Amenable actions and applications

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Abstract. We will give a brief account of (topological) amenable actions and exactness for countable discrete groups. The class of exact groups contains most of the familiar groups and yet is manageable enough to provide interesting applications in geometric topology, von Neumann algebras and ergodic theory.

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1. Introduction

The notion of amenable groups was introduced by J. von Neumann in 1929 in his investigation of the Banach–Tarski paradox. He observed that non-abelian free groups are not amenable and that this fact is the source of the Banach–Tarski paradox. Since then it has been shown that the amenability of a locally compact group is equivalent to many fundamental properties in harmonic analysis of the group: the Følner property, the fixed point property and the weak containment of the trivial representation in the regular representation, to name a few. For a discrete group, amenability of the group is also characterized by nuclearity of its group C^*-algebra, and by injectivity of its group von Neumann algebra. In this note, we are mainly interested in countable discrete groups. The class of amenable groups contains all solvable groups and is closed under subgroups, quotients, extensions and directed unions. As we mentioned before, a non-abelian free group, or any group which contains it, is not amenable. Amenable groups play a pivotal role in the theory of operator algebras. Many significant operator algebra-related problems on groups have been solved for amenable groups. We just cite two of them; the classification of group von Neumann algebras [14] and measure equivalences [16], [58] on the one hand, and the Baum–Connes conjecture [46] on the other hand. In recent years, there have been exciting breakthroughs in both subjects beyond the amenable cases. We refer to [68] and [84] for accounts of this progress. We will also treat the classification of group von Neumann algebras and measure equivalences in Section 4. Since many significant problems, if not all, are already solved for amenable groups, we would like to set out for the world of non-amenable groups. Still, as Gromov's principle goes, no statement about all groups is both non-trivial and true. So we want a *good* class of groups to play with. We consider a class

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as good if it contains many of the familiar examples, is manageable enough so that it maintains non-trivial theorems, and can be characterized in various ways so that it is versatile. We believe that the class of exact groups, which will be introduced in the following section, stands these tests. The study of exactness originates in C*-algebra theory [50], [51], [52] and was propagated to groups. The class of exact groups is fairly large and it contains all amenable groups, linear groups [39] and hyperbolic groups [2], to name a few. It is closed under subgroups, extensions, directed unions and amalgamated free products. (Since every free group is exact and there exists a non-exact group [37], a quotient of exact group needs not be exact unless the normal subgroup is amenable.) Moreover, there is a remarkable theorem that the injectivity part of the Baum–Connes conjecture holds for exact groups [45], [76], [83], [84]. Since this part of the Baum–Connes conjecture has a lot of applications in geometry and topology, including the strong Novikov conjecture, it is an interesting challenge to prove exactness of a given group. We will encounter some other applications in von Neumann algebra theory and ergodic theory in Section 4.

2. Amenable actions and exactness

We first review the definition of and basic facts on amenable actions. We refer to [65] for the theory of amenable groups and to [5], [11] for the theory of amenable actions. The notion of amenability for a group action was first introduced in the measure space setting in the seminal paper [85], which has had a great influence in both ergodic theory and von Neumann algebra theory. In this spirit the study of its topological counterpart was initiated in [3]. In this note, we restrict our attention to continuous actions of countable discrete groups on (not necessarily second countable) compact spaces. All topological spaces are assumed to be Hausdorff and all groups, written as $\Gamma, \Lambda, \dots$, are assumed to be countable and discrete. Let Γ be a group. A (topological) Γ-*space* is a topological space X together with a continuous action of Γ on it; $\Gamma \times X \ni (s,x) \mapsto s.x \in X$. For a group (or any countable set) Γ, we let

$$\operatorname{prob}(\Gamma) = \left\{\mu \in \ell_1(\Gamma) : \mu \geq 0,\ \textstyle\sum_{t\in\Gamma} \mu(t) = 1\right\} \subset \ell_1(\Gamma)$$

and equip $\operatorname{prob}(\Gamma)$ with the pointwise convergence topology. We note that this topology coincides with the norm topology. The space $\operatorname{prob}(\Gamma)$ is a Γ-space with the Γ-action given by the left translation: $(s.\mu)(t) = \mu(s^{-1}t)$.

Definition 2.1. We say that a compact Γ-space X is *amenable* (or Γ *acts amenably* on X) if there exists a sequence of continuous maps

$$\mu_n \colon X \ni x \mapsto \mu_n^x \in \operatorname{prob}(\Gamma)$$

such that for every $s \in \Gamma$ we have

$$\lim_{n\to\infty} \sup_{x\in X} \|s.\mu_n^x - \mu_n^{s.x}\| = 0.$$

When X is a point, the above definition degenerates to one of the equivalent definitions of amenability for the group Γ. Moreover, if Γ is amenable, then every Γ-space is amenable. Conversely, if there exists an amenable Γ-space which carries an invariant Radon probability measure, then Γ itself is amenable. If X is an amenable Γ-space, then X is amenable as a Λ-space for every subgroup Λ. It follows that all isotropy subgroups of an amenable Γ-space have to be amenable. We recall that the isotropy subgroup of x in a Γ-space X is $\{s \in \Gamma : s.x = x\}$. It is also easy to see that if there exists a Γ-equivariant continuous map from a Γ-space Y into another Γ-space X and if X is amenable, then so is Y. Finally, we only note that there are several equivalent characterizations of an amenable action which generalize those for an amenable group.

Many amenable actions naturally arise from the geometry of groups. The following are the most basic examples of amenable actions.

Example 2.2. Let $\mathbb{F}_r = \langle g_1, \dots, g_r\rangle$ be the free group of rank $r < \infty$. Then its (Gromov) boundary $\partial\mathbb{F}_r$ is amenable. We note that the Cayley graph of $\mathbb{F}_r$ w.r.t. the standard set of generators is a simplicial tree and its boundary

$$\partial\mathbb{F}_r \subset \{g_1, g_1^{-1}, \dots, g_r, g_r^{-1}\}^{\mathbb{N}}$$

is defined as the compact topological space of all infinite reduced words, equipped with the relative product topology (see Figure 1). Similarly, with an appropriate topology, $\mathbb{F}_r \cup \partial\mathbb{F}_r$ becomes a compactification of $\mathbb{F}_r$. The free group $\mathbb{F}_r$ acts continuously on $\partial\mathbb{F}_r$ by left multiplication (and rectifying possible redundancy). For $x \in \partial\mathbb{F}_r$ with its reduced form $x = a_1a_2\dots$, we set $x_0 = e$ and $x_k = a_1\dots a_k$. For every $n \in \mathbb{N}$, we let

$$\mu_n \colon \partial\mathbb{F}_r \ni x \mapsto \mu_n^x = \frac{1}{n}\sum_{k=0}^{n-1}\delta_{x_k} \in \operatorname{prob}(\mathbb{F}_r).$$

Thus μ_n^x is the normalized characteristic function of the first n segments of the path in the Cayley graph of $\mathbb{F}_r$, connecting e to x (see Figure 1). It is not hard to see that μ_n is a continuous map such that

$$\sup_{x\in\partial\mathbb{F}_r} \|s.\mu_n^x - \mu_n^{s.x}\| \le \frac{2|s|}{n}$$

for every $s \in \mathbb{F}_r$, where $|s|$ is the word length of s. Indeed, $s.\mu_n^x$ is the normalized characteristic function of the first n segments of the path connecting s to $s.x$, which has a large intersection with the path connecting e to $s.x$ (see Figure 2).

There are generalizations of this construction to groups acting on more general buildings [72] and on hyperbolic spaces [2].

Example 2.3. Let Γ be a discrete subgroup of the special linear group $\mathrm{SL}(n,\mathbb{R})$ (e.g., $\Gamma = \mathrm{SL}(n,\mathbb{Z})$) and $P \subset \mathrm{SL}(n,\mathbb{R})$ be the closed subgroup of upper triangular matrices.

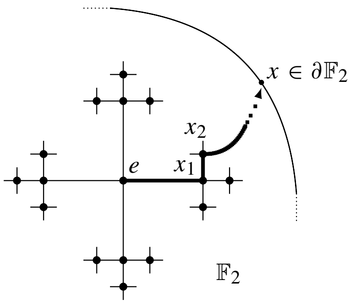

Figure 1. The Cayley graph of $\mathbb{F}_2$ and the boundary $\partial\mathbb{F}_2$.

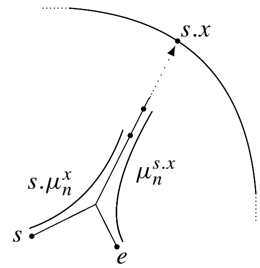

Figure 2. Amenability of $\partial\mathbb{F}_2$.

Then the left multiplication action of Γ on the Furstenberg boundary $\mathrm{SL}(n,\mathbb{R})/P$ is amenable. More generally, if G is a locally compact group with a closed amenable locally compact subgroup P (such that G/P compact), then every discrete subgroup Γ of G acts amenably on G/P.

A far-reaching generalization of this example is given in [39], where it is shown that any linear group admits an amenable action on some compact space. Thus many non-amenable groups admit amenable actions.

Definition 2.4. We say a group Γ is *exact* if there exists a compact Γ-space X which is amenable.

Exact groups are also said to be *boundary amenable*, *amenable at infinity* or to *have the property A*. By definition, all amenable groups are exact. Let X be a compact Γ-space. Then, by the universality of the Stone–Čech compactification $\beta\Gamma$, there exists a Γ-equivariant continuous map from $\beta\Gamma$ into X. It follows that Γ is exact iff $\beta\Gamma$ (or the boundary $\partial^\beta\Gamma = \beta\Gamma \setminus \Gamma$) is amenable. Moreover, whether Γ is exact or not, $\beta\Gamma$ is amenable as a Λ-space for every exact subgroup Λ of Γ since there exists a Λ-equivariant continuous map from $\beta\Gamma$ into $\beta\Lambda$. This observation implies that exactness is preserved under a directed union, i.e., a group Γ is exact iff all of its finitely generated subgroups are exact.

Amenability of the Stone–Čech compactification $\beta\Gamma$ leads to an intrinsic characterization of an exact group Γ. Before stating it, we introduce the notion of coarse metric spaces [36]. Let d be a left translation invariant metric on Γ which is proper in the sense that every subset of finite diameter is finite. Then $l(s) = d(s,e)$ is a length function on Γ, i.e., $l(s^{-1}) = l(s)$, $l(st) \le l(s) + l(t)$ for every $s,t \in \Gamma$, and $l(s) = 0$ iff $s = e$. The length function l is proper in the sense that $l^{-1}([0,R])$ is finite for every $R > 0$. Conversely, every proper length function l gives rise to a proper left translation invariant metric d on Γ such that $d(s,t) = l(s^{-1}t)$. If $\mathcal{S}$ is a finite generating subset of Γ, then the corresponding word metric is defined by

$$d_{\mathcal{S}}(s,t) = \min\{n : s^{-1}t = s_1 \dots s_n,\ s_i \in \mathcal{S} \cup \mathcal{S}^{-1}\}.$$

We note that even when Γ is not finitely generated, there exists a proper left translation invariant metric d on Γ (as we assume that Γ is countable). Two proper length functions l and l' are equivalent in the sense that $l(s_n) \to \infty$ iff $l'(s_n) \to \infty$. Thus we are lead to the notion of coarse equivalence, which is a very loose notion. Two metric spaces (X, d) and (X', d') are *coarsely isomorphic* if there exists a (not necessarily continuous) map $f : X \to X'$ such that $d(z, f(X)) < \infty$ for every $z \in X'$ and

$$\rho_-(d(x,y)) \le d'(f(x), f(y)) \le \rho_+(d(x,y))$$

for some fixed function $\rho_\pm$ on $[0,\infty)$ with $\lim_{r\to\infty} \rho_-(r) = \infty$. Such f is called a coarse isomorphism. We observe that any two proper left translation invariant metrics d and d' on Γ are *coarsely equivalent* in the sense that the formal identity map from (Γ, d) onto (Γ, d') is a coarse isomorphism. A *coarse metric space* is a space together with a coarse equivalence class of metrics. Hence, Γ is provided with a unique coarse metric space structure. Two groups Γ and Γ' are said to be *coarsely isomorphic* if they are coarsely isomorphic as coarse metric spaces. It follows from the following theorem that exactness is a coarse isomorphism invariant. In particular, a group is exact if it has a finite index subgroup which is exact.

Theorem 2.5 ([47], [83]). *For a group Γ, the following are equivalent.*

1. *The group Γ is exact.*
2. *The metric space (Γ, d) has the property A: For every $\varepsilon > 0$ and $R > 0$, there exist a map $\nu : \Gamma \to \mathrm{prob}(\Gamma)$ and $S > 0$ such that $\|\nu_s - \nu_t\| \le \varepsilon$ for every $s, t \in \Gamma$ with $d(s,t) < R$ and $\operatorname{supp} \nu_s \subset \{t : d(s,t) < S\}$ for every $s \in \Gamma$.*
3. *For every $\varepsilon > 0$ and $R > 0$, there exist a Hilbert space $\mathcal{H}$, a map $\xi : \Gamma \to \mathcal{H}$ and $S > 0$ such that $|1 - \langle \xi_t, \xi_s \rangle| < \varepsilon$ for every $s, t \in \Gamma$ with $d(s,t) < R$ and $\langle \xi_t, \xi_s \rangle = 0$ for every $s, t \in \Gamma$ with $d(s,t) \ge S$.*

Moreover, if Γ is exact, then Γ is coarsely isomorphic to a subset of a Hilbert space.

The main result of [83] is the injectivity part of the Baum–Connes conjecture for a group which is coarsely embeddable into a Hilbert space. (See also [45], [76], [84].) This justifies the study of exactness for groups. It is not known whether or not coarse embeddability into a Hilbert space implies exactness (even in the case of groups with the Haagerup property). We recall that a metric space (X, d) has *asymptotic dimension* $\le d$ [36] if for every $R > 0$, there exists a covering $\mathcal{U}$ of X such that $\sup_{U \in \mathcal{U}} \operatorname{diam}(U) < \infty$ and $|\{U \in \mathcal{U} : U \cap B \neq \emptyset\}| \le d + 1$ for any subset $B \subset X$ with $\operatorname{diam}(B) < R$. Asymptotic dimension is a coarse equivalence invariant and hence an invariant for a group. We note that the groups $\mathbb{Z}^d$ and $\mathbb{F}_r^d$ have asymptotic dimension d.

Corollary 2.6 ([47]). *A coarse metric space with finite asymptotic dimension has the property A. In particular, a group with finite asymptotic dimension is exact.*

It was shown in [22] that every Coxeter group has finite asymptotic dimension and hence is exact. We refer to [8] for more information on asymptotic dimension.

We describe a relative version of an amenable action, which is useful in proving various kinds of permanence properties of exactness. There are other approaches [6], [7], [20] which are as well useful. The following is in the spirit of [3].

Proposition 2.7 ([63]). *Let X be a compact Γ-space and K be a countable Γ-space. Assume that there exists a net of Borel maps*

$$\mu_n \colon X \to \operatorname{prob}(K)$$

(*i.e., the function $X \ni x \mapsto \mu_n^x(a) \in \mathbb{R}$ is Borel for every $a \in K$*) *such that*

$$\lim_n \int_X \|s.\mu_n^x - \mu_n^{s.x}\| \, dm(x) = 0$$

for every $s \in \Gamma$ and every Radon probability measure m on X. Then Γ is exact provided that all isotropy subgroups of K are exact. Indeed, if Y is a compact Γ-space which is amenable as a Λ-space for every isotropy subgroup Λ, then $X \times Y$ (*with the diagonal Γ-action*) *is an amenable Γ-space.*

Corollary 2.8 ([52]). *An extension of exact groups is again exact.*

Proof. If $\Lambda \lhd \Gamma$ is a normal subgroup such that Γ/Λ is exact, then Proposition 2.7 is applicable to an amenable compact (Γ/Λ)-space X and $K = \Gamma/\Lambda$ □

We turn our attention to a group acting on a countable simplicial tree T, which may not be locally finite. We will define a compactification $\overline{T} = T \cup \partial T$ of T, to which Proposition 2.7 is applicable. We recall that a *simplicial tree* is a connected graph without non-trivial circuits, and identify T with its vertex set. The boundary ∂T of T is defined as in Example 2.2. Thus ∂T is the set of all equivalence classes of (one-sided) infinite simple paths in T, where two infinite simple paths are equivalent if their intersection is infinite. For every $a \in T$ and $x \in \partial T$, there exists a unique infinite simple path γ in the equivalence class x which starts at a. We say that the path γ connects a to x. It follows that every two distinct points in $\overline{T} = T \cup \partial T$ are connected by a unique simple path (which is a biinfinite path, with the obvious definition, when both points are boundary points). Every edge separates $\overline{T}$ into two components, and every finite subset of edges separates X into finitely many components. Now we equip $\overline{T}$ with a topology by declaring that all such components are open. It turns out that $\overline{T}$ is compact with this topology. We note that T is dense but not open in $\overline{T}$ (unless T is locally finite) and that every automorphism s of T extends to a homeomorphism on $\overline{T}$. Fixing a base point $e \in T$, we define $\mu_n\colon \partial T \to \mathrm{prob}(T)$ exactly as in Example 2.2. It is not hard to see that μ_n is a Borel map such that

$$\sup_{x \in \partial T} \| s.\mu_n^x - \mu_n^{s.x} \| \le \frac{2d(s.e, e)}{n}$$

for every automorphism s of T (cf. Figure 2). We extend μ_n to $\overline{T}$ by simply letting $\mu_n^a = \delta_a \in \mathrm{prob}(T)$ for $a \in T$. Then the sequence of Borel maps $\mu_n\colon \overline{T} \to \mathrm{prob}(T)$ satisfies the assumption of Proposition 2.7 for $X = \overline{T}$, $K = T$ and any group Γ acting on T.

We recall that associated with the fundamental group of a graph of groups there exists a tree, called the *Bass–Serre tree*, on which the group acts. We describe it in the case of an amalgamated free product. Let $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$ be the amalgamated free product of groups Γ_1 and Γ_2 with a common subgroup Λ. Then the associated Bass–Serre tree T is the disjoint union $\Gamma/\Gamma_1 \sqcup \Gamma/\Gamma_2$ of left cosets, where $s\Gamma_1$ and $t\Gamma_2$ are adjacent if $s\Gamma_1 \cap t\Gamma_2 \neq \emptyset$. Thus the edge set of T coincides with Γ/Λ, and an edge $s\Lambda$ connects $s\Gamma_1$ and $s\Gamma_2$. It turns out that T is a tree. The group Γ acts on T from the left in such a way that each vertex stabilizer is conjugate to either Γ_1 or Γ_2 and each edge stabilizer is conjugate to Λ. We note that the tree T is not locally finite unless Λ has finite index in both Γ_1 and Γ_2.

Corollary 2.9 ([25], [78]). *Let Γ be a group acting on a countable simplicial tree T. Then Γ is exact provided that all isotropy subgroups are exact. In particular, an amalgamated free product and an HNN-extension of exact groups are again exact.*

It follows that one-relator groups are exact [38] because they are made up by using HNN-extensions following the McCool–Schupp algorithm. A similar remark applies to a fundamental group of a Haken 3-manifold thanks to the Waldhausen decomposition.

Example 2.2 can be generalized to a hyperbolic space, too. The notion of hyperbolicity was introduced in the very influential paper [35] and has been extensively studied since. A metric space is said to be *hyperbolic* if it is “tree-like” in certain sense, and a finitely generated group Γ is said to be *hyperbolic* if its Cayley graph is hyperbolic. Hyperbolicity is a robust notion and there are many natural examples of hyperbolic groups including the free groups. Every hyperbolic group has a nice compactification, called the Gromov compactification, which is a generalization of that given in Example 2.2. It is shown in [2] that the action of a hyperbolic group on its Gromov compactification is amenable. (See also [9] and the appendix of [5].) The result is generalized in [48], [63] to a group acting on hyperbolic spaces, which are not necessarily locally finite. Compactification of a non-locally-finite hyperbolic graph was considered in [10], where its Bowditch compactification $\overline{K}$ is introduced for a *fine* hyperbolic graph K. A simplicial tree T and its compactification $\overline{T}$ are the simplest non-trivial examples of a uniformly fine hyperbolic graph and its Bowditch compactification. See [10] for details. As in the case for a simplicial tree, the assumption of Proposition 2.7 is satisfied for a uniformly fine hyperbolic graph K, its Bowditch compactification $\overline{K}$ and any group acting on K [63]. By a characterization of a relatively hyperbolic group [10], we obtain the following corollary.

Corollary 2.10 ([20], [59], [63]). *A relatively hyperbolic group is exact provided that all peripheral subgroups are exact. In particular, every hyperbolic group is exact.*

Examples of relatively hyperbolic groups include the fundamental groups of complete non-compact finite-volume Riemannian manifolds with pinched negative sectional curvature (which are hyperbolic relative to nilpotent cusp subgroups) [26] and limit groups (which are hyperbolic relative to maximal non-cyclic abelian subgroups) [1], [21]. Exactness of limit groups also follows from their linearity.

Another interesting case of group actions which implies exactness is a proper and co-compact action on a finite dimensional CAT(0) cubical complex [12].

The mapping class group $\Gamma(S)$ of a compact orientable surface S is also a natural example of an exact group [42], [49]. Indeed, the action of $\Gamma(S)$ on the space of complete geodesic laminations is amenable [42]. In contrast, the more well-known action of $\Gamma(S)$ on the Thurston boundary $\mathcal{PMF}$ of Teichmüller space is not amenable because of non-amenable isotropy subgroups. However, if we denote by K the set of all non-trivial isotopy classes of non-peripheral simple closed curves on S (i.e., K is the vertex set of the curve complex of S), then the assumption of Proposition 2.7 is satisfied for $X = \mathcal{PMF}$ [49]. Since every isotropy subgroup of a point in K is a mapping class group of lower complexity, induction applies and the exactness of $\Gamma(S)$ follows.

So far we have enumerated examples of exact groups as many as we can (the author is sorry for any possible omission). Unfortunately, there does exist a (finitely presented) group which is neither exact nor coarsely embeddable into a Hilbert space [37]. Currently, it is not known whether the following groups are exact or not: Thompson's group F, $\mathrm{Out}(\mathbb{F}_r)$, automatic groups, 3-manifold groups, groups of homeomorphisms (resp. diffeomorphisms) on (say) the circle S^1, (free) Burnside groups and other monstrous groups.

The rest of this section is devoted to the relationship of exactness to operator algebras. Associated with a group, there are the reduced group C*-algebra $C^*_\lambda(\Gamma)$ and the group von Neumann algebra $\mathcal{L}(\Gamma)$. When Γ is abelian, $C^*_\lambda(\Gamma)$ is isomorphic to $C(\widehat{\Gamma})$, while $\mathcal{L}(\Gamma)$ is isomorphic to $L^\infty(\widehat{\Gamma})$, where $\widehat{\Gamma}$ is the Pontrjagin dual of Γ. Hence the study of $C^*_\lambda(\Gamma)$ corresponds to "noncommutative topology" and that of $\mathcal{L}(\Gamma)$ to "noncommutative measure theory" [15]. Amenability of Γ can be read from its operator algebras.

Theorem 2.11 ([41], [54], [74]). *For a group Γ, the following are equivalent.*

1. *The group Γ is amenable.*
2. *The reduced group C*-algebra $C^*_\lambda(\Gamma)$ is nuclear.*

3. *The group von Neumann algebra* $\mathcal{L}(\Gamma)$ *is injective.*

A generalization of this theorem to a group action goes as follows.

Theorem 2.12 ([3]). *For a* (*compact*) Γ*-space* X*, the following are equivalent.*

1. *The* Γ*-space* X *is amenable.*
2. *The reduced crossed product* C^**-algebra* $C^*_\lambda(X \rtimes \Gamma)$ *is nuclear.*
3. *The group-measure-space von Neumann algebra* $\mathcal{L}(X \rtimes \Gamma, m)$ *is injective for any* Γ*-quasi-invariant Radon probability measure* m *on* X.

The nuclear C*-algebras are accessible among the C*-algebras and the classification program of nuclear C*-algebras is a very active area of research in C*-algebra theory [73]. Many C*-algebras $C^*_\lambda(X \rtimes \Gamma)$ arising from various kinds of boundary actions are classifiable via their K-theory [4], [53], [77]. Unlike the group case, a C*-subalgebra of a nuclear C*-algebra needs not be nuclear. The notion of exactness was introduced to give an abstract characterization of subnuclearity and has met a great success [50], [51]. Exactness has a deep connection with operator space theory [51], [69]. A C*-algebra A is called *exact* if taking the minimal tensor product with A preserves short exact sequences of C^*-algebras. The following theorem explains the nomenclature of exact groups.

Theorem 2.13 ([11], [40], [51], [60]). *For a group* Γ *the following are equivalent.*

1. *The group* Γ *is exact.*
2. *The reduced group* C^**-algebra* $C^*_\lambda(\Gamma)$ *is exact.*
3. *The group von Neumann algebra* $\mathcal{L}(\Gamma)$ *is weakly exact.*

We note that a C*-subalgebra of an exact C*-algebra is always exact and that a von Neumann subalgebra of a weakly exact von Neumann algebra is weakly exact provided that there exists a normal conditional expectation. Since a von Neumann algebra with a weakly dense exact C*-algebra is weakly exact, we obtain the following corollary.

Corollary 2.14. *Exactness is closed under measure equivalence.*

We recall that two groups Γ and Λ are *measure equivalent* [36] if there exist commuting measure preserving free actions of Γ and Λ on some Lebesgue measure space (Ω, m) such that the action of each of the groups admits a finite measure fundamental domain. For example, lattices in the same (second countable) locally compact group G are measure equivalent. It is known that measure equivalence coincides with the stable orbit equivalence [29] and hence gives rise to a stable isomorphism of the corresponding group-measure-space von Neumann algebras.

3. Amenable compactifications which are small

We study the "size" of an amenable compactification with its application to von Neumann algebra theory in mind. A compactification of a group Γ is a compact space $\Delta\Gamma$ containing Γ as an open dense subset. We only consider those compactifications which are equivariant; the left multiplication action of Γ on Γ extends to a continuous action of Γ on $\Delta\Gamma$. A group Γ is amenable iff the one-point compactification is amenable, and a group Γ is exact iff the Stone–Čech compactification $\beta\Gamma$ is amenable. Thus we think that the "size" of an amenable compactification of a given group measures the "degree of amenability" of the group. We say that a compactification $\Delta\Gamma$ of Γ is *small at infinity* if for every net (s_n) in Γ with $s_n \to x \in \partial\Gamma$, we have $s_n t \to x$ for every $t \in \Gamma$ [13]. In other words, $\Delta\Gamma$ is small at infinity if every flow in Γ drives Γ to a single point. We note that $\Delta\Gamma$ is small at infinity iff the right multiplication action of Γ extends continuously on $\Delta\Gamma$ in such a way that it is trivial on $\Delta\Gamma \setminus \Gamma$. For instance, the Gromov compactification $\mathbb{F}_r \cup \partial\mathbb{F}_r$ of the free group $\mathbb{F}_r$ (cf. Example 2.2) is small at infinity since the first k segment of st is same as that of s as long as $|s| \geq k + |t|$. The same applies to general hyperbolic groups.

We say that a group Γ *belongs to the class* $\mathcal{S}$ if the compact $(\Gamma \times \Gamma)$-space $\partial^\beta\Gamma = \beta\Gamma \setminus \Gamma$ (with the bilateral action) is amenable. If Γ has an amenable compactification $\Delta\Gamma$ which is small at infinity, then we have $\Gamma \in \mathcal{S}$. It follows that the class $\mathcal{S}$ contains amenable groups and hyperbolic groups (or more generally, any group which is hyperbolic relative to a family of amenable subgroups). The class $\mathcal{S}$ is closed under subgroups and free products (with finite amalgamations). Moreover, the wreath product $\Lambda \wr \Gamma$ of an amenable group Λ by a group $\Gamma \in \mathcal{S}$ again belongs to $\mathcal{S}$ [62]. We observe that an inner amenable group in $\mathcal{S}$ has to be amenable because, by definition, a group Γ is *inner amenable* if $\partial^\beta\Gamma$ carries an invariant Radon probability measure for the conjugation action of Γ (cf. [44]). In general, for a given group Γ and a countable Γ-space K, it is an interesting problem to decide whether or not the compact Γ-space $\partial^\beta K = \beta K \setminus K$ is amenable (cf. [62]). We note the trivial case where the isotropy subgroups are all amenable.

The following is a relative version of smallness.

Definition 3.1. Let $\mathcal{G}$ be a non-empty family of subgroups of Γ. For a net (s_n) in Γ we say that $s_n \to \infty$ *relative to* $\mathcal{G}$ if $s_n \notin s\Lambda t$ for any $s, t \in \Gamma$ and $\Lambda \in \mathcal{G}$. We say that a compactification $\Delta\Gamma$ is *small relative to* $\mathcal{G}$ if for every net (s_n) in Γ with $s_n \to x \in \Delta\Gamma$ and with $s_n \to \infty$ relative to $\mathcal{G}$, we have $s_n t \to x$ for every $t \in \Gamma$.

Suppose that a group Γ acts on a simplicial tree T and let $\overline{T}$ be the compactification defined in the previous section. We recall that the open basis of the topology is given by cutting finitely many edges. We fix a base point $e \in T$ and consider the smallest compactification $\Delta^T\Gamma$ of Γ for which the map $\Gamma \ni s \mapsto s.e \in \overline{T}$ is continuous on $\Delta^T\Gamma$. Then $\Delta^T\Gamma$ is small relative to the family of edge stabilizers. Indeed, suppose that $s_n \to \infty$ relative to edge stabilizers and that $a, b \in T$ are given. Let γ be a path connecting a to b. Since the net $(s_n.\gamma)$ of paths leaves every edge, two end points of $s_n.\gamma$ (i.e., $s_n.a$ and $s_n.b$) converge to the same point (if they converge). The same applies to general fine hyperbolic graphs.

For every non-empty family $\mathcal{G}$ of subgroups of Γ, there exists a compactification $\Delta^{\mathcal{G}}\Gamma$ which is small relative to $\mathcal{G}$ and is largest in the sense that the identity map on Γ extends to a continuous map from $\Delta^{\mathcal{G}}\Gamma$ onto any other compactification which is small relative to $\mathcal{G}$. In the case where $\mathcal{G}$ consists of the trivial subgroup $\{e\}$, a compactification $\Delta\Gamma$ is small relative to $\mathcal{G}$ iff it is small at infinity, and $\Delta^{\mathcal{G}}\Gamma$ is the Higson compactification of the coarse metric space Γ. On the contrary, if $\Gamma \in \mathcal{G}$ then $\Delta^{\mathcal{G}}\Gamma = \beta\Gamma$.

Definition 3.2. Let $\mathcal{G}$ be a non-empty family of subgroups of Γ. We say that $\mathcal{G}$ is *admissible* if there exists an amenable compactification of Γ which is small relative to $\mathcal{G}$, or equivalently if the Γ-space $\Delta^{\mathcal{G}}\Gamma$ is amenable.

From what we have seen, we obtain the following result.

Theorem 3.3. 1. *If* Γ *acts on a uniformly fine hyperbolic graph* K *with amenable isotropy subgroups, then the family of edge stabilizers is admissible. In particular, the trivial family of the trivial subgroup* $\{e\}$ *is admissible for a hyperbolic group.*

2. *Let* $\Gamma = \Gamma_1 \times \Gamma_2$ *be a direct product and suppose that* $\mathcal{G}_i$ *are admissible for* Γ_i. *Then* $\mathcal{G} = \{\Gamma_1\} \times \mathcal{G}_2 \cup \mathcal{G}_1 \times \{\Gamma_2\}$ *is admissible for* Γ.

3. *Let* $\Gamma = \Gamma_1 * \Gamma_2$ *be a free product and suppose that* $\mathcal{G}_i$ *are admissible for* Γ_i. *Then* $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$ *is admissible for* Γ.

4. Application to von Neumann algebra theory

Let Γ be a group and $\mathbb{C}\Gamma$ be its complex group algebra (with the convolution product). The left regular representation λ of $\mathbb{C}\Gamma$ on $\ell_2(\Gamma)$ is given by

$$(\lambda(f)\xi)(t) = (f * \xi)(t) = \sum_{s\in\Gamma} f(s)\xi(s^{-1}t)$$

for $f \in \mathbb{C}\Gamma$ and $\xi \in \ell_2(\Gamma)$. By taking completion w.r.t. an appropriate topology, we obtain the *group von Neumann algebra* [57]

$$\begin{aligned}\mathcal{L}(\Gamma) &= \text{the weak closure of } \{\lambda(f) : f \in \mathbb{C}\Gamma\} \text{ in } \mathbb{B}(\ell_2(\Gamma))\\ &= \{\lambda(f) : f \text{ a function on } \Gamma \text{ such that } \lambda(f) \text{ is bounded on } \ell_2(\Gamma)\},\end{aligned}$$

where $\mathbb{B}(\ell_2)$ is the algebra of all bounded linear operators on $\ell_2(\Gamma)$. We note that $\mathcal{L}$ is functorial w.r.t. inclusions, direct products and free products. The group von Neumann algebra $\mathcal{L}(\Gamma)$ is *finite* in the sense that it has a faithful finite trace

$$\tau : \mathcal{L}(\Gamma) \ni \lambda(f) \mapsto \langle\lambda(f)\delta_e, \delta_e\rangle = f(e) \in \mathbb{C}.$$

If Γ is an infinite abelian group, then we have $\mathcal{L}(\Gamma) = L^\infty(\widehat{\Gamma}) \cong L^\infty[0,1]$ by uniqueness of the Lebesgue measure space without atoms. (Note that we are still assuming that groups are countable.) Thus, group von Neumann algebras of infinite abelian groups are all isomorphic. The center $Z(\mathcal{L}(\Gamma))$ of $\mathcal{L}(\Gamma)$ is easy to describe;

$$Z(\mathcal{L}(\Gamma)) = \overline{Z(\mathbb{C}\Gamma)}^w = \{\lambda(f) : f \text{ is constant on every conjugacy class}\}.$$

A von Neumann algebra with a trivial center is called a *factor*. Since $f = \lambda(f)\delta_e$ belongs to $\ell_2(\Gamma)$ for every $\lambda(f) \in \mathcal{L}(\Gamma)$, the group von Neumann algebra $\mathcal{L}(\Gamma)$ is a factor iff all non-trivial conjugacy classes of Γ are infinite. Such a group Γ is said to be ICC (abbreviation of "Infinite Conjugacy Classes"). Examples of ICC groups include the free group $\mathbb{F}_r$ and the amenable group $S_\infty = \bigcup S_n$ of finite permutations on a countably infinite set. The classification problem of von Neumann factors was raised in [57], where it is shown that $\mathcal{L}(\mathbb{F}_r) \neq \mathcal{L}(S_\infty)$. This result is clarified by Theorem 2.11 that Γ is amenable iff $\mathcal{L}(\Gamma)$ is injective. We note that a von Neumann subalgebra of an injective finite von Neumann algebra is again injective and hence injective finite von Neumann algebras are considered "small". Connes's celebrated theorem [14] asserts that $\mathcal{L}(\Gamma) \cong \mathcal{L}(S_\infty)$ for any amenable ICC group Γ. This can be regarded as uniqueness of the amenable noncommutative measure space. In contrast, it is the biggest open problem in the classification of group factors whether $\mathcal{L}(\mathbb{F}_r) \ncong \mathcal{L}(\mathbb{F}_s)$ for $r \neq s$ or not. Free probability theory was invented [80], [82] to tackle this problem and has revealed deep structures of the free group factors [34], [70], [81]. In particular, the free group factors $\mathcal{L}(\mathbb{F}_r)$ $(2 \leq r < \infty)$ are mutually *stably* isomorphic. Moreover, the following dichotomy is known [24], [71]; the free group factors are all isomorphic or all non-isomorphic.

We briefly review the notion of orbit equivalences, which is the ergodic theory counterpart of that of group von Neumann algebras. Let (X,μ) be a Lebesgue probability measure space with a measurable non-singular action of a group Γ. Then we have a group-measure-space von Neumann algebra $\mathcal{L}(X \rtimes \Gamma, \mu)$ which is generated by $L^\infty(X,\mu)$ and a copy of $\mathcal{L}(\Gamma)$ [57]. The von Neumann algebra $\mathcal{L}(X \rtimes \Gamma, \mu)$ is finite if the Γ-action is m.p. (measure preserving) and is a factor if the Γ-action is e.f. (ergodic and free). For two e.f.m.p. actions $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$, we have

$$(L^\infty(X,\mu) \subset \mathcal{L}(X \rtimes \Gamma, \mu)) \cong (L^\infty(Y,\nu) \subset \mathcal{L}(Y \rtimes \Lambda, \nu))$$

iff they are *orbit equivalent* [27], [57], i.e., there exists an isomorphism $F\colon X \to Y$ of measure spaces such that $F(\Gamma x) = \Lambda F(x)$ for a.e. $x \in X$. Thus the classification of von Neumann algebras and that of orbit equivalences are closely related. We note that it is possible that $\mathcal{L}(X \rtimes \Gamma, \mu) \cong \mathcal{L}(Y \rtimes \Lambda, \nu)$ without being orbit equivalent [17]. We say that two e.f.m.p. actions are *stably orbit equivalent* (or *weakly orbit equivalent*) if they are orbit equivalent "after stabilization", and that two groups Γ and Λ are (resp. *stably*) *orbit equivalent* if they have e.f.m.p. actions which are (resp. stably) orbit equivalent. As we mentioned at the end of Section 2, two groups are stably orbit equivalent iff they are measure equivalent. Connes's aforementioned theorem has the following counterpart [58]; e.f.m.p. actions of amenable groups are all orbit equivalent to each other. Beyond the amenable case, there has been remarkable progress [86] and exciting new developments [28], [29], [30], [31], [33], [43], [56], [67] in this subject. In particular, it is shown that free groups of different ranks are mutually non-orbit equivalent [30]. We do not further elaborate on ergodic theory, but refer to [32], [68], [75] for details. Before leaving this subject, we mention that as far as we know, the following bold conjecture (communicated to us by D. Shlyakhtenko) stands; ICC groups are (stably) orbit equivalent iff they have (stably) isomorphic group von Neumann algebras.

We now focus on von Neumann algebras. Generally speaking, distinguishing group von Neumann algebras is a difficult task. Indeed, most of known invariants for group von Neumann algebras are binary; injectivity, the property (Γ), the property (T), Haagerup's property, etc. A notable exception is the Cowling–Haagerup constant [19]. Free entropy (dimension) [80], [82] and L^2-homology [18], [55] are candidates for invariants. Recently, a breakthrough was obtained in [66], where a longstanding problem from [57] is solved. It is shown that under certain circumstances, one can specify the position of a prescribed von Neumann subalgebra in the ambient von Neumann algebra. This versatile method found several applications [33], [43], [67], [68]. The following result is obtained by combining this device with theory of exact C*-algebras. In the last few pages, we allow ourselves to be more technical.

Theorem 4.1. *Let Γ be a group and $\mathcal{G}$ be an admissible family of its subgroups. Suppose that $\mathcal{N} \subset \mathcal{L}(\Gamma)$ is an injective von Neumann subalgebra whose relative commutant $\mathcal{N}' \cap \mathcal{L}(\Gamma)$ is non-injective. Then there exist $\Lambda \in \mathcal{G}$ and a non-zero projection $p \in \mathcal{N}$ such that $p\mathcal{N}p$ is conjugated into $\mathcal{L}(\Lambda)$ by a partial isometry in $\mathcal{L}(\Gamma)$.*

We recall that $\mathcal{N}' \cap \mathcal{M} = \{a \in \mathcal{M} : [a, \mathcal{N}] = \{0\}\}$. Sometimes we can patch the pieces $p\mathcal{N}p$ together and find a unitary element $u \in \mathcal{L}(\Gamma)$ such that $u\mathcal{N}u^* \subset \mathcal{L}(\Lambda)$. By Theorems 3.3 and 4.1 and a bit more effort, we obtain the following corollaries.

Recall that a von Neumann algebra $\mathcal{M}$ is *prime* if it does not decompose into a tensor product of two infinite dimensional (diffuse) von Neumann algebras. The free group factor $\mathcal{L}(\mathbb{F}_r)$ is the first example of a separable prime factor [34].

Corollary 4.2 ([61]). *Suppose that Γ belongs to the class $\mathcal{S}$. Then $\mathcal{L}(\Gamma)$ is solid, i.e., for any diffuse subalgebra $\mathcal{N} \subset \mathcal{L}(\Gamma)$, the relative commutant $\mathcal{N}' \cap \mathcal{L}(\Gamma)$ is injective. In particular, $\mathcal{L}(\Gamma)$ is prime unless Γ is amenable.*

Indeed, replacing $\mathcal{N}$ with its maximal abelian subalgebra, we may assume that $\mathcal{N}$ is injective. Then $p\mathcal{N}p$ is conjugated into $\mathcal{L}(\{e\}) = \mathbb{C}$ iff the projection p is atomic in $\mathcal{N}$. Thus, if $\mathcal{N}' \cap \mathcal{L}(\Gamma)$ is non-injective, then $\mathcal{N}$ is not diffuse. We note that if a solid group von Neumann algebra $\mathcal{L}(\Gamma) \cong \mathcal{N}_1 \otimes \mathcal{N}_2$ is not prime, then both $\mathcal{N}_i$ have to be injective and hence $\mathcal{L}(\Gamma)$ itself is injective. Similarly, one can prove that a group-measure-space von Neumann algebra $\mathcal{L}(X \rtimes \Gamma, \mu)$ for $\Gamma \in \mathcal{S}$ is prime [62]. This generalizes a result in [2] that an orbit equivalence relation of a hyperbolic group is indecomposable. In passing, we mention that an analogous result is obtained for discrete quantum groups and their von Neumann algebras [79]. The following is a von Neumann algebra analogue of the result in [56].

Corollary 4.3 ([64]). *Let $\Gamma_1, \dots, \Gamma_n \in \mathcal{S}$ and assume that they are all non-amenable and ICC. If $\mathcal{M}_1, \dots, \mathcal{M}_m$ are non-injective factors such that*

$$\bigotimes_{j=1}^{m} \mathcal{M}_j \subset \bigotimes_{i=1}^{n} \mathcal{L}(\Gamma_i),$$

then we have $m \le n$. If in addition $m = n$, then we have "$\mathcal{M}_i \subset \mathcal{L}(\Gamma_i)$" modulo permutation of indices, rescaling, and unitary conjugacy.

We have a Kurosh type theorem for non-prime von Neumann factors. Another version of Kurosh type theorem in presence of rigidity is found in [43], [68].

Corollary 4.4 ([62]). *Let $\Gamma_1, \dots, \Gamma_n$ and $\Lambda_1, \dots, \Lambda_m$ be ICC exact non-amenable groups all of which decompose into non-trivial direct products. Suppose that*

$$\mathcal{L}(\mathbb{F}_\infty * \Lambda_1 * \cdots * \Lambda_m) \cong \mathcal{L}(\mathbb{F}_\infty * \Gamma_1 * \cdots * \Gamma_n).$$

Then $n = m$ and, modulo permutation of indices, $\mathcal{L}(\Lambda_i)$ is unitarily conjugated onto $\mathcal{L}(\Gamma_i)$ for every $i \ge 1$.

It follows that iterated free product factors $\mathcal{L}(\mathbb{F}_\infty * (\mathbb{F}_\infty \times S_\infty)^{*n})$ are mutually non-isomorphic. In contrast, $\mathcal{L}(\mathbb{F}_\infty * (\mathbb{F}_\infty \times \mathbb{Z})^{*n})$ are all isomorphic [23].

We describe one more application of amenable actions in von Neumann algebra theory and ergodic theory. Let Γ be a group acting on a group Λ by automorphisms. Then the semi-direct product $(\Lambda \times \Lambda) \rtimes \Gamma$ naturally acts on Λ, where $\Lambda \times \Lambda$ acts on Λ bilaterally. This action extends to a continuous action on the Stone–Čech compactification $\beta\Lambda$ and then restricts to $\partial^\beta \Lambda = \beta\Lambda \setminus \Lambda$.

Proposition 4.5. *Let Γ and Λ be as above with Λ amenable and assume that the compact $(\Lambda \times \Lambda) \rtimes \Gamma$-space $\partial^\beta \Lambda$ is amenable. Then, for any diffuse von Neumann subalgebra $\mathcal{N} \subset \mathcal{L}(\Lambda) \subset \mathcal{L}(\Lambda \rtimes \Gamma)$, the relative commutant $\mathcal{N}' \cap \mathcal{L}(\Lambda \rtimes \Gamma)$ is injective.*

This proposition applies to the wreath product $\Lambda \wr \Gamma = \left(\bigoplus_\Gamma \Lambda\right) \rtimes \Gamma$ for every amenable group Λ and for every exact group Γ. In the case where Λ is an infinite abelian group, the group factor $\mathcal{L}(\Lambda \wr \Gamma)$ is isomorphic to the group-measure-space factor $\mathcal{L}([0,1]^\Gamma \rtimes \Gamma)$ of the Bernoulli shift action over the base space ([0, 1], Lebesgue). Hence, we obtain the following corollary. We note that the same holds for a noncommutative Bernoulli shift by setting $\Lambda = S_\infty$.

Corollary 4.6 ([62]). *Let Γ be an exact group and $\mathcal{M} = \mathcal{L}([0,1]^\Gamma \rtimes \Gamma)$ be the group-measure-space factor of the Bernoulli shift action. Then, for any diffuse von Neumann subalgebra $\mathcal{A} \subset L^\infty([0,1]^\Gamma)$, the relative commutant $\mathcal{A}' \cap \mathcal{M}$ is injective. In particular, the orbit equivalence relation of a Bernoulli shift action by an exact non-amenable group is indecomposable.*

References

[1] Alibegović, E., A combination theorem for relatively hyperbolic groups. *Bull. London Math. Soc.* **37** (2005), 459–466.

[2] Adams, S., Boundary amenability for word hyperbolic groups and an application to smooth dynamics of simple groups. *Topology* **33** (1994), 765–783.

[3] Anantharaman-Delaroche, C., Systèmes dynamiques non commutatifs et moyennabilité. *Math. Ann.* **279** (1987), 297–315.

[4] Anantharaman-Delaroche, C., Purely infinite C*-algebras arising from dynamical systems. *Bull. Soc. Math. France* **125** (1997), 199–225.

[5] Anantharaman-Delaroche C., Renault, J., *Amenable groupoids.* With a foreword by Georges Skandalis and Appendix B by E. Germain. Monographies de L'Enseignement Mathématique 36, Geneva 2000.

[6] Bell, G. C., Property A for groups acting on metric spaces. *Topology Appl.* **130** (2003), 239–251.

[7] Bell, G., Dranishnikov, A. On asymptotic dimension of groups. *Algebr. Geom. Topol.* **1** (2001), 57–71.

[8] Bell, G., Dranishnikov, A., Asymptotic dimension in Bedlewo. Preprint, 2005.

[9] Biane, P., Germain, E., Actions moyennables et fonctions harmoniques. *C. R. Math. Acad. Sci. Paris* **334** (2002), 355–358.

[10] Bowditch, B. H., Relatively hyperbolic groups. Preprint, 1999.

[11] Brown, N. P., Ozawa, N., *C*-algebras and finite dimensional approximations.* Book in preparation.

[12] Campbell, S. J., Niblo, G. A., Hilbert space compression and exactness for discrete groups. *J. Funct. Anal.* **222** (2005), 292–305.

[13] Carlsson, G., Pedersen, E., Controlled algebra and the Novikov conjectures for K- and L-theory. *Topology* **34** (1995), 731–758.

[14] Connes, A., Classification of injective factors. *Ann. of Math.* (2) **104** (1976), 73–115.

[15] Connes, A., *Noncommutative geometry.* Academic Press, Inc., San Diego, CA, 1994.

[16] Connes, A., Feldman, J., Weiss, B., An amenable equivalence relation is generated by a single transformation. *Ergodic Theory Dynam. Systems* **1** (1981), 431–450.

[17] Connes, A., Jones, V., A II_1 factor with two nonconjugate Cartan subalgebras. *Bull. Amer. Math. Soc.* **6** (1982), 211–212.

[18] Connes, A., Shlyakhtenko, D., L^2-Homology for von Neumann Algebras. Preprint, 2003.

[19] Cowling, M., Haagerup, U., Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one. *Invent. Math.* **96** (1989), 507–549.

[20] Dadarlat, M., Guentner, E., Uniform embeddability of relatively hyperbolic groups. Preprint, 2005.

[21] Dahmani, F., Combination of convergence groups. *Geom. Topol.* **7** (2003), 933–963.

[22] Dranishnikov, A., Januszkiewicz, T., Every Coxeter group acts amenably on a compact space. In *Proceedings of the 1999 Topology and Dynamics Conference* (Salt Lake City, UT). Topology Proc. **24** (1999), Spring, 135–141.

[23] Dykema, K., Free products of hyperfinite von Neumann algebras and free dimension. *Duke Math. J.* **69** (1993), 97–119.

[24] Dykema, K., Interpolated free group factors. *Pacific J. Math.* **163** (1994), 123–135.

[25] Dykema, K. J., Exactness of reduced amalgamated free product C^*-algebras. *Forum Math.* **16** (2004), 161–180.

[26] Farb, B., Relatively hyperbolic groups. *Geom. Funct. Anal.* **8** (1998), 810–840.

[27] Feldman, J., Moore, C. C., Ergodic equivalence relations, cohomology, and von Neumann algebras. I, II. *Trans. Amer. Math. Soc.* **234** (1977), 289–359.

[28] Furman, A., Gromov's measure equivalence and rigidity of higher rank lattices. *Ann. of Math.* (2) **150** (1999), 1059–1081.

[29] Furman, A., Orbit equivalence rigidity. *Ann. of Math.* (2) **150** (1999), 1083–1108.

[30] Gaboriau, D., Coût des relations d'équivalence et des groupes. *Invent. Math.* **139** (2000), 41–98.

[31] Gaboriau, D., Invariants l^2 de relations d'équivalence et de groupes. *Inst. Hautes Études Sci. Publ. Math.* **95** (2002), 93–150.

[32] Gaboriau, D., On orbit equivalence of measure preserving actions. In *Rigidity in dynamics and geometry* (Cambridge, 2000), Springer-Verlag, Berlin 2002, 167–186.

[33] Gaboriau, D., Popa, S., An uncountable family of nonorbit equivalent actions of $\mathbb{F}_n$. *J. Amer. Math. Soc.* **18** (2005), 547–559.

[34] Ge, L., Applications of free entropy to finite von Neumann algebras. II. *Ann. of Math.* (2) **147** (1998), 143–157.

[35] Gromov, M., Hyperbolic groups. In *Essays in group theory*, Math. Sci. Res. Inst. Publ., **8**, Springer-Verlag, New York 1987, 75–263.

[36] Gromov, M., Asymptotic invariants of infinite groups. In *Geometric group theory* (Sussex, 1991), Vol. 2, London Math. Soc. Lecture Note Ser. 182, Cambridge University Press, Cambridge 1993, 1–295.

[37] Gromov, M., Random walk in random groups. *Geom. Funct. Anal.* **13** (2003), 73–146.

[38] Guentner, E., Exactness of the one relator groups. *Proc. Amer. Math. Soc.* **130** (2002), 1087–1093.

[39] Guentner, E., Higson N., Weinberger, S., The Novikov Conjecture for Linear Groups. Preprint, 2003.

[40] Guentner, E., Kaminker, J., Exactness and the Novikov conjecture. *Topology* **41** (2002), 411–418.

[41] Hakeda, J., Tomiyama, J., On some extension properties of von Neumann algebras. *Tôhoku Math. J.* (2) **19** (1967) 315–323.

[42] Hamenstädt, U., Geometry of the mapping class groups I: Boundary amenability. Preprint, 2005.

[43] Ioana, A., Peterson, J., Popa, S., Amalgamated free products of w-rigid factors and calculation of their symmetry groups. Preprint, 2005.

[44] de la Harpe, P., Groupes hyperboliques, algèbres d'opérateurs et un théorème de Jolissaint. *C. R. Acad. Sci. Paris Sér. I Math.* **307** (1988), 771–774.

[45] Higson, N., Bivariant K-theory and the Novikov conjecture. *Geom. Funct. Anal.* **10** (2000), 563–581.

[46] Higson, N., Kasparov, G., E-theory and KK-theory for groups which act properly and isometrically on Hilbert space. *Invent. Math.* **144** (2001), 23–74.

[47] Higson N., Roe, J., Amenable group actions and the Novikov conjecture. *J. Reine Angew. Math.* **519** (2000), 143–153.

[48] Kaimanovich, V., Boundary amenability of hyperbolic spaces. In *Discrete geometric analysis*, Contemp. Math. 347, Amer. Math. Soc., Providence, RI, 2004, 83–111.

[49] Kida, Y., The mapping class group from the viewpoint of measure equivalence theory. Preprint, 2005.

[50] Kirchberg, E., On nonsemisplit extensions, tensor products and exactness of group C^*-algebras. *Invent. Math.* **112** (1993), 449–489.

[51] Kirchberg, E., Exact C^*-algebras, tensor products, and the classification of purely infinite algebras. In *Proceedings of the International Congress of Mathematicians* (Zürich, 1994), Vol. 2, Birkhäuser, Basel 1995, 943–954.

[52] Kirchberg, E., Wassermann, S., Permanence properties of C^*-exact groups. *Doc. Math.* **4** (1999), 513–558.

[53] Laca, M., Spielberg, J., Purely infinite C^*-algebras from boundary actions of discrete groups. *J. Reine Angew. Math.* **480** (1996), 125–139.

[54] Lance, C., On nuclear C^*-algebras. *J. Funct. Anal.* **12** (1973), 157–176.

[55] Mineyev, I., Shlyakhtenko, D., Non-microstates free entropy dimension for groups. *Geom. Funct. Anal.* **15** (2005), 476–490.

[56] Monod, N., Shalom, Y., Orbit equivalence rigidity and bounded cohomology. *Ann. of Math.* (2), to appear.

[57] Murray, F. J., von Neumann, J., On rings of operators. IV. *Ann. of Math.* (2) **44** (1943), 716–808.

[58] Ornstein, D. S., Weiss, B., Ergodic theory of amenable group actions. I. The Rohlin lemma. *Bull. Amer. Math. Soc.* **2** (1980), 161–164.

[59] Osin, D. V., Asymptotic dimension of relatively hyperbolic groups. Preprint, 2004.

[60] Ozawa, N., Amenable actions and exactness for discrete groups. *C. R. Acad. Sci. Paris Sér. I Math.* **330** (2000), 691–695.

[61] Ozawa, N., Solid von Neumann algebras. *Acta Math.* **192** (2004), 111–117.

[62] Ozawa, N., A Kurosh type theorem for type II_1 factors. Preprint, 2004.

[63] Ozawa, N., Boundary amenability of relatively hyperbolic groups. *Topology Appl.*, to appear.

[64] Ozawa, N., Popa, S., Some prime factorization results for type II_1 factors. *Invent. Math.* **156** (2004), 223–234.

[65] Paterson, A. L. T., *Amenability*. Math. Surveys Monogr. 29, Amer. Math. Soc., Providence, RI, 1988.

[66] Popa, S., On the fundamental group of type II_1 factors. *Proc. Natl. Acad. Sci. USA* **101** (2004), 723–726.

[67] Popa, S., Strong rigidity of II_1 factors arising from malleable actions of w-rigid groups, I, II. Preprint, 2003.

[68] Popa, S., Deformation and rigidity for group actions and von Neumann algebras. In *Proceedings of the International Congress of Mathematicians* (Madrid, 2006), Volume I, EMS Publishing House, Zürich 2006/2007.

[69] Pisier, G., *Introduction to operator space theory.* London Math. Soc. Lecture Note Ser. 294, Cambridge University Press, Cambridge 2003.

[70] Rădulescu, F., The fundamental group of the von Neumann algebra of a free group with infinitely many generators is $\mathbf{R}_+ \setminus \{0\}$. *J. Amer. Math. Soc.* **5** (1992), 517–532.

[71] Rădulescu, F., Random matrices, amalgamated free products and subfactors of the von Neumann algebra of a free group, of noninteger index. *Invent. Math.* **115** (1994), 347–389.

[72] Robertson, G., Steger, T., C^*-algebras arising from group actions on the boundary of a triangle building. *Proc. London Math. Soc.* (3) **72** (1996), 613–637.

[73] Rørdam, M., Structure and classification of C^*-algebras. In *Proceedings of the International Congress of Mathematicians* (Madrid, 2006), Volume II, EMS Publishing House, Zürich 2006, 1581–1598.

[74] Sakai, S., On the hyperfinite II_1-factor. *Proc. Amer. Math. Soc.* **19** (1968) 589–591.

[75] Shalom, Y., Measurable Group Theory. In *European Congress of Mathematics* (Stockholm, 2004), European Mathematical Society Publishing House, Zürich 2005, 391–424.

[76] Skandalis, G., Tu, J. L., Yu, G., The coarse Baum-Connes conjecture and groupoids. *Topology* **41** (2002), 807–834.

[77] Spielberg, J., Free-product groups, Cuntz-Krieger algebras, and covariant maps. *Internat. J. Math.* **2** (1991), 457–476.

[78] Tu, J. L., Remarks on Yu's "property A" for discrete metric spaces and groups. *Bull. Soc. Math. France* **129** (2001), 115–139.

[79] Vaes, S., Vergnioux, R., The boundary of universal discrete quantum groups, exactness and factoriality. Preprint, 2005.

[80] Voiculescu, D., Free probability theory: random matrices and von Neumann algebras. *Proceedings of the International Congress of Mathematicians* (Zürich, 1994), Vol. 1, Birkhäuser, Basel 1995, 227–241.

[81] Voiculescu, D., The analogues of entropy and of Fisher's information measure in free probability theory. III. The absence of Cartan subalgebras. *Geom. Funct. Anal.* **6** (1996), 172–199.

[82] Voiculescu, D., Free probability and the von Neumann algebras of free groups. *Rep. Math. Phys.* **55** (2005), 127–133.

[83] Yu, G., The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. *Invent. Math.* **139** (2000), 201–240.

[84] Yu, G., Higher index theory of elliptic operators and geometry of groups. In *Proceedings of the International Congress of Mathematicians* (Madrid, 2006), Volume II, EMS Publishing House, Zürich 2006, 1623–1639.

[85] Zimmer, R. J., Amenable ergodic group actions and an application to Poisson boundaries of random walks. J. Funct. Anal. **27** (1978), 350–372.

[86] Zimmer, R. J., *Ergodic theory and semisimple groups.* Monogr. Math. 81, Birkhäuser, Basel 1984.

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