

Higher index theory of elliptic operators and geometry of groups

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Abstract. The Atiyah–Singer index theorem has been vastly generalized to higher index theory for elliptic operators in the context of noncommutative geometry. Higher index theory has important applications to problems in differential topology and differential geometry such as the Novikov Conjecture on homotopy invariance of higher signatures and the existence problem of Riemannian metrics with positive scalar curvature. In this article, I will give a survey on recent development of higher index theory, its applications, and its fascinating connection to the geometry of groups and metric spaces.

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1. Introduction

An elliptic differential operator D on a compact manifold M is Fredholm in the sense that the kernel and cokernel of D are finite dimensional and the image of D is closed. The Fredholm index of D is defined to be

$$\text{index}(D) = \dim(\ker D) - \dim(\text{coker } D).$$

Roughly speaking, $\text{index}(D)$ measures the size of the solution space of a certain system of differential equations associated to D . The Fredholm index has the following fundamental properties: (1) it is an obstruction to invertibility of D ; (2) it is invariant under homotopy equivalence. These are essential properties for the purpose of applications. The celebrated Atiyah–Singer index theorem computes the Fredholm index of elliptic differential operators on compact manifolds and has important applications [4].

Elliptic differential operators on noncompact manifolds are in general not Fredholm in the usual sense but Fredholm in a generalized sense. The generalized Fredholm index for such operators is called the higher index. Higher index theories for elliptic operators in various noncompact situations have been successfully developed

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by Kasparov [51], [52], Mishchenko–Fomenko [60], Baum–Connes [6], Connes–Skandalis [23], Connes–Moscovici [22], and Roe [67]. In this survey, we will focus on higher index theory in the following two cases: (1) noncompact manifolds with proper and free cocompact actions of discrete groups, (2) noncompact complete Riemannian manifolds.

In the case of a manifold \tilde{M} with a proper and free cocompact action of a discrete group Γ , let M be the compact quotient \tilde{M}/Γ , let D be an elliptic differential operator on M and let \tilde{D} be the lifting of D to \tilde{M} . \tilde{D} is a generalized Fredholm operator and its generalized Fredholm index is an element in the K -theory of the reduced group C^* -algebra $C_r^*(\Gamma)$ [51], [52], [60], [6]. The Fredholm index of D is essentially the 0-dimensional information of the generalized Fredholm index of \tilde{D} . For this reason, the generalized Fredholm index of \tilde{D} is called the higher index of \tilde{D} . The higher index of \tilde{D} , denoted by $H\text{-index}(\tilde{D})$, has properties similar to that of the classical Fredholm index: (1) $H\text{-index}(\tilde{D})$ is an obstruction to invertibility of \tilde{D} ; (2) $H\text{-index}(\tilde{D})$ is invariant under homotopy equivalence. These properties are crucial for purpose of applications. For example, property (1) implies that if $H\text{-index}(\tilde{D})$ is non-zero for the Dirac operator, then the manifold M can not carry a Riemannian metric of positive scalar curvature (as a consequence of the Lichnerowicz formula, positive scalar curvature implies that the Dirac operator \tilde{D} is invertible) [69]. The Baum–Connes Conjecture provides an algorithm to compute the higher index of \tilde{D} . The Baum–Connes Conjecture was proved by Higson–Kasparov when Γ has the Haagerup property (e.g. all amenable groups) [43] and by Lafforgue for a large class of groups with Property T [56]. I should also mention important work by Puschnigg [66] and Chabert–Echterhoff–Nest [14]. The general problem of computing the higher index of \tilde{D} is still wide open. However, for the purpose of applications to geometry and topology, it is often enough to determine when the higher index is non-zero. The Strong Novikov Conjecture is an algorithm of determining non-vanishing of the higher index. Currently much more is known about this conjecture than the Baum–Connes Conjecture. In this article, we will focus on the recent development of the Strong Novikov Conjecture.

In the case of a general noncompact complete Riemannian manifold M , Roe has introduced a higher index theory for elliptic differential operators on M [67]. The Coarse Baum–Connes Conjecture is an algorithm to compute the higher index of elliptic differential operators on noncompact complete Riemannian manifolds. This conjecture has been proved for a large class of interesting spaces. In general, there are counter-examples to the Coarse Baum–Connes Conjecture [80], [44]. The Coarse Strong Novikov Conjecture is an algorithm of determining non-vanishing of the higher index. There is an unbounded geometry counter-example to this conjecture [80]. This conjecture is still open for spaces with bounded geometry. In this article, we will also report recent progress on the Coarse Baum–Connes Conjecture and the Coarse Strong Novikov Conjecture.

There is a beautiful link between higher index theory and a certain aspect of metric geometry. At this stage, this part of metric geometry is mostly uncharted territory.

One purpose of this survey is to advertise this aspect of metric geometry.

We remark that all manifolds in this article are smooth.

2. Higher index theory of elliptic operators

In this section we briefly review the higher index theory of elliptic operators. The first part of this section is devoted to higher index theory for noncompact manifolds with a proper and free cocompact action of a discrete group [51], [52], [60], [6]. In the second part of this section, we discuss higher index theory for noncompact complete Riemannian manifolds [67].

2.1. Higher index theory and discrete groups. Let Γ be a discrete group acting properly and freely on a manifold \tilde{M} with compact quotient $M = \tilde{M}/\Gamma$.

We first recall the concept of the reduced group C^* -algebra for any countable discrete group Γ .

Let $l^2(\Gamma)$ be the Hilbert space defined by

$$l^2(\Gamma) = \{\xi : \Gamma \rightarrow \mathbb{C} \mid \sum_{\gamma \in \Gamma} |\xi(\gamma)|^2 < \infty\}.$$

For each $g \in \Gamma$ we define a unitary operator U_g acting on $l^2(\Gamma)$ by

$$(U_g \xi)(\gamma) = \xi(g^{-1}\gamma)$$

for all $\xi \in l^2(\Gamma)$ and $\gamma \in \Gamma$.

We define the group algebra $\mathbb{C}\Gamma$ by

$$\mathbb{C}\Gamma = \left\{ \sum_{g \in \Gamma} c_g U_g : c_g \in \mathbb{C} \right\},$$

where $\sum_{g \in \Gamma} c_g U_g$ is a finite sum. Observe that $\mathbb{C}\Gamma$ is an algebra over \mathbb{C} .

Definition 2.1. The *reduced* group C^* -algebra $C_r^*(\Gamma)$ is the closure of $\mathbb{C}\Gamma$ under operator norm.

In general, $C_r^*(\Gamma)$ is a highly noncommutative C^* -algebra and is a typical example of a “noncommutative space” in Connes’ noncommutative geometry [16].

Let D be an elliptic differential operator on the compact manifold M . Let \tilde{D} be the lifting of D to \tilde{M} . Recall that a classical theorem in functional analysis says that an operator is Fredholm if and only if it is invertible modulo K , the algebra of all compact operators. \tilde{D} is in general not Fredholm. However, \tilde{D} is a generalized Fredholm operator in the sense that \tilde{D} is invertible modulo $C_r^*(\Gamma) \otimes K$. By a standard procedure in K -theory, one can define the higher index of \tilde{D} , denoted by H -index(\tilde{D}), as an element of the K -group $K_0(C_r^*(\Gamma))$. When D is a self-adjoint elliptic differential operator, we can define the higher index of \tilde{D} , denoted by H -index(\tilde{D}), as an element of the K -group $K_1(C_r^*(\Gamma))$. The higher index of \tilde{D} has the following important

properties: (1) it is an obstruction to invertibility of \tilde{D} , (2) it is invariant under homotopy equivalence of \tilde{D} .

Let $\text{tr}: C_r^*(\Gamma) \rightarrow \mathbb{C}$ be the canonical trace defined by

$$\text{tr}(T) = \langle T\delta_e, \delta_e \rangle$$

for every $T \in C_r^*(\Gamma)$, where $\delta_e \in l^2(\Gamma)$ is the Dirac function at the identity element e of the group Γ . This trace induces a homomorphism

$$\text{tr}_*: K_0(C_r^*(\Gamma)) \rightarrow \mathbb{C}.$$

Atiyah's L^2 -index theorem in [3] implies the following identity:

$$\text{tr}_*(H\text{-index}(\tilde{D})) = \text{index}(D).$$

It follows that the Fredholm index of D is the 0-dimensional information of the higher index of \tilde{D} .

Let Z be a locally compact space with a proper cocompact Γ -action. The Γ -equivariant K -homology group $K_0^\Gamma(Z)$ is generated by all Γ -invariant “abstract elliptic operators” on Z [51], [52]. Similarly, the Γ -equivariant K -homology group $K_1^\Gamma(Z)$ is generated by all Γ -invariant self-adjoint “abstract elliptic operators” on Z .

Let $\mathcal{E}\Gamma$ be the universal space for proper Γ -actions [7]. The Γ -equivariant K -homology group $K_*^\Gamma(\mathcal{E}\Gamma)$ is defined to be the inductive limit

$$\lim_{Z \subseteq \mathcal{E}\Gamma} K_*^\Gamma(Z),$$

where the inductive limit is taken over all Γ -invariant and Γ -cocompact subsets of $\mathcal{E}\Gamma$.

The Baum–Connes map μ associates each Γ -invariant “abstract elliptic operator” to its higher index:

$$\mu: K_*^\Gamma(\mathcal{E}\Gamma) \rightarrow K_*(C_r^*(\Gamma)).$$

Conjecture 2.2 (*The Baum–Connes Conjecture* [6], [7]). Let Γ be a countable discrete group. The Baum–Connes map $\mu: K_*^\Gamma(\mathcal{E}\Gamma) \rightarrow K_*(C_r^*(\Gamma))$ is an isomorphism for Γ .

Let Γ be a discrete group acting properly and freely on a manifold \tilde{M} with compact quotient $M = \tilde{M}/\Gamma$. Let D be an elliptic differential operator on M and \tilde{D} be its lifting to \tilde{M} . Denote by $[\tilde{D}]$ the K -homology class of \tilde{D} in $K_*^\Gamma(\tilde{M})$. By the universality of $\mathcal{E}\Gamma$, there exists a Γ -invariant classifying map $f: \tilde{M} \rightarrow \mathcal{E}\Gamma$.

We have

$$H\text{-index}(\tilde{D}) = \mu(f_*[\tilde{D}]).$$

It follows that the Baum–Connes Conjecture would reduce the computation of the higher index $H\text{-index}(\tilde{D})$ to that of $f_*[\tilde{D}]$ in $K_*^\Gamma(\mathcal{E}\Gamma)$, which is in principle computable.

Conjecture 2.3 (*The Strong Novikov Conjecture*). Let Γ be a countable discrete group. The Baum–Connes map $\mu: K_*^\Gamma(\mathcal{E}\Gamma) \rightarrow K_*(C_r^*(\Gamma))$ is injective for Γ .

The Strong Novikov Conjecture would reduce the non-vanishing problem of the higher index H -index(\tilde{D}) to that of $f_*[\tilde{D}]$ in $K_*^\Gamma(\mathcal{E}\Gamma)$, which is in principle decidable.

If M is an aspherical compact manifold (i.e. its universal cover is contractible), the Strong Novikov Conjecture implies the Gromov–Lawson Conjecture which claims that an aspherical compact manifold can not carry a Riemannian metric with positive scalar curvature [67]. Let Γ be the fundamental group of M and \tilde{M} be the universal cover of M . In this case f_* is an isomorphism and the Strong Novikov Conjecture implies that the higher index of the Dirac operator is non-zero [67]. However, by the Lichnerowicz formula, positive scalar curvature would imply invertibility of the Dirac operator. In general, the Strong Novikov Conjecture implies a stable version of the Gromov–Lawson–Rosenberg Conjecture [73]. This stable version of the Gromov–Lawson–Rosenberg Conjecture provides a complete answer to the question when a compact manifold can stably carry a Riemannian metric with positive scalar curvature.

Another very important corollary of the Strong Novikov Conjecture is the Novikov Conjecture on homotopy invariance of higher signatures. The Novikov Conjecture is a central problem in the classification of higher dimensional compact manifolds (see [32] for a historic account of the Novikov Conjecture). In the case of aspherical compact manifolds, the Novikov Conjecture states that rational Pontryagin classes are homotopy invariants. In this case the Novikov Conjecture can be considered as an infinitesimal version of the Borel Conjecture, which claims that any aspherical compact manifold M is rigid in the sense that if another compact manifold M' is homotopy equivalent to M , then M' is homeomorphic to M . The Strong Novikov Conjecture implies an integral version of the Novikov Conjecture in L -theory after inverting 2 [70]. This integral version of the Novikov Conjecture implies the stable Borel Conjecture which states that if a compact manifold M is aspherical and another compact manifold M' is homotopy equivalent to M , then $M' \times \mathbb{R}^n$ is homeomorphic to $M \times \mathbb{R}^n$ for some large n [31].

2.2. Higher index theory for noncompact complete Riemannian manifolds. We shall briefly review Roe's higher index theory for noncompact complete Riemannian manifolds [67].

We first recall the concept of Roe algebra. Let Γ be a locally finite discrete metric space (recall that a discrete metric space is said to be locally finite if every ball has finitely many elements). Let H be a separable and infinite dimensional Hilbert space. We decompose

$$l^2(\Gamma) \otimes H = \bigoplus_{\gamma \in \Gamma} (\delta_\gamma \otimes H),$$

where $\delta_\gamma \in l^2(\Gamma)$ is the Dirac function at γ . For each bounded linear operator $T: l^2(\Gamma) \otimes H \rightarrow l^2(\Gamma) \otimes H$, we have a corresponding decomposition:

$$T = (T_{x,y})_{x,y \in \Gamma},$$

where $T_{x,y}$ is a bounded linear operator from $\delta_y \otimes H$ to $\delta_x \otimes H$.

Definition 2.4 (Roe [67]). Let Γ be a locally finite discrete metric space.

- (1) A bounded linear operator $T: l^2(\Gamma) \otimes H \rightarrow l^2(\Gamma) \otimes H$ is said to be *locally compact* if $T_{x,y}$ is compact for all $x, y \in \Gamma$;
- (2) A bounded linear operator $T: l^2(\Gamma) \otimes H \rightarrow l^2(\Gamma) \otimes H$ is said to have *finite propagation* if there exists $R \geq 0$ such that $T_{x,y} = 0$ for all $x, y \in \Gamma$ satisfying $d(x, y) > R$.
- (3) The *Roe algebra* $C^*(\Gamma)$ is defined to be the operator norm closure of all locally compact operators acting on $l^2(\Gamma) \otimes H$ with finite propagation.

An important feature of the Roe algebra is that, up to $*$ -isomorphism, it depends only on the quasi-isometry type (or more generally the coarse type) of the locally finite discrete metric space.

If Z is a locally compact metric space, we choose a net Γ in Z and define the Roe algebra $C^*(Z)$ by $C^*(\Gamma)$. Recall that a locally finite discrete subspace $\Gamma \subseteq Z$ is said to be a net if there exists $c \geq 0$ satisfying $d(z, \Gamma) \leq c$ for every $z \in Z$. We observe that the definition of $C^*(Z)$ is independent of the choice of Γ up to $*$ -isomorphism.

If M is a noncompact complete Riemannian manifold and D is a geometric elliptic operator on M , then D is a generalized Fredholm operator in the sense that it is invertible modulo $C^*(M)$. One can define a higher index, denoted by H -index(D), as an element of the K -group $K_0(C^*(M))$ [67]. Similarly when D is a self-adjoint geometric elliptic operator on M , one can define a higher index, denoted by H -index(D), as an element of the K -group $K_1(C^*(M))$. This higher index is an obstruction to invertibility and is invariant under homotopy equivalence.

Definition 2.5. Let Γ be a locally finite metric space. For each $d \geq 0$, the *Rips complex* $P_d(\Gamma)$ is the simplicial polyhedron where the set of all vertices is Γ , and a finite subset $\{\gamma_0, \dots, \gamma_n\} \subseteq \Gamma$ spans a simplex iff $d(\gamma_i, \gamma_j) \leq d$ for all $0 \leq i, j \leq n$.

Let Z be a locally compact space. Recall that the K -homology group $K_0(Z)$ is generated by all “abstract elliptic operators” on Z [51], [52]. Similarly, the K -homology group $K_1(Z)$ is generated by all self-adjoint “abstract elliptic operators” on Z .

Let Γ be a locally finite discrete metric space. The coarse Baum–Connes map μ associates every “abstract elliptic operator” to its higher index:

$$\mu: \lim_{d \rightarrow \infty} K_*(P_d(\Gamma)) \rightarrow K_*(C^*(\Gamma)).$$

Conjecture 2.6 (*The Coarse Baum–Connes Conjecture* [45], [78]). If Γ is a locally finite discrete metric space, then the coarse Baum–Connes map μ is an isomorphism.

If Γ is a countable discrete group with a length metric and its classifying space $B\Gamma$ has the homotopy type of a finite CW -complex, then the descent principle says that the Coarse Baum–Connes Conjecture for Γ as a metric space implies the Strong Novikov Conjecture for Γ as a group [68].

Recall that a function $l: \Gamma \rightarrow [0, \infty)$ is said to be a length function if

- (1) $l(g) = 0$ for some $g \in \Gamma$ if and only if g is the identity element;
- (2) $l(g) = l(g^{-1})$ for all $g \in \Gamma$ and $l(g_1 g_2) \leq l(g_1) + l(g_2)$ for all g_1 and g_2 in Γ ;
- (3) l is proper in the sense that $l^{-1}(K)$ is a finite subset of Γ for every compact subset K of $[0, \infty)$.

We remark that such a length function always exists for any countable discrete group. If Γ is a finitely generated group, we can construct a word length for any finite generating set. For any length function l on a countable group Γ , we can associate a length metric d on Γ by: $d(g_1, g_2) = l(g_1^{-1} g_2)$ for all g_1 and g_2 in Γ . We remark that a countable discrete group with a length metric has bounded geometry (recall that a discrete metric space Γ is said to have bounded geometry if for all $r > 0$ there exists $N(r) > 0$ such that the number of elements in every ball with radius r is at most $N(r)$).

A counterexample to the injectivity of the coarse Baum–Connes map is found in [81]. However, this example does not have bounded geometry. A bounded geometry counterexample to surjectivity is found in [44].

The following conjecture is still wide open.

Conjecture 2.7 (*The Coarse Strong Novikov Conjecture*). If Γ is a discrete metric space with bounded geometry, then the coarse Baum–Connes map μ is injective.

If M is a noncompact complete Riemannian manifold with bounded geometry (i.e. M has bounded curvature and positive injectivity radius), then there exists a net Γ in M such that Γ has bounded geometry. Let $\{U_\gamma\}_{\gamma \in \Gamma}$ be an open cover of M such that $\gamma \in U_\gamma$ for each $\gamma \in \Gamma$ and $\{\text{diameter}(U_\gamma)\}_{\gamma \in \Gamma}$ is uniformly bounded. Let $\{\phi_\gamma\}_{\gamma \in \Gamma}$ be a partition of unity subordinate to $\{U_\gamma\}_{\gamma \in \Gamma}$. We define a continuous map $\phi: M \rightarrow P_d(\Gamma)$ for some large d by

$$\phi(x) = \sum_{\gamma \in \Gamma} \phi_\gamma(x) \gamma$$

for all $x \in M$.

Let D be a geometric elliptic operator on M and $[D]$ be its K -homology class in $K_*(M)$.

We have

$$H\text{-index}(D) = \mu(\phi_*[D]).$$

It follows that the Coarse Strong Novikov Conjecture would reduce the non-vanishing problem of $H\text{-index}(D)$ to that of $\phi_*([D])$ in $\lim_{d \rightarrow \infty} K_*(P_d(\Gamma))$, which is in principle decidable.

Recall that a Riemannian manifold M is said to be uniformly contractible if for every $r > 0$, there exists $R \geq r$ such that for each $x \in M$, the ball with radius r and center x can be contracted to a point within the ball with radius R and center x . If M is a uniformly contractible Riemannian manifold with bounded geometry, then the Coarse

Strong Novikov Conjecture implies a conjecture of Gromov which states that M can not have uniform positive scalar curvature [36]. In this case ϕ_* is an isomorphism and therefore the Coarse Strong Novikov Conjecture implies that the higher index of the Dirac operator is non-zero. However, by the Lichnerowicz formula, uniform positive scalar curvature would imply invertibility of the Dirac operator.

3. Geometry of groups and metric spaces

In this section we briefly discuss several useful concepts from metric geometry. These concepts play important roles in higher index theory.

The following concept plays an important role in the study of Novikov type conjectures.

Definition 3.1 (Gromov [35]). Let Γ be a metric space; let X be a Banach space. A map $f: \Gamma \rightarrow X$ is said to be a (coarse) uniform embedding if there exist non-decreasing functions ρ_1 and ρ_2 from $\mathbb{R}_+ = [0, \infty)$ to \mathbb{R} such that

- (1) $\rho_1(d(x, y)) \leq \|f(x) - f(y)\| \leq \rho_2(d(x, y))$ for all $x, y \in \Gamma$;
- (2) $\lim_{r \rightarrow +\infty} \rho_i(r) = +\infty$ for $i = 1, 2$.

Intuitively (coarse) uniform embeddability of a metric space Γ into a Banach space X means that we can draw a “nice” picture of Γ in X which reflects the large scale geometry of Γ . If Γ is a countable discrete group with a length metric, the concept of (coarse) uniform embeddability of Γ into a Banach space X does not depend on the choice of the length metric.

The following construction implies that every discrete metric space admits a (coarse) uniform embedding into some Banach space, and we need to consider reasonably nice Banach spaces to do anything interesting.

Let Γ be a discrete metric space, let $X = l^\infty(\Gamma)$. Fix $x_0 \in \Gamma$. We define an isometric embedding of Γ into X by

$$(f(x))(\gamma) = d(\gamma, x) - d(\gamma, x_0)$$

for every $x, \gamma \in \Gamma$.

The Hilbert space case is particularly important. Examples of groups which admit (coarse) uniform embedding into Hilbert space include

- (1) amenable groups (Bekka–Cherix–Valette [8]);
- (2) groups with finite asymptotic dimension (e.g. a certain mapping class groups [9]), more generally groups with polynomial asymptotic dimension growth [26];
- (3) hyperbolic groups [71], more generally a class of relatively hyperbolic groups [62], [65], [25];
- (4) countable subgroups of any almost connected Lie groups (Guentner–Higson–Weinberger [39]);

- (5) Coxeter groups [29], more generally certain diagram groups (including R. Thompson's group F) [30], [12], [1];
- (6) semi-direct products of groups of the above types.

In [34], Gong–Yu proved that a noncompact Riemannian manifold with subexponential volume growth admits a (coarse) uniform embedding into Hilbert space.

The following concept provides the most effective method to construct (coarse) uniform embedding into Hilbert space.

Definition 3.2 ([81]). A locally finite discrete metric space Γ is said to have *Property A* if for any $r > 0$, $\varepsilon > 0$, there exist a family of finite subsets $\{A_\gamma\}_{\gamma \in \Gamma}$ of $\Gamma \times \mathbb{N}$ (\mathbb{N} is the set of all natural numbers) such that

- (1) $(\gamma, 1) \in A_\gamma$ for all $\gamma \in \Gamma$;
- (2) if γ and γ' are two points in Γ satisfying $d(\gamma, \gamma') \leq r$, then

$$\frac{\#(A_\gamma - A_{\gamma'}) + \#(A_{\gamma'} - A_\gamma)}{\#(A_\gamma \cap A_{\gamma'})} < \varepsilon,$$

where $\#S$ is the number of elements in S for any finite set S ;

- (3) there exists $R > 0$ such that if $(x, m) \in A_\gamma$, $(y, n) \in A_{\gamma'}$ for some $\gamma \in \Gamma$, then $d(x, y) \leq R$.

The set of natural numbers \mathbb{N} in the above definition allows one to count points in Γ with multiplicity. We remark that the concept of Property A is invariant under quasi-isometry (more generally it is invariant under coarse equivalence). If Z is a locally compact metric space, we say that Z has Property A if a net Γ of Z has Property A.

Proposition 3.3 (Yu [81]). *If a locally compact metric space Z has Property A, then Z admits a (coarse) uniform embedding into Hilbert space.*

In the case of a countable discrete group Γ with a length metric, Higson–Roe proved that Γ has Property A if and only if Γ acts amenably on some compact space [46]. Guentner–Kaminker proved that a certain exactness of the reduced group C^* -algebra $C_r^*(\Gamma)$ implies that Γ admits a (coarse) uniform embedding into Hilbert space [40]. Ozawa further proved that exactness of the reduced group C^* -algebra $C_r^*(\Gamma)$ is equivalent to Property A of Γ [63]. In [15], Chen–Wang established a very nice connection between Property A and the ideal structure of the Roe algebra.

Many of the known examples of locally finite discrete metric spaces (coarsely) uniformly embeddable into Hilbert space have Property A. In a recent paper [61], Nowak constructed the first example of a locally finite discrete metric spaces which does not have Property A, but admits a (coarse) uniform embedding into Hilbert space.

Based on ideas of Enflo, Dranishnikov–Gong–Lafforgue–Yu found the first example of a locally finite discrete metric space which does not admit a (coarse) uniform

embedding into Hilbert space [28]. But this example does not have bounded geometry. In [37], Gromov used expanding graphs to construct examples of discrete metric spaces with bounded geometry and also finitely generated groups which do not admit a (coarse) uniform embedding into Hilbert space.

For purpose of the Novikov Conjecture, we need to introduce the following convexity conditions for Banach spaces.

Definition 3.4. Let X be a Banach space.

- (1) X is said to be *strictly convex* if

$$\left\| \frac{x+y}{2} \right\| < 1$$

whenever $x, y \in S(X)$ and $x \neq y$, where $S(X) = \{x \in X, \|x\| = 1\}$;

- (2) X is called *uniformly convex* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in S(X)$ and $\|x - y\| \geq \varepsilon$, then

$$\left\| \frac{x+y}{2} \right\| < 1 - \delta.$$

Examples of uniformly convex Banach spaces include l^p , L^p , and C_p for all $1 < p < \infty$, where C_p is the Banach space of all Schatten- p class operators on a Hilbert space, i.e. $C_p = \{T : H \rightarrow H \mid \text{tr}(T^*T)^{\frac{p}{2}} < \infty\}$ (H is a Hilbert space).

N. Brown and E. Guentner proved that every countable discrete metric space admits a (coarse) uniform embedding into a strictly convex and reflexive Banach space [11]. W. B. Johnson and N. L. Randrianarivony showed that l^p ($p > 2$) does not admit a (coarse) uniform embedding into a Hilbert space [50]. More recently, M. Mendel and A. Naor proved that l^p does not admit a (coarse) uniform embedding into l^q if $p > q \geq 2$ [57].

Problem 3.5. Let $p > q \geq 2$. Construct a discrete metric space with bounded geometry which admits a (coarse) uniform embedding into l^p but not into l^q ;

Problem 3.6. Let $p > q \geq 2$. Construct a countable discrete group which admits a (coarse) uniform embedding into l^p but not into l^q .

Such a group would be fascinating in the world of group theory since it is neither among Gromov's random groups [37] nor among the familiar class of groups constructed algebraically using amenable groups, hyperbolic groups and linear groups.

Question 3.7. Does every discrete metric space with bounded geometry admit (coarse) uniform embedding into a uniformly convex Banach space?

A negative answer to this question might reveal a geometric phenomenon leading to counter-examples of the Novikov Conjecture. Of course, a positive answer to the

question would imply the Novikov Conjecture for every countable discrete group [55]. We should mention that Ozawa proved that if X is a uniformly convex Banach space with an unconditional basis, then a sequence of expanding graphs does not admit a (coarse) uniform embedding into X [64].

The following concept plays a very useful role in my work with Kasparov on the Baum–Connes Conjecture (in preparation).

Definition 3.8. Let X be a Banach space and Γ be a countable discrete group. An affine and isometric action α of Γ on X is said to be *proper* if

$$\lim_{l(g) \rightarrow \infty} \|\alpha(g)\xi\| = \infty$$

for every $\xi \in X$, where l is a length metric on Γ .

Roughly speaking, if a group Γ admits a proper and affine isometric action on X , it means that Γ can be effectively realized as symmetries of the Banach space X . If α is a proper affine isometric action of Γ on a Banach space X , then the map $\gamma \rightarrow \alpha(\gamma)0$ is a (coarse) uniform embedding of Γ into X , where 0 is the zero vector in X .

We remark that the concept of proper affine isometric action does not depend on the choice of the length function. If Γ admits a proper isometric affine action on Hilbert space, then Γ is said to satisfy the Haagerup property [41] or to be a-T-menable [35].

The following example shows that every countable discrete group admits a proper affine isometric action on some Banach space.

Let $X = l^\infty(\Gamma)$. Let π be the regular action of Γ on X defined by

$$((\pi(g))\xi)(\gamma) = \xi(g^{-1}\gamma)$$

for every $\xi \in X$, all g and γ in Γ . We define an affine and isometric action α on X by:

$$\alpha(g)\xi = \pi(g)\xi + \pi(g)l - l$$

for every $\xi \in X$ and $g \in \Gamma$, where l is a length function on Γ . It is not difficult to verify that α is a proper action.

It is well known that an infinite Property T group does not admit a proper affine isometric action on Hilbert space. A theorem of Zuk says that most (infinite) hyperbolic groups have Property T and therefore do not admit a proper affine isometric action on Hilbert space [83]. The following result shows the usefulness of considering more general uniformly convex Banach spaces.

Proposition 3.9 ([82]). *If Γ is a hyperbolic group, then there exists $2 \leq p < \infty$ such that Γ admits a proper affine isometric action on an l^p -space.*

A very recent result of Bader–Furman–Gelder–Monod says that $SL(n, \mathbb{Z})$ does not admit a proper affine isometric action on l^p -spaces if $n \geq 3$ [5]. A natural question is the following.

Question 3.10. Does $SL(n, \mathbb{Z})$ admit a proper affine isometric action on some uniformly convex Banach space for $n \geq 3$?

A positive answer to this question would have interesting applications to the Baum–Connes Conjecture.

4. Main results

In this section, we briefly discuss several recent results in higher index theory and their applications.

Theorem 4.1 ([81]). *Let Γ be a discrete metric space with bounded geometry. If Γ admits a (coarse) uniform embedding into Hilbert space, then the Coarse Baum–Connes Conjecture is true for Γ .*

The special case of finite asymptotic dimension was first proved in [80] by a controlled operator K -theory method.

Corollary 4.2. *Let M be a uniformly contractible Riemannian manifold with bounded geometry. If M admits a (coarse) uniform embedding into Hilbert space, then M can not have uniform positive scalar curvature.*

Theorem 4.3 ([81], [74]). *Let Γ be a countable discrete group. If Γ admits a (coarse) uniform embedding into Hilbert space, then the Strong Novikov Conjecture holds for Γ .*

By the descent principle, Theorem 4.3 follows from Theorem 4.1 under an additional finiteness assumption on the homotopy type of the classifying space $B\Gamma$ [81]. This finiteness assumption was removed in [74] with the help of a technique introduced by Higson [42] and a theorem of Tu [75].

Corollary 4.4. *The Novikov Higher Signature Conjecture is true if the fundamental group admits a (coarse) uniform embedding into Hilbert space.*

By discussions in Section 2, the Novikov Conjecture is essentially a topological “recognition” problem for compact manifolds. Roughly speaking, Corollary 4.4 says if we can draw a “nice” picture of the fundamental group of a compact manifold in Hilbert space, then we can “recognize” the manifold topologically.

The following result is a generalization of the injectivity part of Theorem 4.1.

Theorem 4.5 ([54]). *Let Γ be a discrete metric space with bounded geometry. If Γ admits a (coarse) uniform embedding into a uniformly convex Banach space, then the Coarse Strong Novikov Conjecture is true for Γ .*

Corollary 4.6. *If a uniformly contractible Riemannian manifold M with bounded geometry admits a (coarse) uniform embedding into a uniformly convex Banach space, then M can not have uniform positive scalar curvature.*

The following result is a generalization of Theorem 4.3.

Theorem 4.7 ([55]). *Let Γ be a countable discrete group. If Γ admits a (coarse) uniform embedding into a uniformly convex Banach space, then the Strong Novikov Conjecture is true for Γ .*

A key ingredient in the proofs of Theorems 4.5 and 4.7 is the construction of an infinite dimensional Bott bundle over the Banach space. Uniform convexity is used to show that this Bott bundle is almost flat in a certain Banach sense.

Corollary 4.8. *The Novikov Higher Signature Conjecture is true if the fundamental group admits a (coarse) uniform embedding into a uniformly convex Banach space.*

We end this survey with the following folklore conjecture.

Conjecture 4.9. Let M be a compact manifold and $\text{Diff}(M)$ be the group of all diffeomorphisms of M . If Γ is a countable subgroup of $\text{Diff}(M)$, then Γ admits a uniform embedding into C_p for some $1 < p < \infty$, where p depends on the dimension of M and C_p is the Banach space of all Schatten- p class operators on Hilbert space.

It is an open question whether every countable subgroup of $\text{Diff}(M)$ for a compact manifold M admits a proper affine isometric action on C_p for some $1 < p < \infty$. This question and Conjecture 4.9 are open even for the case of the circle. If Conjecture 4.9 is true, then Theorem 4.7 implies the Strong Novikov Conjecture for every countable subgroup of $\text{Diff}(M)$ for a compact manifold M .

Note added in proof. Y. Kida has proved that a mapping class group admits a (coarse) uniform embedding into Hilbert space in his recent preprint “The mapping class group from the viewpoint of measure equivalence theory”. U. Hamenstädt has independently obtained the same result in her recent preprint “Geometry of the mapping class groups I: Boundary amenability”.

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