

On spectral invariants in modern ergodic theory

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Abstract. This is a short survey of recent developments in one of the oldest areas of ergodic theory, sometimes called the spectral theory of dynamical systems. We mainly discuss the spectral realization problem in the rich class of all invertible measure preserving dynamical systems, a “behavior” of different spectral invariants in natural subclasses of dynamical systems, and a complete solution of Rokhlin’s problem on homogeneous spectrum in ergodic theory.

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1. Introduction

Ergodic theory, also called *measurable dynamics*, studies iterations of a map (a collection of maps forming an image of some (semi) group G with respect to a homomorphism) equipped with at least one (quasi) invariant measure. The maps are called *transformations*, *automorphisms*, or *dynamical systems* (the collection of maps is called a G -action, or a *dynamical system*, where G plays a role of “time”).

One of the main classes of dynamical systems is the group of all invertible measure-preserving maps (automorphisms of a σ -algebra of measurable sets) defined on a non-atomic Lebesgue space (X, μ) , $\mu(X) = 1$, where the Lebesgue space (X, μ) can be viewed as an interval $[0, 1]$ with the Lebesgue measure. This group is a Polish (complete metrizable separable) topological group, denoted by $\text{Aut}(\mu)$, with respect to the weak (coarse) topology defined by

$$T_n \rightarrow T \Leftrightarrow \mu(T_n^{-1} A \Delta T^{-1} A) \rightarrow 0 \text{ for each measurable } A$$

(we identify transformations if they coincide up to a measure zero set). We say that a property holds for a *typical* element from a topological space D if the set of elements with this property contains a dense G_δ -subset of D ; in particular, for Polish spaces D a property holds for a *typical* element if the set of such elements is a comeager set. Iterations of $T \in \text{Aut}(\mu)$ define a \mathbb{Z} -action denoted by the same

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symbol T . More generally, given a discrete countable group G , the set of all G -actions, i.e. homomorphisms $T: G \rightarrow \text{Aut}(\mu)$, equipped with the weak topology defined by

$$T(n) \rightarrow T \Leftrightarrow (\forall g \in G) T_g(n) \rightarrow T_g,$$

forms a Polish space denoted by Ω_G .

We are interested in *metrical* properties (equivalently, *metrical invariants*) of dynamical systems, i.e. in properties which are invariant with respect to measurable isomorphisms. Since every Polish space equipped with a Borel probability measure is measurably isomorphic to a Lebesgue space (X, μ) (not necessarily non-atomic, but usually we have a very easy dynamic on an atomic part), a “large” class of dynamical systems can be viewed as elements of $\text{Aut}(\mu)$ or Ω_G .

If we wish to consider non-discrete groups G (the most famous example here is the case of \mathbb{R} , called a *flow*), then by G -action we mean only continuous homomorphisms (and their images) into $\text{Aut}(\mu)$. As a rule it is no problem to find an invariant measure for a more or less natural map preserving some space. So dynamical systems are sufficiently common in different areas of mathematics. We are not able to discuss every result on this very rich theory. So we restrict ourselves to the main ‘classical’ subclass of one-to-one dynamical systems, i.e. to $\text{Aut}(\mu)$ or Ω_G including, for example, shifts and automorphisms of commutative compact groups, interval exchanges, and billiards.

There exist many natural (or not so natural) ways to associate operators with dynamical systems. We will discuss the historically first classical operator introduced by Koopman [26]. A unitary operator $\hat{T}f(x) = f(Tx)$ induced on $L_2(X, \mu)$ by a transformation $T \in \text{Aut}(\mu)$ is called *Koopman operator*. Unitary isomorphisms of Hilbert spaces define a spectral equivalence relation on the set of Koopman operators and, consequently, on $\text{Aut}(\mu)$ or Ω_G .

Spectral invariants of dynamical systems, i.e. properties depending only on classes of the spectral equivalence relation, have become classical in the theory of dynamical systems after the celebrated work of John von Neumann [29] on classical mechanics.

Let us recall the properties

1. *to be ergodic* $\Leftrightarrow [\forall \text{ measurable } A[(\forall g \in G)[T_g A = A] \Rightarrow \mu(A)\mu(X \setminus A) = 0]]$,
2. *to be weakly mixing* $\Leftrightarrow [\exists g_n \rightarrow \infty[\hat{T}_{g_n} \rightarrow \int]]$,
3. *to be mixing* $\Leftrightarrow [\forall g_n \rightarrow \infty[\hat{T}_{g_n} \rightarrow \int]]$,

where we consider the weak operator topology and \int is an operator of the orthogonal projection onto the space of constants. These are natural examples of (non-complete) spectral invariants, because each of them divides $\text{Aut}(\mu)$ (resp. Ω_G) into only two subsets consisting of full classes of the spectral equivalence relation.

Consider a pair (ν, M) , where ν is a Borel measure on \mathbb{T} and M is a ν measurable function $M: \mathbb{T} \rightarrow \mathbb{N} \cup \{\infty\}$. Let $H^{\nu, M}$ be the subspace of the ν measurable square integrable functions $\varphi: \mathbb{T} \rightarrow l_2$ such that at a point $\lambda \in \mathbb{T}$ all but the first $M(\lambda)$

coordinates of $\varphi(\lambda)$ vanish. The space $H^{\nu, M}$ is a separable Hilbert space with respect to the inner product

$$\langle \varphi, \psi \rangle = \int_{\mathbb{T}} (\varphi(\lambda), \psi(\lambda))_{l_2} d\nu,$$

and the group \mathbb{Z} acts unitarily on $H^{\nu, M}$ by the natural scalar multiplications:

$$U_n^{\nu, M} \varphi(\lambda) = \lambda^n \varphi(\lambda).$$

Due the spectral theorem for any unitary representation of \mathbb{Z} , say U , in a separable Hilbert space, there exists a pair (ν, M) such that $U^{\nu, M}$ is unitarily equivalent to U . This pair (ν, M) is unique in the sense that representations U^{ν_1, M_1} and U^{ν_2, M_2} are unitarily equivalent if and only if the measures ν_1 and ν_2 are equivalent and $M_1 = M_2$ almost everywhere with respect to ν_1 (or ν_2). Therefore a pair $([\nu], M(\hat{T}))$ is an example of a complete spectral invariant, where the *maximal spectral type* $[\nu]$ of \hat{T} (or T) is a class of mutually equivalent Borel measures on \mathbb{T} and $M(\hat{T})$ modulo ν -zero sets is the *spectral multiplicity function*. More concerning definitions of different spectral properties common in dynamics can be found in [13].

The setting of “freeness”, where one says that a G -action is *free* if $\mu\{x \in X : (\exists g \neq 1) T_g x = x\} = 0$, can be viewed as the first trivial example of a non-spectral metrical invariant. However, usually in ergodic theory one considers only ergodic dynamical systems, because the “typical” problem in this area can be naturally reduced to the “ergodic” one. Since every ergodic transformation is obviously free, this setting is trivial in the class of ergodic $T \in \text{Aut}(\mu)$. However there are many theorems going back to Anzai who showed that the spectral invariant is not a complete metrical invariant even in sufficiently “small” subclasses of $\text{Aut}(\mu)$, and there are a few non-spectral metrical invariants. Let us especially remind here that the notion of *entropy* introduced by Kolmogorov [25] is one of those. In spite of these facts, in modern ergodic theory spectral invariants form a very rich class, and their study sometimes has essential influence on different branches of modern mathematics.

2. General classical problems of the spectral ergodic theory

1. *Spectral realization problem*: to determine precise conditions on the spectrum, i.e. spectral invariants, of a unitary operator under which it can be realized by a dynamical system.
2. *Spectral computation problem*: to calculate spectral invariants for every transformation of the Lebesgue space.
3. *Spectral isomorphism problem*: to describe the “gap” between spectral and metrical properties, i.e., to realize what kind of extra information should be added to spectral invariants for the purpose of having a complete metrical invariant.

There are numerous theorems, examples and counterexamples to different conjectures, which provides evidence of the extreme difficulty of handling these problems, because $\text{Aut}(\mu)$ is a very rich class from the spectral point of view. This implies the importance of studying of general spectral problems in natural subclasses of dynamical systems, where such problems have not been solved yet.

3. Weak operator convergence

From time to time in the spectral ergodic theory was a pulsing activity stimulated by coming of new ideas. In the 60s and 70s some very important problems were solved by approximations. We will concentrate our attention on $M(T)$ (i.e. $M(\hat{T})$). All known constructions of transformations T with calculations of nontrivial $M(T)$ are quite complicated. Oseledec [31] constructed a first example of an ergodic transformation T with $2 \leq m(T) < 30$ where

$$m(T) = \sup\{n : n \in M(T)\}.$$

For a given number n , Robinson [33] used approximations for calculations and found an ergodic (even weakly mixing) transformation T with $m(T) = n$. More information about the history of the spectral multiplicity in ergodic theory can be found in [33], [27].

The main problem of the realization of possible values of the spectral multiplicity function was the homogeneous spectrum problem:

For any $n > 1$, does there exist an ergodic transformation T with $M(T) = \{n\}$ in the orthogonal complement of the space of constants?

This is a so-called Rokhlin problem on homogeneous spectrum. There is no written confirmation signed by V. Rokhlin of the fact that he suggested this problem. However in the memory of many (former) Russian mathematicians this long standing problem is associated with V. Rokhlin who posed this question in many discussions. Moreover, he even asked for less, namely

whether an ergodic transformation can have homogeneous spectrum with multiplicity different from 1 or infinity in the orthogonal complement of the space of constants.

The main goal of this section is to advertise a new sufficiently recent approach which uses weak operator convergence. This approach recently allowed the possibility to solve (Ageev, Ryzhikov) Rokhlin's old problem and to answer some other well-known questions, for example, Katok's question.

3.1. The case $1 \in M(T)$

Theorem 3.1. *For any $M \subseteq \mathbb{N} \cup \{\infty\}$ ($1 \in M$) there exists an ergodic automorphism T with $M(T) = M$.*

Theorem 3.1 was proved in [27] for natural factors of transformations constructed in [19]. However the proof of the main result in [19] used, in fact, limit points (with respect to the weak operator convergence) of iterations of \hat{T} in \hat{T} -invariant subspaces of $L_2(X, \mu)$. We remark here that Theorem 3.1 was also proved independently in [3] in the framework of a new direct construction.

Theorem 3.1 can be viewed as the complete solution of the realization problem of $M(T)$ in $L_2(X, \mu)$ for ergodic transformations T , because an easy corollary of the fact that the space of invariant functions is one dimensional is that $1 \in M(T)$.

3.2. Cartesian powers. For a typical T , Katok (see [21] and a renewed version [22]) showed using some approximation methods that

$$M(T \times T) \in \{\{2\}, \{2, 4\}\}$$

and conjectured that

$$M(\underbrace{T \times \dots \times T}_n) = \{n, n(n-1), \dots, n!\}.$$

It was the unique known construction, where a solution of Rokhlin’s problem (only for $n = 2$) was expected.

This conjecture was confirmed completely by Ageev [2] (see also [6]) and, independently, by Ryzhikov [35] for $n = 2$, thereby giving a positive answer to Rokhlin’s problem for $n = 2$. The solution of this version of Rokhlin’s problem was a sufficiently simple application of the very unexpected fact that, for a typical transformation $T \in \text{Aut}(\mu)$, all polynomials $P_n(\hat{T})$ forming a convex combination of some \hat{T}^k ’s are limit points (with respect to the weak operator convergence) of iterations of \hat{T} .

This approach via so-called limit polynomials gave a lot of new additional information about $M(T)$. However, it did not give a possibility to solve the homogeneous spectrum problem for arbitrary n .

4. The homogeneous spectrum problem of arbitrary multiplicity

The homogeneous spectrum problem was solved completely in [7].

Theorem 4.1. *For any n there exists an ergodic transformation with homogeneous spectrum of multiplicity n in the orthogonal complement of the constant functions.*

One of the new guesses to prove this theorem was to apply the inner group symmetry coming from appropriate identities to symmetry in the spectrum of elements of corresponding group actions. Consider the following (quite exceptional for our purposes) group:

$$G_n = \text{gr}\langle t, s; t_i t_j t_i^{-1} t_j^{-1} = 1 = t_0 \dots t_{n-1} \text{ for all } i, j \rangle,$$

where $t_i = s^i t s^{-i}$, $t_0 = t$.

Theorem 4.1 is a natural corollary of the following main theorem:

Theorem 4.2 ([7]). *For a typical G_n -action T , T_{s^n} is a weakly mixing transformation with homogeneous spectrum of multiplicity n in the orthogonal complement of the constant functions.*

4.1. Sketch of Proof. By our choice G_n contains a normal subgroup G of finite index so that G is a free commutative group in generators t_0, \dots, t_{n-2}, s^n . The spectral theorem applied for G -subaction of a G_n -action T implies that the Hilbert space $L_2(X, \mu)$ can be decomposed into the direct sum of mutually orthogonal T_{s^n} -invariant subspaces $H_i, i = 1, \dots, n$, and a remaining part L , so that both restrictions of T_{s^n} to H_i are mutually unitarily equivalent and L is a subspace of $L_2(X, \mu)$ spanned by all eigenfunctions of $T_g, g \in G$. Therefore if L is trivial, i.e. it consists of constant functions, then the values of the spectral multiplicity function of T_{s^n} are multiples of n in the orthogonal complement of the constant functions.

By our choice G_n is sufficiently “close” to the class of commutative groups. This implies the guess that, given $g \in G_n$, the topological status in Ω_{G_n} of different spectral invariants of T_g is “almost” the same as in $\text{Aut}(\mu)$. More precisely, we found counterparts in Ω_{G_n} of well-known classical results that, for a typical T in $\text{Aut}(\mu)$, the transformations T^k ($k \neq 0$) are weakly mixing and have a simple spectrum. Namely, for a typical G_n -action we prove two statements: (I) Transformations T_s have a simple spectrum. (II) T_g ($g \neq 1$)¹ are weakly mixing (or, equivalently, have no eigenfunctions in the orthogonal complement of the constant functions).

To show (II) we follow a classical method to prove that some property A is typical in the space of all actions of some group. Namely, we find a free action having the property A . Rokhlin’s lemma usually implies that the set of conjugations of any fixed free action is dense. Since we only consider properties which are invariant with respect to any metrical isomorphism (in particular, any conjugation), the set of actions having the property A is dense. Finally, we use convenient approximations by elements of this dense set to show that A is valid for a typical action. This method explains the importance to construct examples of dynamical systems having different metrical properties in ergodic theory.

A priori, we have no example of a free G_n -action T ($n \geq 3$) such that T_s has a simple spectrum. Therefore to show (I) we applied another (more delicate) method that used cyclic approximations [24]². Namely, we proved that a typical G_n -action T can be approximated by appropriate finite G_n -actions $T(k)$ so that T_s is approximated sufficiently fast by a cyclic $T_s(k)$.

¹Formally, (II) was proved in [7] only for a certain subset of elements of G . This was sufficient to establish Theorem 4.2. However, following the same method of the proof, we can then deduce almost automatically that (II) holds.

²A very interesting recent preprint of Danilenko contains a few technical improvements. Using our method, he proved Theorem 4.2 for a simplified version G^* of G , and, in particular, he replaced arguments of cyclic approximations by algebraic ones. He also constructed explicit examples of ergodic transformations with homogeneous spectrum of multiplicity n on L_2^0 using (not directly) the well-known fact that transformations admitting a sufficiently fast approximation can be constructed explicitly. Proofs for G^* are simpler, but my choice of G_n in [7] was made due to different reasons, for example, to show that even in the well-studied class of transformations conjugated to their inverses (i.e. $T_t, t \in G_2$) we have a few new interesting results.

Finally, to have a homogeneous spectrum of multiplicity n , we note that L is trivial, because there is no non-constant eigenfunction for any T_g ($g \neq 1$) if the G_n -action T is typical, and the upper bound n of $M(T_{g^n})$ follows from simplicity of the spectrum of T_g .

I would like to stress that an analogy between Ω_{G_n} and $\text{Aut}(\mu)$ is not always certain, because, for example, for a typical G_n -action T , T_s^n does not have a simple spectrum.

4.2. Some spectral properties of typical G_n -actions. Using a technique from [7], we can say somewhat more concerning elements of a typical G_n -action. More precisely, the first simple observation in the proof of the main theorem in [7] is the remark that for a typical T from Ω_{G_n} , T_s (equivalently, T_s^n) is a *rigid* (non-mixing) transformation, i.e. $T_t^{k_i} \rightarrow E$ for some $k_i \rightarrow \infty$ (E the identity map). In particular, this means that a σ_{T_s} (or $\sigma_{T_s^n}$) is singular, where σ_T is the measure of the maximal spectral type of T .

It is easy to check that for the transformations in Theorem 4.2, $\sigma'_{T_s} \sim \varphi \sigma'_{T_s}$, where $\varphi: \mathbb{T} \rightarrow \mathbb{T}$ is a map defined by $\varphi(\lambda) = \exp(2\pi i/n)\lambda$, and σ'_S is the measure of the maximal spectral type of \hat{S} on $\{\text{const}\}^\perp$. Therefore we have the following corollary:

Corollary 4.3 ([7]). *For a typical G_n -action T , if $k|n$ then T_s^k is a weakly mixing transformation with homogeneous spectrum of multiplicity k in the orthogonal complement of the constant functions.*

Analogous results for an element T_t are somewhat surprising.

Theorem 4.4 ([7]). *For a typical T from Ω_{G_n} , the following properties hold:*

1. T_t is weakly mixing.
2. T_t is rigid.
3. If $n = 2$, then T_t has a homogeneous spectrum of multiplicity 2 on $\{\text{const}\}^\perp$.
4. If $n > 2$, then T_t has a simple spectrum.

5. Spectral rigidity of group actions

Our discussion in this section focuses on values of the spectral multiplicity function of elements of typical dynamical systems.

It is easy to see that $M(T_g) = \{\infty\}$ if g has a finite order.

Remark 5.1. Let G be a countable abelian group. Then, for a typical G -action T , $M(T_g) = \{1\}$ if g has infinite order.

The proof is more or less a standard application of approximation arguments.

All discussion above suggests us to conclude that, as a rule, groups fulfill some (new) effect which I call spectral rigidity.

Definition 5.2. Following [7] we say that an element h of a group H has *spectral rigidity* if for a typical H -action T the set of essential values $M(\hat{T}_h)$ of the spectral multiplicity function of T_h is constant. If every element of H has this property we say that the group H has *spectral rigidity*.

It is easy to see that the notion of spectral rigidity can be considered as an invariant with respect to group or metrical isomorphisms.

In fact, the main theorem on the homogeneous spectrum could be proved because certain elements of G_n have spectral rigidity. Let us mention a partial result valid for every countable group.

Proposition 5.3. *Suppose g is an element of a countable group G ; then $m(T_g)$ is independent of our choice of T from Ω_G except for a meager set of actions.*³

This proposition is an easy corollary of Proposition 3 in [7] and the fact that every countable group G has the *weak Rokhlin property*, i.e. $\{T \in \Omega_G : T \cong T'\}$ is dense for some $T' \in \Omega_G$, where \cong is a metrical isomorphism established by Glasner, Thouvenot, and Weiss in [16].

Let us also mention that an easy application of the technique to study spectral rigidity [7] to solvable Baumslag–Solitar groups is the following theorem (cf. [9]).

Theorem 5.4. *For a typical G -action T , T_t is a weakly mixing rank one transformation, where $G = \text{gr}\langle t, s; ts = st^2 \rangle$.*

This theorem gives a positive answer to a well-known question (see, for example, [18]).

6. Spectral invariants in natural subclasses of dynamical systems

6.1. Finite rank case. Let us recall one of the metrical invariants introduced by Ornstein and Chacon and intensively studied over the last three decades.

Definition 6.1. The transformation T has *rank n* if n is the smallest number such that, for any k , there exist towers (columns)

$$A_{k,i}, T A_{k,i}, \dots, T^{h_{k,i}} A_{k,i}, \quad i = 1, \dots, n$$

such that all levels $T^j A_{k,i}$ and the remaining set form a measurable partition ξ_k of X and $\xi_k \rightarrow \varepsilon$, i.e., for any measurable set A there are ξ_k -measurable sets A_k such

³Recently, this proposition was extended in [10] to the case $M(T_g)$, i.e. every countable group has spectral rigidity. However, the calculation of values of $M(T_g)$ for a typical group action T remains an interesting unsolved problem even in the class of countable groups.

that $\mu(A\Delta A_k) \rightarrow 0$ as $k \rightarrow \infty$. A transformation has *infinite* rank if there is no such number. Finally, a transformation has *uniform* rank n if the towers above can be chosen in such a way that $h_{k,i}$ is independent of i .

Let us mention a few results on spectral invariants in the case of finite rank transformations. Chacon [14] proved that this class does not contain transformations with an unbounded spectral multiplicity function since always

$$m(T) \leq \text{rank } T.$$

Ornstein [30] proved that there exist rank one mixing transformations (see also another staircase construction of mixing rank one transformations [1]). Bourgain [12] showed that Ornstein’s transformations have a singular spectrum. In [4] it was proved that there exist finite rank mildly mixing transformations with a Lebesgue component of any even multiplicity in the spectrum (see also [32],[28]). Every finite subset of \mathbb{N} (with 1) can be a spectral multiplicity function of finite rank transformations (see [3]).

Observe that the setting to be of finite rank can be viewed as the easiest geometrical version of an appropriate approximation. Therefore the solution of the homogeneous spectrum problem in the class of finite rank transformations was a natural addition to the main result of [7]. More precisely, we have the following result.

Theorem 6.2 ([7]). *For a typical G_n -action T , T_{S^n} is a weakly mixing transformation with homogeneous spectrum of the multiplicity n in the orthogonal complement of the constant functions. Moreover, T_{S^n} has uniform rank n .*

Remark 6.3. All transformations of the form $T \times T$ have infinite rank (Ryzhikov [36]). Therefore the transformations in Theorem 4.2 (and 4.4) are from a new class with homogeneous spectra.

6.2. Mixing case In spite of the fact that in the set of all ergodic transformations we have almost full information concerning possible values of the spectral multiplicity function, the case of mixing transformations is still weakly studied, because many useful methods working for typical transformations are not applicable to the subset of mixing transformations.

The complete calculation of values of the spectral multiplicity function of natural factors of the Cartesian power of a transformation (under certain conditions on a transformation) was given in [5]. This construction can be considered as a source of mixing transformations having new (highly nonhomogeneous) spectral multiplicity functions, and covers all currently known non-trivial (i.e. $M(\hat{T}) \neq \{1\}, \{\infty\}$) examples of $M(\hat{T})$ for mixing transformations.

Let G be any subgroup of the symmetric group \mathfrak{S}_n acting on $\{1, \dots, n\}$ by permutations. For any $1 \leq k \leq n$ denote by the same symbol G the diagonal action of G on $I_k = \{1, \dots, n\}^k$. Consider a restriction of the G -action to a G -invariant subset

$$I'_k = \{i_k = (i_k(1), \dots, i_k(k)) \in I_k : i_k(l) = i_k(m) \text{ iff } l = m\}.$$

Suppose \sim is an orbital equivalence relation naturally defined by G on I'_k . Define

$$D_k = \#I'_k / \sim \quad (1 \leq k \leq n) \quad \text{and} \quad M_G(n) = \{D_1, \dots, D_n\}.$$

Theorem 6.4 ([5]). *For any subgroup G of \mathfrak{S}_n there exists a mixing (of all orders) transformation T satisfying*

$$M(\hat{T}|_{L_2^0}) = M_G(n),$$

where $L_2^0 = \{f \in L_2 : \int f d\mu = 0\}$.

Corollary 6.5. *For any n there exists a mixing (of all orders) transformation T satisfying*

$$M(\hat{T}) = \{2, 3, \dots, n\} \quad \text{on } L_2^0.$$

This gives

$$M(\hat{T}) = \{1, 2, 3, \dots, n\} \quad \text{on } L_2.$$

Let us remark that for the case $n = 2$ this was proved earlier by Ryzhikov (see [37]), and that Corollary 6.5 also gives an answer to a question of Robinson (see [34] and [17], 5.3) about the set of possible values of $m(T) = \max M(\hat{T})$ for mixing transformations.

6.3. Interval exchanges Let us remind the definition of interval exchange (transformation). Let $n > 1$ and let π be an irreducible permutation of $\{1, \dots, n\}$, where a permutation π is called *irreducible* if $\pi\{1, \dots, m\} \neq \{1, \dots, m\}$, $1 \leq m < n$. Let Δ be the simplex in \mathbb{R}^n ,

$$\lambda = (\lambda_1, \dots, \lambda_n), \quad \lambda_i \geq 0, \quad \sum_i \lambda_i = 1.$$

The unit interval $I = [0, 1)$ is divided into semi-open intervals

$$I_m = \left[\sum_{i < m} \lambda_i, \sum_{i \leq m} \lambda_i \right), \quad 1 \leq m \leq n.$$

Definition 6.6. The *interval exchange transformation* $T_{\pi, \lambda}$ is uniquely defined on every I_m as a *shift*, i.e. a map $T_m: x \rightarrow x + \alpha_m$ such that that the intervals I_m are rearranged according to the permutation π , where α_m is

$$\sum_{i: \pi\{i\} \leq \pi\{m\}} \lambda_i - \sum_{i \leq m} \lambda_i.$$

If π is a *rotation*, i.e. $\pi(i + 1) \equiv \pi(i) + 1 \pmod{n}$, then $T_{\pi, \lambda}$ has a very easy dynamical behavior, because it is a shift on $I \equiv \mathbb{R}/\mathbb{Z}$. Given an irreducible permutation π which is not a rotation, we restrict ourselves to the study of the topological status of subsets of transformations with fixed spectral invariants. Then the class of all interval

exchanges $T_{\pi,\lambda}$, say Ω_π , looks like a small model of $\text{Aut}(\mu)$. However, due a natural parametrization of Ω_π by elements of Δ , it is more natural to consider the setting “to be typical” from the measure theoretical point view, i.e. to say that a property A is “typical” in Ω_π if the set of $\lambda \in \Delta$ such that $T_{\pi,\lambda}$ satisfies A forms a set of full Lebesgue measure. Indeed, analogs of the very well-known results that a typical transformation $\text{Aut}(\mu)$ is not mixing, weakly mixing, and has a simple spectrum are the following. Katok [23] showed that there do not exist mixing interval exchanges. Veech [38] proved that a “typical” interval exchange has a simple spectrum. And only recently Avila and Forni [11] solved the long-standing problem that a “typical” interval exchange is weakly mixing.

As an analog to Theorem 3.1 it was proved in [8] that for any bounded M , $1 \in M \subseteq \mathbb{N}$, there exists an ergodic interval exchange T with $M(T) = M$ in the class of all interval exchanges. Using Oseledec’s [31] result that the upper bound of the spectral multiplicity of an ergodic interval exchange of n intervals is at most n , we see that this is a complete solution of the problem of possible values of the spectral multiplicity function ($1 \in M(T)$) in the subclass of ergodic interval exchanges.

No ergodic interval exchange T is known with $1 \notin M(T)$.

7. Some more problems

Spectral invariants were studied in ergodic theory for more than seven decades and a lot of unsolved problems have been accumulated. Let me formulate some of them.

7.1. (One of the oldest problems in ergodic theory associated with Banach.) Does there exist a (mixing) transformation with simple Lebesgue spectrum in the orthogonal complement of constant functions?

7.2. (A well-known problem.) Is it true that every rank one transformation has a singular spectrum?

7.3. Is the property to be mixing of all orders spectral (i.e. a spectral invariant)?

7.4. (Another well-known problem.) Is the property to have a fixed rank spectral?

7.5. Given $n > 1$, find the set of pairs (g, G) , where G is a countable group and $g \in G$ such that, for a typical G -action T , a transformation T_g has a homogeneous spectrum of multiplicity n in the orthogonal complement of the constant functions.

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