Almost everywhere convergence and divergence of Fourier series

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Abstract. The aim of this expository paper is to demonstrate that there are several challenging problems concerning the behavior of trigonometric Fourier series almost everywhere.

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1. Introduction

We write $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ for the one-dimensional torus considered as the real line with $x+2\pi$ identified with x. Let $L(\mathbb{T})$ denote the class of all Lebesgue integrable functions $\mathbb{T} \to \mathbb{C}$. We associate with any function $f \in L(\mathbb{T})$ its Fourier series

$$f \sim \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx},$$

where

$$\hat{f}(k) = \frac{1}{2\pi} \int_{\mathbb{T}} f(x) \exp(-ikx)\, dx,$$

The m-th partial sum of the trigonometric Fourier series of f is

$$S_m(f, x) = \sum_{k=-m}^{m} \hat{f}(k) \exp(ikx).$$

Unfortunately, the Fourier series of f does not necessarily converge to f. It is known from Du Bois-Reymond [8] that the Fourier series of a continuous function can unboundedly diverge at some point. Fejér [10] and Lebesgue [27] constructed other examples of such functions. Almost one century ago it was proven that a trigonometric series with coefficients tending to 0 can diverge everywhere. Lusin [30] constructed

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such a series of power type $\sum_{k=0}^{\infty} c_k e^{ikx}$, and Steinhaus [38] gave an example of everywhere divergent real series

$$\sum_{k=1}^{\infty}(a_k \cos(kx) + b_k \sin(kx))$$

such that $a_k \to 0, b_k \to 0$ as $k \to \infty$. The central problem was: can the Fourier series of an integrable function diverge almost everywhere (everywhere)? G. H. Hardy [13] proved that if it is the case the divergence is not too fast:

$$S_m(f,x) = o(\log m) \quad (m \to \infty) \quad \text{almost everywhere.} \tag{1}$$

A. N. Kolmogoroff [17] constructed his famous example of a function $f \in L(\mathbb{T})$ such that $S_m(f,x)$ diverges unboundedly almost everywhere. In another paper [19] he constructed an everywhere divergent Fourier series.

These prominent results caused two type of problems.

I. To find a large class of functions with almost everywhere convergent Fourier series.

II. To show that a Fourier series of every integrable function converges in some sense to the function.

Some versions of these problems are discussed in the paper. We will deal not only with the trigonometric system but also with the Walsh system. The Walsh system is the following system of functions defined on [0, 1):

$$w_0(x) = 1, \quad w_k(x) = \prod_{j=0}^{r} \big(\operatorname{sign} \sin 2^{j+1}\pi x\big)^{\varepsilon_j} \quad (k \in \mathbb{N}),$$

where $\varepsilon_0, \dots, \varepsilon_r$ are the digits in the representation of k in the dyadic system

$$k = \sum_{j=0}^{r} \varepsilon_j 2^j, \quad \varepsilon_j \in \{0,1\},\ \varepsilon_r = 1. \tag{2}$$

This is a complete orthonormal functional system, and any function $f \in L[0,1)$ has the Fourier–Walsh representation

$$f \sim \sum_{k=0}^{\infty} \hat{f}(k) w_k(x), \quad \hat{f}(k) = \int_0^1 f(x) w_k(x)\, dx.$$

The Fourier–Walsh partial sums are defined as

$$S_m(f,x) = \sum_{k=0}^{m-1} \hat{f}(k) w_k(x).$$

The theory and the history of Fourier–Walsh series can be found in [11].

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2. Convergence of the sequence of all partial sums

Let $\phi\colon [0,+\infty) \to [0,+\infty)$ be a nondecreasing function, $\phi(0) = 0$. Denote

$$\phi(L) = \Big\{ f \in L(\mathbb{T}) : \int_{\mathbb{T}} \phi(|f(x)|)\,dx < \infty \Big\}.$$

For example, if $\phi(u) = u$, then $\phi(L) = L(\mathbb{T})$; if $\phi(u) = u^2$, then $\phi(L) = L^2(\mathbb{T})$.

Problem 2.1. For what functions ϕ the trigonometric Fourier series of any function $f \in \phi(L)$ converges almost everywhere?

In particular, it was not clear whether the Fourier series of a continuous function can diverge everywhere. Carleson [6] showed that if $f \in L^2(\mathbb{T})$ (i.e., f is measurable and $\int_{\mathbb{T}} |f^2(x)|dx < \infty$), then $S_m(f,x) \to f(x)$ as $m \to \infty$ almost everywhere. The condition $f \in L^2(\mathbb{T})$ in the Carleson theorem was weakened by Hunt [15], Sjölin [36]. Antonov [2] proved that if f is a measurable function and

$$\int_{\mathbb{T}} |f(x)| \log^+ |f(x)| \log^+ \log^+ \log^+ |f(x)|\,dx < \infty,$$

then $S_m(f,x) \to f(x)$ as $m \to \infty$ for almost all $x \in \mathbb{T}$. (We denote $\log^+ u = \log u$ if $u \geq 1$ and $\log^+ u = 0$ if $u < 1$.) The results in [36] were proven for Fourier–Walsh series, but the proofs can be rewritten for trigonometric series as well. The analog of Antonov's result for Fourier–Walsh series was established by Sjölin and Soria [37]. Arias-de-Reyna [1] constructed a rearrangement invariant space QA of functions with almost everywhere convergent Fourier series such that QA strictly contains Antonov's space $L \log^+ L \log\log\log^+ L$.

On the other hand, Körner [26] proved that if

$$\phi(u) = o(u \log\log u) \quad (u \to \infty) \tag{3}$$

then there is a function $f \in \phi(L)$ such that $\limsup_{m\to\infty} |S_m(f,x)| = \infty$ for all $x \in \mathbb{T}$. This result was improved by the author [22], [23].

Theorem 2.2. *If*

$$\phi(u) = o(u\sqrt{\log u/\log\log u}) \quad (u \to \infty)$$

then there is a function $f \in \phi(L)$ such that

$$\limsup_{m\to\infty} S_m(f,x) = \infty \quad \textit{for all } x \in \mathbb{T}.$$

Also, in [22] and [23] it was proven the existence of a function $f \in L(\mathbb{T})$ satisfying

$$S_m(f,x) > \psi(m) \quad \text{for all } x \text{ and infinitely many } m$$

provided that

$$\psi(m) = o(\sqrt{\log m/\log\log m}) \quad (m \to \infty).$$

This improved the result of Chen [7] asserting the existence of such a function f under a stronger supposition

$$\psi(m) = o(\log\log m) \quad (m \to \infty).$$

However, it is still unknown whether Hardy's inequality (1) can be improved. More precisely: does there exist a function $\psi(u) = o(u)$ as $u \to \infty$ such that for any $f \in L(\mathbb{T})$ the inequality

$$S_m(f,x) = o(\psi(\log m)) \quad (m \to \infty)$$

holds everywhere?

Conjecture 2.3. For any $f \in L(\mathbb{T})$

$$S_m(f,x) = o(\sqrt{\log m}) \quad (m \to \infty) \quad \text{almost everywhere.}$$

Due to Bochkarev, we know the existence of a function with almost everywhere divergent Fourier–Walsh series from a smaller functional class $\phi(L)$ than in the case of trigonometric Fourier series. In fact, the following result was proven in [5].

Theorem 2.4. *If*

$$\phi(u) = o(u\sqrt{\log u}) \quad (u \to \infty)$$

then there is a function $f \in L[0,1)$ such that

$$\int_0^1 \phi(|f(x)|)dx < \infty$$

and the Fourier–Walsh partial sums of f unboundedly diverge almost everywhere.

Antonov extended his result to convergence of cubic partial sums for multiple trigonometric Fourier series ([3], [4]; see also [37]). An extension of Körner's result to multiple trigonometric Fourier series was made in [21].

3. Convergence of subsequences of the sequence of partial sums

Gosselin [12] proved that for any increasing sequence $\{m_j\}$ there is $f \in L(\mathbb{T})$ such that

$$\sup_j |S_{m_j}(f,x)| = \infty \tag{4}$$

for almost all $x \in \mathbb{T}$. Totik [39] established the existence of f such that (4) holds for all $x \in \mathbb{T}$.

The problem is to determine under which conditions on a sequence $\{m_j\}$ and a function ϕ the partial sums $S_{m_j}(f)$ converge to f almost everywhere for any function $f \in \phi(L)$. In particular, is it true that for enough sparse sequence $\{m_j\}$ we can claim the almost everywhere convergence of $S_{m_j}(f)$ to f for a wider functional class $\phi(L)$ than in the case of taking the full sequence of the partial sums?

Let $\{m_j\}$ be a lacunary sequence, that is, $\inf_j m_{j+1}/m_j > 1$. If the Fourier series of a function g is a power-type series, namely,

$$g \sim \sum_{k=0}^{\infty} \hat{g}(k) \exp(ikx),$$

then the sequence $\{S_{m_j}(g)\}$ converges to g almost everywhere ([42], Chapter 15, Theorem 5.11). Combining the last result with Theorem 5.11 of Chapter 7 from [42] we get that if a measurable function f satisfies the condition

$$\int_{\mathbb{T}} |f(x)| \log^+ |f(x)|\, dx < \infty, \tag{5}$$

and $\{m_j\}$ is a lacunary sequence, then $\{S_{m_j}(f)\}$ converges to f almost everywhere. Let us recall that we do not know whether (5) is sufficient for the almost everywhere convergence of the Fourier series of the function f.

The following theorem [25] contains the above-mentioned results of [26] and [39].

Theorem 3.1. *For any increasing sequence $\{m_j\}$ of positive integers and any nondecreasing function $\phi\colon [0, +\infty) \to [0, +\infty)$ satisfying condition* (3)*, there is a function $f \in \phi(L)$ such that*

$$\sup_j |S_{m_j}(f, x)| = \infty \quad \textit{for all } x \in \mathbb{T}.$$

For the proof the construction suggested by Heladze [14] is used.

However, the technique of [22] and [23] employed the partial sums S_{m_j} for a quite rich subsequence $\{m_j\}$ and did not allow to weaken condition (3) in Theorem 3.1. Moreover, it is quite possible that this condition is sharp.

Conjecture 3.2. For any lacunary increasing sequence $\{m_j\}$ of positive integers and any measurable function f such that

$$\int_{\mathbb{T}} |f(x)| \log^+ \log^+ |f(x)|\, dx < \infty,$$

we have $S_{m_j}(f, x) \to f(x)$ as $j \to \infty$ for almost all $x \in \mathbb{T}$.

There is no a direct analog of Gosselin's theorem for Fourier–Walsh series. For a positive integer k denote

$$s(k) = 1 + \sum_{j=0}^{r-1} |\varepsilon_j - \varepsilon_{j+1}|,$$

where $\varepsilon_0, \ldots, \varepsilon_r$ are defined by (2). It is well-known that if $\{m_j\}$ is an increasing sequence of positive integers and

$$\sup_j s(m_j) < \infty, \tag{6}$$

then the Fourier–Walsh sums $S_{m_j}(f)$ converge to f almost everywhere for any $f \in L[0,1)$. In particular, this holds for $m_j = 2^j$. Moreover, condition (6) is not necessary for almost everywhere convergence of $S_{m_j}(f)$ to f for all functions $f \in L[0,1)$ [20].

Problem 3.3. Find a necessary and sufficient condition on a sequence $\{m_j\}$ of positive integers under which the partial Fourier–Walsh sums $S_{m_j}(f)$ converge to f almost everywhere for every function $f \in L[0,1)$.

One of the most remarkable results in this subject belongs to Lukomskij [29] who settled the problem for multiple Fourier–Walsh series of any dimension greater than one [29].

4. Ul'yanov's problem

Kolmogoroff [18] established the weak-type estimate for conjugate functions. One of the corollaries of his result is the convergence of trigonometric Fourier series of any integrable function in $L^p(0 < p < 1)$ and, therefore, in measure. Hence, for any function $f \in L(\mathbb{T})$ there exists an increasing sequence $\{m_j\}$ of positive integers such that the partial sums $S_{m_j}(f)$ converge to f almost everywhere. Gosselin's theorem demonstrates that such a sequence $\{m_j\}$ must depend on a function f.

Ul'yanov [40] asked the following question.

Problem 4.1. Does there exist a sequence $\{M_j\}$ such that the Fourier series of any $f \in L(\mathbb{T})$ possesses a subsequence $\{S_{m_j}(f)\}$ of its partial sums converging almost everywhere to f such that $m_j \le M_j$ for all j?

The following results have been proven in [24].

Theorem 4.2. *If $\phi(u)/u \to \infty$ as $u \to \infty$ then there exists a sequence $\{M_j\}$ such that for every function $f \in \phi(L)$ there is an increasing sequence $\{m_j\}$ such that $m_j \le M_j$ for all j and $S_{m_j}(f) \to f$ almost everywhere.*

Theorem 4.3. *There exists a sequence $\{M_j\}$ such that for every function $f \in L(\mathbb{T})$ there is an increasing sequence $\{m_j\}$ such that $m_j \le M_j$ for infinitely many j and $S_{m_j}(f) \to f$ almost everywhere.*

The sequence $\{M_j\}$ from Theorem 4.3 must grow very rapidly, faster than any multiple iteration of the exponent. Define the function $\exp^*(k)$, $k \in \mathbb{N}$, as the following: $\exp^*(0) = 0$, $\exp^*(k) = e^{\exp^*(k-1)}$ $(k \ge 1)$. It turns out that for any $\varepsilon > 0$ any sequence $\{M_j\}$ with

$$M_j = O\left(\exp^*([(\log\log j)^{1-\varepsilon}])\right) \quad (j \ge 20)$$

does not satisfy the requirements of Theorem 4.3. (Here $[u]$ is the greatest integer not exceeding u.)

Now we consider Ul'yanov's problem for measures. Let μ be a Borel measure on $\mathbb{T}$ with $\int_{\mathbb{T}} |d\mu| < \infty$. Denote

$$\hat{\mu}(k) = \frac{1}{2\pi} \int_{\mathbb{T}} \exp(-ikx) d\mu(x), \quad S_m(\mu, x) = \sum_{k=-m}^{m} \hat{\mu}(k) \exp(ikx).$$

In general, one cannot find an increasing sequence $\{m_j\}$ such that for almost all $x \in \mathbb{T}$ a sequence $\{S_{m_j}(\mu, x)\}$ converges, but, as follows from [18], the latter sequence is bounded for some $\{m_j\}$ and for almost all $x \in \mathbb{T}$. Ul'yanov's problem for measures can be formulated as follows.

Problem 4.4. Does there exist a sequence $\{M_j\}$ such that for any measure μ there is an increasing sequence $\{m_j\}$ such that $m_j \le M_j$ and for almost all $x \in \mathbb{T}$ the sequence $\{S_{m_j}(\mu, x)\}$ is bounded?

Let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence of points in $\mathbb{T}$ and $\{a_k\}_{k\in\mathbb{N}}$ be a sequence of positive numbers satisfying the conditions $a_1 \ge a_2 \ge \cdots$, and $\sum_k a_k < \infty$. Let $\sigma = \{\sigma_k\}_{k\in\mathbb{N}}$ be a sequence of independent (real or complex) uniformly bounded random variables with the zero expectations. We define the random measure $\mu = \sum_k \sigma_k a_k \delta_{x_k}$, where δ_x is the Dirac unit point mass at x. The following assertion has been proved by the author and F. L. Nazarov.

Theorem 4.5. *Let $\eta > 0$ and $M_{\eta,j} = M_j = \exp^*[\eta \log\log(j+2))]$ for $j \in \mathbb{N}$. Then for any sequences $\{x_k\}$ and $\{a_k\}$ there exists an increasing sequence $\{m_j\}$ such that $m_j \le M_j$ for all sufficiently large j and, moreover, for any σ, almost all $\omega \in \Omega$ the inequality $sup_j |S_{m_j}(d\mu(\omega))| < \infty$ holds almost everywhere on $\mathbb{T}$.*

I can prove that in general the estimate for the growth of $\{M_j\}$ in the theorem is sharp.

5. Strong summability

There are several ways to reconstruct the values of an integrable function via its partial Fourier sums almost everywhere. By the classical theorem of Lebesgue [28], the Fejér means $\frac{1}{M+1}\sum_{m=0}^{M} S_m(f)$ converge to f almost everywhere for every integrable f. This fact can be rewritten as

$$\lim_{M\to\infty} \frac{1}{M+1} \sum_{m=0}^{M} (S_m(f,x) - f(x)) = 0$$

almost everywhere. Marcienkiewicz [31] discovered that the convergence is not connected with oscillation of positive and negative terms, but reflects the fact that for most values m the difference $S_m(f,x) - f(x)$ is small. He proved that

$$\lim_{M\to\infty} \frac{1}{M+1} \sum_{m=0}^{M} |S_m(f,x) - f(x)|^2 = 0$$

holds almost everywhere and thus created the theory of strong summability. Let $\phi\colon [0,+\infty) \to [0,+\infty)$ be a nondecreasing function, $\phi(0) = 0$. There was a problem: for which ϕ we have

$$\lim_{M\to\infty} \frac{1}{M+1} \sum_{m=0}^{M} \phi(|S_m(f,x) - f(x)|) = 0 \tag{7}$$

almost everywhere for all $f \in L(\mathbb{T})$? As faster the growth of ϕ, as stronger the result. Thus, Marcienkiewicz [31] proved (7) for $\phi(u) = u^2$, and Zygmund [41] extended it to $\phi(u) = u^p$ for any $p > 0$.

If, moreover, $\phi(u) > 0$ for $u > 0$, then (7) implies the following property of the sequence $\{s_m\} = \{S_m(f,x)\}$ and the number $s = f(x)$: there exists an increasing sequence $\{m_j\}$ such that $\lim_j m_j/j = 1$ and $\lim_{j\to\infty} s_{m_j} = s$. This property is called almost everywhere ([42], Chapter 13, (9.1)), or statistical, convergence. There are many recent papers on statistical convergence; see, e.g., [32] and [33].

So, for almost all $x \in \mathbb{T}$ there exists an increasing sequence $\{m_j\}$ with $m_j = O(j)$ such that $\lim_{j\to\infty} S_{m_j}(f,x) = f(x)$. Recall that if we are looking for the same $\{m_j\}$ for almost all $x \in \mathbb{T}$ then in general its growth must be much faster.

To describe the contemporary approach to the strong summability, we recall the notion of the bounded mean oscillation (BMO); see, e.g., ([9], Chapter 6, §2). For a nondegenerate interval $I \subset \mathbb{R}$ and a function $f \in L(I)$ denote $f_I = |I|^{-1}\int_I f(x)dx$. The set BMO$[0,\infty)$ is defined as the class of all locally integrable on $[0,\infty)$ functions f such that

$$\|f\|_* = \sup_{I\subset[0,\infty)} \left\{|I|^{-1}\int_I |f(x) - f_I|\,dx\right\} < \infty.$$

Next, for a function $f \in L(\mathbb{T})$, $x \in \mathbb{T}$, $t \in [0,\infty)$ denote

$$g_x(t) = S_{[t]}(f,x) - f(x).$$

Let $\mu(E)$ be the Lebesgue measure of a set E. The following result was proven by Rodin in [34], [35].

Theorem 5.1. *For any function $f \in L(\mathbb{T})$ and for almost all $x \in \mathbb{T}$ we have $g_x \in$ BMO$[0,\infty)$. Moreover, there exists an absolute constant $C > 0$ such that for any $\alpha > 0$ the following inequality holds*

$$\mu\left\{x \in \mathbb{T} : \|g_x\|_* > \alpha \int_{\mathbb{T}} |f(y)|\,dy\right\} \le C\alpha^{-1}.$$

Corollary 5.2. *Let $\lambda > 0$ and $\phi(u) = \exp(\lambda u) - 1$. Then equality (7) holds almost everywhere for all $f \in L(\mathbb{T})$.*

Corollary 5.2 easily follows from Theorem 5.1 and the John–Nirenberg inequality ([9], Chapter 6, §4).

The condition on the function ϕ in Corollary 5.2 is sharp. Karagulyan [16] proved that if

$$\limsup_{u\to\infty} \log\phi(u)/u = \infty$$

then (7) fails for some $f \in L(\mathbb{T})$ for all $x \in \mathbb{T}$.

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