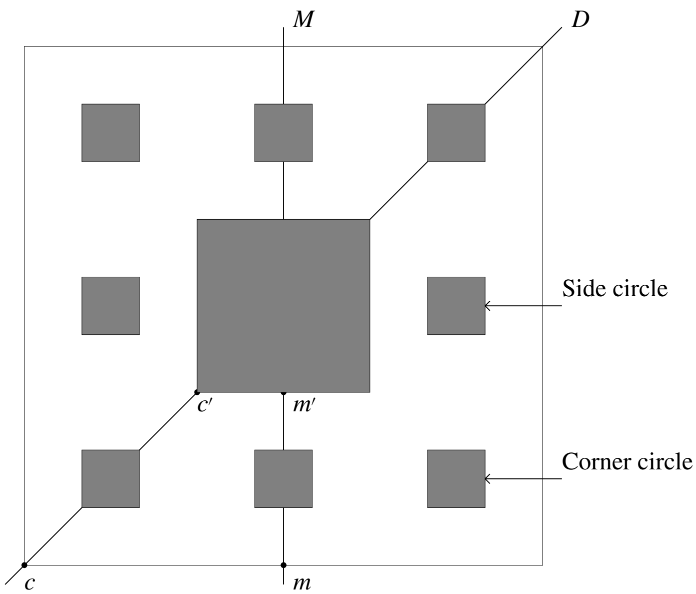

Figure 3. Corner and side circles of S_3.

$R_M \circ f = f \circ R_D$ by using the Three-Circle Theorem 8.3. This leads to $f(c) = m$ and $f(c') = m'$. Blowing up the carpet at c and m, the map f induces a quasisymmetric equivalence between the weak tangents W_c and W_m of S_3 at these points (with a suitable normalization these weak tangents are uniquely determined up to isometry). Similarly, the weak tangents $W_{c'}$ and $W_{m'}$ are quasisymmetrically equivalent. Since W_m and $W_{m'}$ are isometric, one concludes that W_c and $W_{c'}$ are quasisymmetrically equivalent. Since $W_{c'}$ essentially consists of three copies of W_c, one can get the desired contradiction by using the modulus invariant $M_X(\Gamma)$ and its monotonicity properties. The last step in the proof could be simplified, if one knew that W_c and W_m

are not quasisymmetrically equivalent. This is likely to be true, but an open problem at the moment.

The following result was also proved in [BMe].

Theorem 8.4. *Let $p, q \geq 3$ be odd integers. Then S_p and S_q are quasisymmetrically equivalent if and only if $p = q$.*

Using the known estimate

$$\dim_{\mathrm{c}} S_p \geq 1 + \frac{\log(p-1)}{\log p}$$

for the conformal dimension of S_p, it is not hard to see that S_p cannot be quasisymmetrically equivalent to S_q if p is much larger than q. The full result Theorem 8.4 is much harder to establish and uses ideas similar to the ones just described. An interesting open problem in this connection is to determine $\dim_{\mathrm{c}} S_p$.

9. Conclusion

It is evident from the preceding discussion that we are still far from a full understanding of the quasiconformal geometry of fractal 2-spheres and Sierpiński carpets. An obstacle in the solution of Cannon's conjecture is the lack of examples that could reveal some hidden structures. All known examples of Gromov hyperbolic groups with 2-sphere boundary arise from the standard Kleinian group situation, and Cannon's conjecture predicts that there are no others. In this sense the fractal 2-spheres that arise in the dynamics of post-critically finite maps exhibit more interesting phenomena, because sometimes they are quasisymmetrically equivalent to the standard 2-sphere and sometimes not. By investigating these spaces one may discover some general obstruction (formed by a "large" path family for example) that prevents a self-similar 2-sphere from obtaining a minimum for its conformal dimension. One may speculate that in the situation of Cannon's conjecture the group action prevents the existence of such an obstruction. This would lead to a solution of the conjecture according to Theorem 5.3.

The Kapovich–Kleiner conjecture looks somewhat more accessible due to the additional features given by the geometry of the peripheral circles of a group carpet. It has to be explored whether a modulus invariant similar to the invariant $M_X(\Gamma)$ for carpets in the plane can be used to prove general uniformization theorems for metric carpets.

The picture is sketchiest for the rigidity results on square carpets. Here it would be desirable to put isolated facts such as Theorems 8.1 and 8.4 into a general framework. A possible venue here is to develop an analytic theory of quasisymmetrically invariant "harmonic" functions. Similarly as in the definition of $M_X(\Gamma)$ this can be based on the minimization of energies of discrete weights $\rho = \{\rho_i\}$ which play the role of "upper" gradients of the functions. Such a theory could also lead to the solution of problems about weak tangents of carpets such as the one mentioned in Section 8.

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