Some recent progress in geometric methods in the instability problem in Hamiltonian mechanics

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Abstract. We discuss some geometric structures that lead to instability in Hamiltonian systems arbitrarily close to integrable.

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1. Introduction

Suppose that we have a system (without friction) and that we perturb it periodically. Will the effect of the forces accumulate or, on the contrary, will the forces average out?

Versions of this question have appeared since ancient times. One of the more ancients versions is the solar system. The Kepler model results from ignoring the interactions between the planets. This model can be solved explicitly and the system persists forever. We can think of the real solar system as a small modification of the Kepler model and wonder if the constant push and pull between the planets will eventually accumulate or whether they would average out and the orbits will remain bounded. See [LP66], [Arn63b] for surveys of the problem at different times. Similar problems appear in many areas of applications. For example, in the study of plasma confinement, or accelerator physics, one can rather easily exhibit confinement for idealized models which ignore mutual interactions, imperfections, etc. One can wonder whether including back the approximations of the model will spoil the confinement.

In other areas of applications, the instabilities are features that have to be sought out and exploited. (Indeed, one of the main goals of human kind has been to get as far as possible with as little effort as possible.) For example, in spacecraft dynamics, there is a great interest in finding ways of moving satellites using the (free!) gravitational forces rather than the (rather expensive!) forces generated by the engines [Bel04]. In theoretical chemistry an important problem is to understand whether the vibrations

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of one part of a molecule will affect other parts or whether small perturbations will lead to dissociation [JVMU96].

Given the importance and difficulty of the problem, it is no surprise that it has received substantial attention from mathematicians and more applied scientists and that it has been studied by a variety of methods, both rigorous and heuristic.

In this lecture we cannot hope to do justice to this extremely rich history or to survey the recent developments in this very active area. Among several others, we mention the papers [Ber04], [BB02], [BBB03], [Bes96], [BCV01], [BT99], [BK04], [CY04b], [CY04a], [CG94], [CG98], [CP02], [CG03], [Cre03], [Dou88], [DLC83], [FM01], [FM03], [Itu96], [Kal03, LM05], [MS02], [MS04], [Moe96], [Moe02], [Tre02], [Tre04] and the announcements [Xia98], [Mat95], [Mat02]. This list is clearly incomplete!

The only goal of this lecture is to present a concrete point of view, namely, to explain the results in the papers [DdlLS00], [DdlLS03a], [DdlLS03b], [dlL02], [DdlLS05], [GdlL06b] as well as some work in progress (as of Dec. 05) based on the same circle of ideas. We cannot attempt here the much needed systematic survey of the area. We will not even attempt to cover the area of geometric methods and will omit topics such as the *separatrix map*, the study of phenomena that happen in adiabatic phenomena at resonances [Nei86], [NSV03], [IdlLNV02] or the generation of unbounded orbits in Newtonian systems taking advantage of the singularities [Ger91], [Xia92].

Our only goal of this lecture is to present the milestones in a particular approach to the problem. The passage from one milestone to the next is accomplished using a toolkit of standard techniques in the geometric theory of dynamical systems (normal hyperbolicity, averaging methods, KAM theory, Melnikov theory, obstruction theory, and correctly aligned windows). A less standard tool is the scattering map for a normally hyperbolic manifold, which we describe in Section 4.1). Many of the steps could be achieved in different ways. We certainly expect that more techniques will be incorporated in the near future to make the proofs sharper or shorter.

Since we are mainly covering material which is already published or available in much more detailed form, we have omitted important details and assumptions, hence we have not stated theorems. For precise statements and detailed proofs we refer to the references quoted.

2. A mathematical formulation of the instability problem

We will consider a mechanical system. That is a Hamiltonian system defined on a exact symplectic manifold. In some of the models we will discuss, it will be in fact, the product of a torus times and an Euclidean space.[1]

[1] If one considers Hamiltonian systems defined in more general symplectic manifolds, the problems of stability may be very different [Her91].

We will be concerned with situations where our system is close to "integrable." That is, our system can be written as

$$H = H_0(I, \varphi) + \varepsilon H_1(I, \varphi, t) \tag{1}$$

where H_0 is supposed to be integrable and H_1 is periodic (or quasi-periodic) in t.

"Integrable" is often an ill-defined term. In this lecture, we will consider two notions of integrable:

$$H_0 = \sum_{i=1}^{d} h_i(I_i), \tag{2}$$

$$H_0 = \sum_{i=1}^{d} \tilde{h}_i(I_i, \varphi_i). \tag{3}$$

Typical examples are $h_i(I_i) = \frac{1}{2} m_i I_i^2$, $\tilde{h}_i(I_i) = \pm\big(\frac{1}{2m_i} I_i^2 + V_i(\varphi_i)\big)$ with V_i functions with non-degenerate critical points (we will assume $V_i'(0) = 0$, $V_i''(0) > 0$). It is common usage to refer to (2) as "a-priori stable" and (3) as "a-priori unstable", at least when there are hyperbolic fixed points in (3). In the classical terminology used in [AKN88], they are called, respectively, "completely integrable" and "integrable with separable variables". Note that for the systems H_0, the quantities h_i, $\tilde{h}_i$ are conserved quantities. Nevertheless, in (3) the quantities $\tilde{h}_i$ have critical points and indeed, it is impossible to transform them into action-angle variables across the separatrices.

The problem we will consider is to give conditions on H_0 and on the ε-jet of H_1 which guarantee that for $0 < \varepsilon \ll 1$, there are orbits for which some of the variables I experience changes of order 1. This is not the only problem formulated in the literature (see e.g. [Arn68], [Arn04]) but it is the one we will considerer here. The author remembers a round table in S'Agaro [Sim99] in which the organizers asked a distinguished and broad panel to make standard a precise definition of Arnol'd instability. The unanimous consensus was that it was better to let each author make a precise definition of the problems considered in the paper. We note that many papers starting with [HM82] give the name Arnol'd diffusion to the existence of intersections between whiskered tori in (3). This is weaker than the problem we consider here since it ignores the *large gap problem*. See Section 5.

We would also like to describe quantitatively and qualitatively the orbits found. In particular, we would like to describe them in geometric terms and provide geometric features such as estimates on the time they take to move, Hausdorff measure of the orbits described and statistical properties.

Remark. We point out that the distinction between a-priori stable and a-priori unstable models makes sense only for models with one parameter. The model (6) is close to a-priori stable systems, but if $\varepsilon \ll \mu$ we can treat it by methods associated with a-priori unstable systems.

Another important set of models that will be described in Section 4 are systems described by the Hamiltonian

$$\begin{aligned} &H : T^*M \times \mathbb{T}^d \to \mathbb{R}, \\ &H = H_0(p,q) + V(q, \omega t) \end{aligned} \tag{4}$$

where (p, q) denotes a point in T^*M and ω is an external perturbation. We will assume that $H_0(\lambda p, q) = \lambda^2 H_0(p, q)$. Note that the potential term is homogeneous of degree zero.

In the models (4), we would like to find conditions on for which H_0 – which is conserved when $\varepsilon = 0$ – changes by amounts of order 1 whenever $\varepsilon > 0$.

The models (4) were introduced in [Mat95] with H_0 a Riemannian metric and $M = \mathbb{T}^2$, the method presented in [Mat95] is variational. The discussion presented here in section 4 will be based on [DdlLS00], [DdlLS05], [dlL02], [GdlL06b]. Related results are in [BT99]. We note that the geometric methods do not require that the metric is positive definite, so that they apply to Lorenz metrics too. A more detailed comparison between the related results is in [DdlLS05].

Closely related to the models (4) are models the form

$$H(p, q, t) = \frac{1}{2} p^2 + V_n(q) + V_m(q, t) \tag{5}$$

where $p, q \in \mathbb{R}^d$, $d \geq 2$, V_n, V_m are homogeneous of degree n, m respectively, $n > m$, $n > 2, V_n > 0$, V_m periodic or quasi-periodic in t. The fact that different terms have different homogeneities makes the geometric analysis similar to that of the models (4). Nevertheless, there are some differences, since the energy surface of (5) has less topology.

The models (5) are extensions to higher dimensions of the models introduced in [Lit66a], [Lit66b] for $d = 1$. A detailed survey of the results on $d = 1$ and important simplifications and corrections to the proofs in the literature is [Lev92] (the original paper [Lit66a], contains a serious error). For $d = 1$, if the terms are sufficiently close to polynomial, [LZ95] show that the orbits remain bounded.

Remark. The main reason to consider periodic or quasi-periodic perturbations is that this is the case that appears in many applications and also as intermediate models after averaging. (Some excellent references for averaging theory are [AKN88], [LM88].)

If one considers more general dependence on time, there is no reason to expect that all orbits average out and indeed, in many cases, it has recently been shown that one should expect instability [Ort04].

2.1. Some early mathematical results on stability. In spite of extremely insightful pioneering works [Poi99], [Lya92] it is not unfair to say that the first definitive and systematic mathematical results to deal with the stability problem arrived in the late 1950s.

The KAM theorem [Kol79], [Arn63a], [Mos66b], [Mos66a] – see also [dlL01] for a modern review – showed that, for a set of initial conditions of large measure, the oscillations of actions remain $O(\varepsilon^{1/2})$ for all time. The applications to celestial mechanics are particularly subtle. (see [Arn63b], [Féj04]) because the system – as many systems of physical interest – fails to satisfy the "generic" assumptions made in many results.

The result [Neh77], [Loc92], [DG96] showed that under "steepness" properties on H_0, one gets stability of $O(\varepsilon^a)$ for very long times $O(\exp(-1/\varepsilon^b))$ for some positive a, b. Hence, in many systems, one can only expect oscillations of the actions of size $\varepsilon^{1/2}$ for all conditions for exponentially long times or for all times in sets of large measure.

It is important to realize that in the a-priori unstable models, the hypotheses of KAM and Nekhoroshev theorem fail in neighborhoods of the separatrices where action-angle coordinates cannot be introduced.

2.2. Some early mathematical results on instability. The first real progress in the mathematical treatment of the problem of instability is the paper [Arn64]. This paper introduces the remarkable example

$$H = \frac{1}{2}p^2 + \frac{1}{2}I^2 + \mu(\cos q - 1) + \varepsilon(\cos q - 1)(\sin\varphi + \cos t), \tag{6}$$

and shows that for $0 < \varepsilon \ll \mu \ll 1$ there are orbits for which the action changes order 1. The mechanism introduced there has served as the basis of much progress. I guess that it will be hard to find a 4 page paper that has generated so much.

The striking example (6) lead to the problems of instability being termed "Arnol'd diffusion". The use of "diffusion" should not imply that it has anything to do with the heat equation.

Some other early results on instability which we will not be able to discuss are [Pus95], [Dou89], [Sit60], [Ale71]. The last two papers consider problems in celestial mechanics and use methods of normally hyperbolic manifolds, so are somewhat related to the methods described here.

2.3. Heuristic studies. Starting in the late 1960s there were extensive numerical studies on the instability problem by many authors. Notably, among many others, we will just mention the surveys [Chi79], [ZZN+89], [TLL80], [Ten82].

Even if these studies were not rigorous, they identified several possible geometric mechanisms for instability and gave empirical mechanisms for their existence. Perhaps the most important fact uncovered by the numerical experiments was that the diffusion is caused by resonances. That is, the trajectories move along the regions where $k \cdot \nabla H_0 = 0$ for $k \in \mathbb{Z}^d$. These curves form the so-called "Arnold web" [Chi79], [ZZN+89], [LR02]. It also became clear that there are different mechanisms at play. A very lucid – albeit non-rigorous – explanation of different mechanisms at play in instability is [Ten82].

3. The example of [Arn64] and the large gap problem

The analysis of [Arn64] is based on a few easy geometric observations, which we will review briefly now in language that we will use later.

We observe that for $0 < \mu$ the manifold $\Lambda = \{A_2 = 0\}$ is invariant. Moreover, Λ is foliated by invariant tori. The stable and unstable manifolds of these tori coincide.

The crucial point of the example is that the ε-dependent term vanishes together with its gradient in Λ. Therefore Λ and the dynamics on it remain unaltered when we make ε positive (but sufficiently small).

Even if the ε term does not have any effect on the manifold Λ it does have an effect on the stable and unstable manifolds. As it turns out, this effect can be computed perturbatively. The idea of the calculation is that the stable and unstable manifolds depend smoothly on parameters. Therefore, the first order term can be computed simply by integrating the variational equations. Some of these calculations in particular cases appear already in [Poi99]. Nowadays these calculations are referred to in the literature as *Melinkov method*.

The last step of the argument of [Arn64] is to show that given any sequence of tori τ_i such that the motion on them is minimal, and such that $W^u_{\tau_i} \pitchfork W^s_{\tau_{i+1}}$ then there is an orbit that follows them.

We note that, because of the exponential convergence, any point which converges to the torus converges to a concrete orbit: $W^{s,u}_{\tau_i} = \bigcup_{x_i \in \tau_i} W^{s,u}_{x_i}$. Therefore, given a sequence of tori, whose manifolds have intersections, there is a sequence of points $x_i \in \tau_i$ such that $W^u_{\tilde{x}_i} \pitchfork W^s_{\tilde{x}_{i+1}}$.

If we start in τ_i, we can wait long enough to move δ-close to $\tilde{x}_i$. Then, we can just move to $W^u_{\tilde{x}_i}$, and then arrive ε-close to τ_{i+1}. Then, a jump allows us to get into τ_{i+1} so that we can get to the next transition point.

If we follow the procedure sketched above, the orbits for each δ are different, so that one needs a separate argument to conclude that there is a true orbit. In the original paper, this is accomplished by a method, which is essentially topological – the obstruction method – but which uses some mild differentiability properties (some version of the λ lemma).

In the words of J. Moser's review in Mathscinet on the 4 page gem is: "The details of the proof must be formidable, although the idea of the proof is clearly outlined". By now, all the details are clearly in print. A modern exposition of the method, including some generalizations is [FM00]. A rather different approach to the analysis of the example in [Arn64] is in [Bes96], which uses variational methods rather than geometric to obtain orbits that shadow the connections between the tori. Unfortunately, we cannot describe the deep and numerous developments generated by [Arn64]. We refer to the comments in [Arn04]. The reader should be warned that many of these papers differ in technical – but crucial! – details. For example, many of the implementations of the obstruction argument on the literature differ on whether or not the method applies to infinitely long orbits. Several important papers assume that one of the terms of the normal form around the torus is zero or that the stable and unstable bundles are trivial. A sharp version of the argument of [AA67] with references to several other variants is in [DdlLS05]. Other arguments will be described in Sections 7, 9.

3.1. The large gap problem. If rather than taking a perturbation which vanishes on Λ, we had taken a generic perturbation, the tori in Λ that have a rational frequency (resonances) would break, [Poi99], [Tre91], [dlLW04]. These destroyed tori will leave a gap of size $\varepsilon^{1/2}a_i + O(\varepsilon)$ where a_i is a coefficient that depends on the resonance. The fact that outside of these gaps one can find whiskered tori is proved in [Val00].

Since the first order calculations can only predict connections $O(\varepsilon)$ – with a coefficient that is exponentially small in μ – we see that the argument in [Arn64] does not conclude that there are orbits that transverse the gaps generated by a resonance. This is what has been called the "large gap problem". A very lucid discussion of this problem can be found in [Moe96].

This is somewhat unfortunate because the heuristic and numerical explorations suggest that diffusion is more intense near resonance.

Remark. The above discussion has been restricted to second order resonances for flows. That is, $\{k \in \mathbb{Z}^d \mid \frac{\partial H_0}{\partial A} \cdot k = 0\}$ is a 2-dimensional module. One can also wonder about what happens near higher order resonances.

For higher order resonances, the gaps between whiskered tori are larger and the normal forms are not "integrable", so that the discussion of Section 5 does not apply.

Nevertheless, we note that these resonances happen in sets of higher codimension so that using the methods of Section 5, it is possible to construct trajectories that detour around them [DdlLS06a]. A heuristic description of the role of higher order resonances can be found in [Chi79].

4. The role of normally hyperbolic manifolds

One of the main observations of [DdlLS00] (which was crucial for [DdlLS05], [DdlLS03a], [DdlLS03b], [GdlL06b], [CY04b], [CY04a], [Kal03], [BK04] is that the main geometric structure organizing the instability is the presence of a normally hyperbolic invariant manifold Λ whose stable and unstable manifolds intersect transversely.

This invariant manifold Λ acts like a distribution center or a hub. Orbits come to it and get rearranged to exit in a different direction. By returning to Λ again and again, the orbits can change the action systematically. As we will see, in some mechanisms, the orbits gain energy while staying close to Λ and in others, the main gain of energy during the homoclinic excursions.

4.1. The scattering map. A particularly useful tool to keep track of homoclinic excursions to a normally hyperbolic manifold is the scattering map introduced in

[DdlLS00]. Essentially the same idea in a slightly more restrictive context appeared in [Gar00].

Given a family of homoclinic excursions, the scattering map – very similar to the scattering matrix in quantum mechanics – gives the behavior in the distant future as a function of the behavior in the distant past.

The main idea of the scattering map is that the theory of normally hyperbolic invariant manifolds provides a very efficient geometric description of the orbits with a certain asymptotic behavior. By reducing the description of the excursions to the theory of normally hyperbolic manifolds, we obtain a theory which is analytically well behaved. Moreover, the scattering map has very nice geometric properties. By taking the geometry in consideration, it is possible to develop very efficient perturbative calculations [DdlLS06b].

4.1.1. The theory of normally hyperbolic invariant manifolds. We recall some of the results of [Fen72], [Fen74], [HPS77], [Pes04] on normally hyperbolic manifolds. Given a normally hyperbolic manifold Λ, the papers above give an efficient description of the orbits that converge (with a good exponential rate) to Λ, either in the future or in the past. These papers show that these orbits lie on manifolds, which we will denote as W^s_Λ, W^u_Λ, W^s_x, W^u_x. Moreover, these manifolds depend smoothly on parameters. Therefore, by expressing as much as possible the excursions in terms of intersections of invariant manifolds, we obtain remarkable geometric properties and smooth dependence on parameters that will be used to obtain rather explicit perturbative calculations.

Denoting by $W^{s,u}_x$ the set of points whose iterates converge exponentially fast – with an appropriate exponential rate – to the iterates of x,

$$W^s_\Lambda = \bigcup_{x\in\Lambda} W^s_x\ , \quad W^u_\Lambda = \bigcup_{x\in\Lambda} W^s_x \tag{7}$$

and that the decompositions in (7) are a foliation. That is, if $W^s_x \cap W^s_{\tilde{x}} \neq \emptyset$ then $x = \tilde{x}$.

The leaves $W^{s,u}_x$ are as smooth as the flow (or map). The map $x \to W^{s,u}_x$ may be significantly less differentiable than the flow (or map) depending on ratios of rates of contraction and expansions. In the applications here, we will have regularities as high as desired since the manifolds have motions close to integrable.

4.1.2. Definition of the scattering map. Let $y \in W^s_\Lambda \cap W^u_\Lambda$ satisfy

$$T_y W^s_\Lambda + T_y W^u_\Lambda = T_y M. \tag{8}$$

By the implicit function theorem, we can find a locally unique manifold $\Gamma \subset W^s_\Lambda \cap W^u_\Lambda$ such that all its points satisfy also (8). Then $T_y(W^s_\Lambda \cap W^u_\Lambda) = T_y\Gamma$.

Given $y \in W^s_\Lambda$, we can associate to it $\Omega_+(y) \in \Lambda$ determined uniquely by $y \in W^s_{\Omega_+(y)}$. Analogously, given $y \in W^u_\Lambda$ we associate to it $\Omega_-(y) \in \Lambda$ determined uniquely by $y \in W^u_{\Omega_-(y)}$.

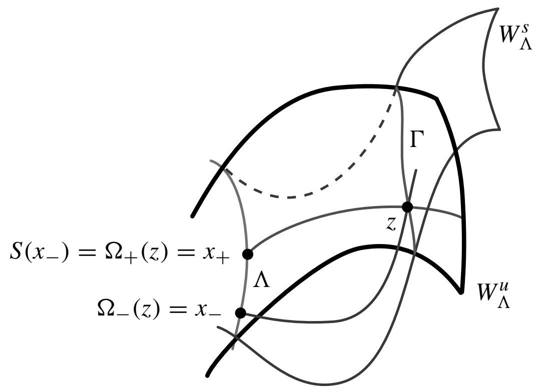

Figure 1. Illustration of the scattering map associated to an intersection Γ.

It is easy to see counting dimensions that, then, the stable and unstable manifolds have a clean intersection along Γ. That is,

$$T_y W^s_{\Omega_+(y)} \oplus T_y\Gamma = T_y W^s_\Lambda; \quad T_y W^u_{\Omega_-(y)} \oplus T_y\Gamma = T_y W^u_\Lambda.$$

Moreover, by taking Γ sufficiently small if necessary, the implicit function theorem ensures that $\Omega_\pm$ are local diffeomorphisms from Γ to their images.

When Ω_- is a local diffeomorphism from Γ to its image we define the mapping S^Γ by

$$S^\Gamma(x_-) = \Omega_+ \circ (\Omega_-)^{-1}. \tag{9}$$

The domain of S^Γ is $H^\Gamma_- \equiv \Omega_-(\Gamma)$. We denote the range of S^Γ as H^Γ_+. We will assume that S^Γ is a diffeomorphism from H^Γ_- to H^Γ_+. By the implicit function theorem, this is true if we have (8) and we take Γ small enough.

Note that, under the assumption that Γ is small enough, that we use to obtain the local uniqueness, it could happen that H^Γ_- is a strict subset of Λ. Nevertheless, for the applications in [DdlLS03b], it is important to observe that the domain of the scattering map can be chosen uniformly for $|\varepsilon|$ small enough.

4.1.3. Geometric properties of the scattering map. Many properties of the scattering map studied in [DdlLS00], [Gar00], [DdlLS03b] are systematized in [DdlLS06b].

Among the properties of the scattering map established in [DdlLS00], [Gar00], [DdlLS03a], [DdlLS03b], [DdlLS05] we mention somewhat informally, ignoring regularity requirements and some subtle points about domains of definition etc. A detailed discussion appears in [DdlLS06b].

- The map S^Γ is (exact) symplectic when f,Γ,Λ are (exact) symplectic diffeomorphisms and manifolds.

- The map S^Γ depends smoothly on parameters when f depends smoothly on parameters.

- There are rather explicit formulas for the derivative of the scattering map with respect to a parameter. These formulas are particularly explicit in the case of symplectic mappings.

- The scattering map associated to an intersection is locally unique. It may happen that when we propagate around a loop, the scattering map has a monodromy.

The scattering map provides a generalization and a more geometric interpretation of the more customary Melinkov theory, which generally assumes that the limiting behavior is of a specific type (e.g., quasi-periodic).

This generality is crucial for the applications in [DdlLS03a], [DdlLS03b] in which transitions to orbits of different topological types are considered. Much more so in [GdlL06a], [GdlL06b] which consider global phenomena.

We also note that the explicit perturbative formulas for the scattering map developed in [DdlLS06b] are always bona fide convergent integrals.

Remark 4.1. The corresponding discussion of the convergence for the Melinkov integrals is rather subtle [Rob96]. Unfortunately, the literature on Melnikov functions is often wrong because it omits a geometric term and the indefinite integrals have a quasi-periodic integrand. The usual explanation that one can take the limit along a subsequence (which one?) is meaningless and, of course, the real reason for this problem is that the argument presented is incorrect. This affects quite a number of papers in the literature. Of course, some of the conclusions may still be true, but they need separate arguments. The reader is alerted to check for this point in the literature.

5. Overcoming the large gap problem by the method of [DdlLS03b]

The work [DdlLS03b] is concerned with one parameter families of the form (2). To simplify the geometric intuition, we will consider f, the time-1 map of the flow.

The first observation is that by appealing to the theory of normally hyperbolic manifolds, in the models (1), there is a normally hyperbolic manifold Λ and that the time-1 map is exact symplectic. This *"inner"* map can be computed to high enough orders in perturbation theory.

A second observation is that, inside Λ, we can perform averaging procedures to high enough order outside of the resonances, so that, outside the resonances, the system can be considered, in an appropriate system of coordinates, as integrable up to order ε^m and m as large as we want. Therefore, outside the resonances, applying the KAM theorem to the averaged system, we can find KAM tori $\varepsilon^{m/2}$ close with m large.

Remark. For the sake of simplicity, the paper [DdlLS03b] includes the hypothesis that the perturbation is a trigonometric polynomial. This allowed to consider uniform several constants appearing in the analysis of the resonances and, therefore, simplified the detailed calculations.

This assumption can be removed by performing a more delicate analysis of the resonances that takes into account that, if the system is differentiable enough, the size of the resonances decreases rapidly and that, for a fixed ε, only a finite number of resonances have width bigger than ε and need to use secondary tori. This is accomplished in [DH06] which contains very detailed analysis of the geometry around resonances. The hypothesis of polynomial perturbations in [DdlLS03b] to conclude topological instability can also be eliminated using topological methods as in [GdlL06a]. See the discussion in Section 7.

As we will see later, the only resonances that play a role in the argument are the resonances that appear during the first averaging procedure and those that appear during the second averaging procedure. These resonances have size $O(\varepsilon^{1/2})$ and $O(\varepsilon)$ respectively.

One key observation for the method of [DdlLS03b] is that, within the resonances, we can find secondary tori – i.e. contractible tori which were not present in the unperturbed system – which dovetail very precisely in the gaps of the foliation of KAM tori. So that, up to precision $O(\varepsilon^{3/2})$ the dynamics on the invariant manifold Λ can be understood as in Figure 2. We emphasize that the invariant sets are, very approximately, the level sets of an averaged energy.

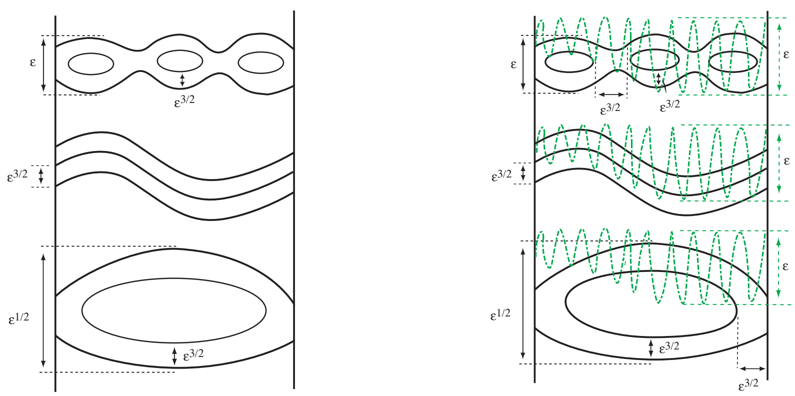

Figure 2. Illustration of the primary and secondary tori in Λ and their images under the scattering map.

If the image under the scattering map of an invariant circle $\tau_1 \subset \Lambda$ crosses transversally another invariant circle $\tau_2 \subset \Lambda$ then $W^u_{\tau_1}$ crosses transversally $W^s_{\tau_2}$.

Lets fix a section in the set of tori (both primary and secondary), say $\theta = \theta_0$ contained in the domain of the scattering map. In [DdlLS03b] there are perturbative calculations of $\frac{d}{d\theta}\bar{H} \circ S|_{\theta=\theta_0}$, where $\bar{H}$ is the averaged energy. If the first order term happens to be different from zero in an open set (notice it can be chosen independent of ε) then, one can show that given τ_1, it intersects transversally all the tori in a neighborhood of size $O(\varepsilon)$. Using that bounded sets are compact and all the quantities involved are continuous, the order of magnitude estimates have coefficients uniform across a set of size independent of ε. In such a way, one obtains increases of the action in sets of order 1. See a pictorial illustration in Figure 2.

The calculation of the expressions of the angles of intersections is different depending on the types of the tori (primary or secondary) that are involved in the intersections. The hypothesis H5 of [DdlLS03b] is precisely imposing that all the leading terms in the expansion of these angles are different from zero. The geometric meaning of the hypothesis is a transversality condition between the scattering map and the inner dynamics. Since both of them are affected in different ways by the perturbations, it is intuitively clear that the hypothesis should hold for many systems. Given a concrete systems, the conditions can be verified by a finite computation.

Remark. Inside the resonances, one can also find (weakly) hyperbolic periodic orbits for the inner map. Their (weak) stable and unstable manifolds can play the same role as the secondary tori in the construction of transition chains. We refer to [DdlLS03b].

6. Perturbations of geodesic flows and of superlinear oscillators

Variations of the method described in Section 5 can be applied to establish the existence of orbits of unbounded energy in (4) and (5).

In the Mather models (4) we will refer to H_0 as the main term and in the Littlewood models (5) we refer to $\frac{1}{2}p^2 + V_n(q)$ as the main term. The other terms are referred to as perturbative terms.

The reason is that, for high energies, if we scale the time, and the coordinates p, q appropriately, we can map one energy surface of the main term into another. When we perform these scalings, the perturbative terms become small and slow. So that the perturbative parameter ε is just a negative power of the energy. Note that the main term is autonomous and, therefore, the energy is conserved.

The main assumptions (we omit several assumptions on regularity etc. but we refer to the detailed papers) are:

H1 Considering the system generated by the main part restricted to its unit energy surface, there exists a hyperbolic periodic orbit. Its stable and unstable manifolds cross transversally in the unit energy surface.

H2 There are some integrals of the perturbation along the unperturbed orbit that do not vanish identically

H3 The frequency of the external perturbation, ω is Diophantine.

Under these assumptions, the papers [DdlLS00], [DdlLS05] establish the existence of orbits of unbounded energy in (4). The adaptation of these methods to (5) is work in progress. As we will see, there are other methods that allow the elimination of H3, at the price of changing slightly H2.

The idea of the proof is very simple. We observe that, given a hyperbolic orbit in the unit energy, by the scaling properties of the geodesic flow, there are corresponding hyperbolic orbits in any energy surface. The union of all these orbits is a normally hyperbolic manifold for the whole system. If we take the product with the manifold of the phases of the external perturbation, we obtain a normally hyperbolic manifold in the extended system.

In the scaled variables, the external perturbation is slow and small. The fact that the perturbation is slow allows us to conclude that the normally hyperbolic invariant manifold persists, and the fact that the perturbation is slow allows us to average an arbitrarily large number of times. Hence, we can conclude that the invariant manifold contains invariant circles very densely spaced, so that these models do not present the large gap problem and we are in a situation very similar to that of [Arn64].

The calculation of the intersections of the tori can be done rather comfortably using the scattering map. The assumption H2 alluded above is just that the scattering map is non-trivial.

It is quite remarkable that the leading term of the scattering map as the energy grows to infinity is the same expression that was found in [Mat95]. The work [Kal03] uses the geometric approach described above but at some stages of the argument uses variational methods.

It is quite clear that once we choose an orbit and a homoclinic intersection in the main term, the set of lower order terms which verify H2 is a submanifold of infinite codimension.

The assumption of existence of periodic orbits with transversal homoclinic intersections has been studied in great detail for Riemannian manifolds. It holds in great generality. For example, it holds for *all* $C^{2+\alpha}$ Riemannian metrics in a surface of genus greater or equal than 2 and it is known to be generic in many other cases. The paper [Lev97] shows that the hypothesis H1 is verified in the classical Hedlund example.

The paper [BT99] uses instead of periodic orbits the existence of whiskered tori with one dimensional stable and unstable manifolds and with homoclinic intersections. In the case of geodesic flows on surfaces, this coincides with periodic orbits, but in more general cases, both hypothesis are very different.

In the Littlewood models, the hypothesis of the method presented here can be verified when $V_n(q)$ is a small perturbation of $|q_1|^4 + |q_2|^4$, a model for which the conclusions are false.

Remark. Another model that can also be treated by similar methods is billiards with periodically moving boundaries. These models were considered in [KMKPdC96]. In these models, the scaling with the energy is rather different from that in the previous cases. The motion of the boundary becomes slow but not small, so that, to apply the methods discussed here, one has to add the smallness of the motion as an extra assumption.

Under the hypothesis that the unperturbed billiard has a homoclinic intersection, that the perturbation is small and satisfies a non-degeneracy assumption, using the methods described here, it is possible to show that there are orbits of unbounded energy. Similar results using variational methods have been announced in [Lev05]. It will be quite interesting to develop detailed comparisons between these methods.

7. The method of correctly aligned windows

The method of correctly aligned windows originated in [Eas75], [Eas78], [EM79] and was extended in [ZG04], [GZ04], [GR03], [GR04]. The main tool of the windowing method is the result that shows that if there is a sequence of *well aligned windows*, there is an orbit that follows all of them. As we will see, the construction of such a sequence of well aligned windows follows from an analysis of the geometric structures discussed before. This leads to alternative methods for several steps of the models described in Section 5 which, eliminate some of the hypothesis and provide explicit estimates on the time. They also allow to analyze some other models.

In some ways, the method of well aligned windows can be considered as a topological version of hyperbolicity since it allows shadowing. Nevertheless, an important difference is that the construction of well aligned windows only uses information for a finite number of iterates. As it is well known, hyperbolicity is very delicate to verify since hyperbolicity built up over arbitrarily long times can be destroyed at longer times. Also, one can use windows that are quite extended in some directions. These two advantages are crucial for the applications to diffusion considered in [GdlL06b], [GdlL06a]. Another important advantage is that the fact that well aligned windows are stable under C^0 perturbations. This allows us to use quite comfortably approximately invariant objects.

The following version of the method is taken from [GdlL06b], which in turn relied on [GR03], [GR04].

Definition 7.1. An (n_1,n_2)-window in an n-dimensional manifold M is a compact subset W of M together with a parametrization given by a homeomorphism $\chi^W : [0,1]^{n_1} \times [0,1]^{n_2} \to W$, where $n_1 + n_2 = n$. The set $W^- = \chi^W\left(\partial[0,1]^n_1 \times [0,1]^{n_2}\right)$ is called the 'exit set' and the set $W^+ = \chi^W\left([0,1]^{n_1} \times \partial[0,1]^{n_2}\right)$ is called the 'entry set' of W. Here ∂ denotes the topological boundary of a set.

Two windows are correctly aligned under some map, provided that the image of the first window under the map can be stretched out and flattened down to a disk crossing the second window all the way through its exist set, so that the induced map on that disk has non-zero degree.

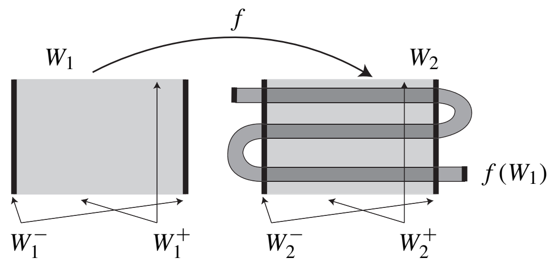

Figure 3. A pair of correctly aligned windows in the plane.

Definition 7.2. Let W_1, W_2 be two (n_1, n_2)-windows in M, and f be a continuous map on M with $f(\mathrm{im}(\chi^{W_1})) \subseteq \mathrm{im}(h^{W_2})$. Denote $f_\chi = (\chi^{W_2})^{-1} \circ f \circ \chi^{W_1}$. We say that W_1 is correctly aligned with W_2 under f provided that the following conditions are satisfied:

(i) $f_\chi((W_1^-)_\chi) \cap (W_2)_\chi = \emptyset, \quad f_\chi((W_1)_\chi) \cap ((W_2)_\chi^+ = \emptyset.$

(ii) there exists a point $y_0 \in [0,1]^{n_2}$ such that

(ii.a) $f_\chi([0,1]^{n_1} \times \{y_0\}) \subseteq \mathrm{int}\,([0,1]^{n_1} \times [0,1]^{n_2} \cup (\mathbb{R}^{n_1} \setminus (0,1)^{n_1}) \times \mathbb{R}^{n_2}),$

(ii.b) If $n_1 = 0$, then $f_\chi((W_1)_\chi) \subseteq \mathrm{int}((W_2)_\chi)$. If $n_1 > 0$, then the map $A_{y_0} : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$ defined by $A_{y_0}(x) = \pi_{n_1}\left(f_\chi(x, y_0)\right)$ satisfies

$$A_{y_0}\left(\partial[0,1]^{n_1}\right) \subseteq \mathbb{R}^{n_1} \setminus [0,1]^{n_1}, \quad \deg(A_{y_0}, 0) \neq 0.$$

Here π_x denotes the projection $(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to x \in \mathbb{R}^{n_1}$.

The main tool of the method is that *"one can see through a sequence of correctly aligned windows"*. Namely, that if we have a sequence of windows $\{W_i\}_{i\in\mathbb{Z}}$, each one of them correctly aligned with the next under the map f, then, there is a point x such that $f^i(x) \in W_i$ for all $i \in \mathbb{Z}$.

In the applications to the models in Sections 5 and 6 considered in [GdlL06b], the windows are constructed as products of intervals in the hyperbolic variables and intervals in the angles and the averaged energy. The windows are laid out around a pseudo-orbit which stays around a constant energy surface but performs jumps at appropriate times.

If we take a window which is an interval slightly offset in the unstable variables and along a homoclinic connection, it will perform a homoclinic excursion and come back as an rectangle very similar to the image under the scattering map of the circle corresponding to a level set of the averaged energy. The transversality between the scattering map and the surfaces of constant averaged energy imply that these windows are well aligned. To construct well aligned windows around the orbits that stay around an invariant circle, the paper [GdlL06b] uses the fact that the inner map has the twist property. We refer to [GdlL06b] for some possible choices of the widths and the choices of exit sets that make the sequence of windows correctly aligned. Compared with the methods in [DdlLS03b], the method of correctly aligned windows uses the same transversality assumptions but does not need to appeal to the KAM theorem – it suffices to use approximately invariant objects – and it also provides explicit estimates of the time it takes the orbits considered to move order 1 in the action variables.

The paper [GdlL06a] provides with a different choice of windows for the large gap problem (and, quite possibly, different orbits) than those in [GdlL06b]. The main idea is to choose windows which are very wide in the action variables. Indeed, they are wider than the resonances in Λ. As a consequence, the resonances in Λ do not cause any problem in the construction and the paper does not need the transversality hypothesis between the scattering map and the constant average energy foliation. It also can eliminate the hypothesis of the perturbation being a trigonometric polynomial and it suffices just that the problem is differentiable enough. As a further advantage, the orbits constructed move rather fast. The amount it takes them to gain $O(1)$ of energy is $\varepsilon|\log\varepsilon|$, which agrees – up to a multiplicative constant – with lower bounds obtained in [BBB03] for certain models (models without large gaps).

One can hope that the method can be developed much more. Indeed, given the robustness of the windows, it is not necessary to appeal to the theory of normally hyperbolic invariant manifolds. It would suffice to have approximately invariant manifolds which may not be hyperbolic in the strict sense but that expand and contract some directions enough for some finite time. This could be useful in applications since numerical calculations (or fits of dynamical systems obtained from measurements of physical systems) can easily give good information on expansiveness for finite time. Nevertheless, assessing true hyperbolicity is only possible under much more restrictive circumstances.

8. The method of normally hyperbolic laminations

This method is developed in [dlL02]. Some antecedents are the *modulational diffusion* of [Chi79], [Ten82]. On the mathematical side, topological versions in dimension two appear in [Moe02], [EMR01]. For simplicity, we will discuss here only the models (4), but as we will see, they apply to all the models discussed in Section 6.

The main assumptions (again ignoring differentiability assumptions, etc.) are:

H1 There exists a transitive hyperbolic set (e.g. a horseshoe) containing $N \geq 2$ hyperbolic periodic orbits γ_i.

H2 For each of the orbits γ_i above, define $G_i(t) = \frac{1}{|\gamma_i|}\int_0^{|\gamma_i|} \partial_2 V(\gamma_i(s), t)\, ds$. Assume that there exist $0 = a_0 < a_1 < \cdots < a_N = 1$ such that

$$\mathcal{A} \equiv \int_{a_0}^{a_1} G_1(t)\, dt + \int_{a_1}^{a_2} G_2(t)\, dt + \cdots + \int_{a_{N-1}}^{a_N} G_N(t)\, dt \neq 0.$$

Using that $\int_0^1 G_i(t)\, dt = 0$, we will assume without loss of generality that $\mathcal{A} > 0$.

Then, we can find a set of orbits with positive Hausdorff dimension such that for an orbit $x(t)$ in this set,

$$H_0(x(t)) \geq \mathcal{A}t - C.$$

We note that the hypotheses are very abundant. For example, H1 is true for *all* $C^{2+\delta}$ Riemannian metrics in a surface of genus bigger than 1. If we consider a metric of negative curvature (with some pinching conditions in dimensions greater than 2), using results of [GK80] we conclude that H2 is verified for all the potentials which are not of the form $V(q,t) = V_1(q) + V_2(t)$. For potentials of the form above, the conservation of energy shows that there are no orbits of unbounded H_0. For negative curvature metrics, these trivial potentials are the only potentials which fail to have a set of positive Hausdorff dimension of orbits whose energy grows linearly.

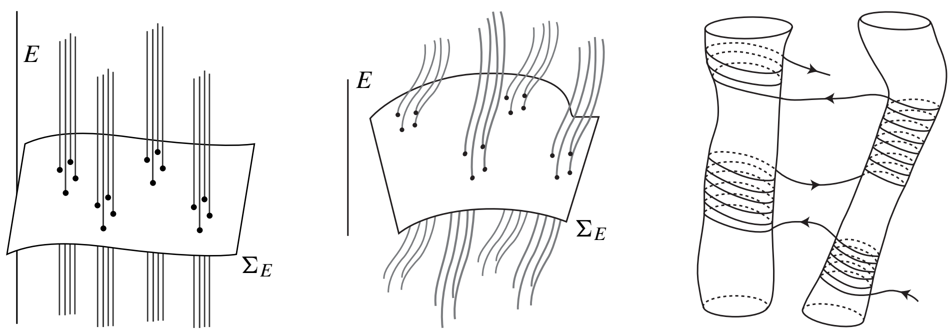

Figure 4. Illustration of the persistence of the invariant laminations and the orbit gaining energy by staying close to periodic orbits.

The idea of the proof is very simple: We observe that, by the scaling properties, for each energy surface there is a transitive hyperbolic set of the geodesic flow. This hyperbolic set satisfies specification and there is a symbolic dynamics that allows us to prescribe times spent in a neighborhood of each of the periodic orbits and connect them by jumps which happen in a scaled time which is uniformly bounded.

If we consider the union of all the hyperbolic sets for all sufficiently large energy, we obtain a normally hyperbolic lamination in the sense of [HPS77].

For high enough energy, the normally hyperbolic lamination persists (one needs to take care of some technicalities such as that the leaves of the foliation have boundary). Indeed, it is possible to find a Hölder map between the new invariant manifold and the old one.

Using the symbolic dynamics given by the normally hyperbolic manifold, we conclude that there are transitions between the periodic orbits happening at times very similar to the a_i in the assumption 2. As can be seen using averaging theory, the meaning of G_i is approximately the gain of energy of orbits that remain close to γ_i but move at high energy. Hence, we can construct orbits which sail along the periodic orbits so that the gains are, on the average $\mathcal{A}$.

Even if the assumptions in Section 6 imply the assumptions in this method, the orbits are very different. Note that the orbits constructed here gain energy near the periodic orbits while the orbits in Section 6 gain energy in the homoclinic excursions.

The method can be adapted to the models (5) and to the billiards with moving boundary. The only differences are that the rate of growths of energy are polynomial and exponential respectively. The adaptation to the time-dependent billiards models requires also the assumption that the motion of the boundary is small. (For these models related results are obtained by variational methods [Lev05].)

9. The scattering map and the obstruction mechanism

By using at the same time arguments similar to the obstruction mechanism of [Arn64], [AA67] it is possible to obtain a mechanism that includes only assumptions on scattering maps. This is work in progress based on preliminary discussions with A. Delshams, M. Gidea and V. Kaloshin. We will only report the simplest case.

We will assume that the manifold Λ is invariant and that it has some homoclinic connections. We will assume without loss of generality that Poincaré's recurrence theorem applies.

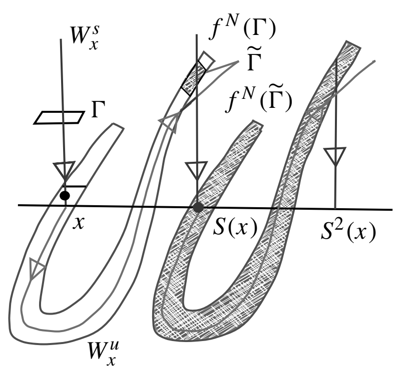

Figure 5. Illustration of the argument in Section 9.

The main inductive lemma starts by considering that Γ is an invariant manifold transverse to W^s_x. It is clear that for all N, we have that $f^N(\Gamma) = \widetilde{\Gamma}$ is transversal to $W^s_{f^N(x)}$. Using the Poincaré recurrence we can assume without loss of generality that $f^N(x)$ comes close to x. On the other hand, by the λ-lemma, the iterates of Γ will be getting aligned with W^u_x. Hence, we can conclude that Γ will intersect transversally the stable manifold of $S(x)$ where S is any of the scattering maps that can be obtained.

In particular, we obtain that if there is a scattering map such that its iterates move order 1, then there is instability in our sense. In the models of the form (1), (3), it suffices that some of the Melnikov integrals do not vanish. Similar results can be obtained for models of the form (4).

10. Conclusions

We have described several different geometric structures that lead to instability in near integrable dynamical systems. These methods rely on normally hyperbolic manifolds acting as a hub for homoclinic excursions. There are other methods, geometric or variational, which we have not covered.

The orbits generated by these different methods have different quantitative properties. It therefore seems that Arnol'd diffusion is not just one phenomenon, but rather a variety of phenomena. It seems that several of the mechanisms are intermingled. When one can find one, one can also find several others.

It seems that the broad array of methods currently under development by a large group of people can lead to answers in different areas. One can hope that even more methods will be brought to bear in a near future.

One can hope that some parts of the very rich heuristic studies – often without a clear statement of conditions of validity – can be clarified by theorems with precise conditions. In particular, it would be quite interesting to develop a rigorous statistical theory of instability.

Relatedly, one can hope that some of the mechanisms can be verified in concrete systems of practical interest or used to design systems with useful properties. From the mathematical point of view, it would also interesting to discuss properties of generic systems.

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