Braids and differential equations

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Abstract. Forcing theorems based on topological features of invariant sets have played a fundamental role in dynamics and differential equations. This talk focuses on the recent work of Vandervorst, Van den Berg, and the author using braids to construct a forcing theory for scalar parabolic PDEs, second-order Lagrangian ODEs, and one-dimensional lattice dynamics.

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This talk covers a particular type of forcing theory for parabolic dynamics which uses the topology of braids in an index theory.

1. Topological forcing theorems

Throughout the last century of work in dynamical systems, forcing theorems have played a substantial role in establishing coarse minimal conditions for complicated dynamics. Forcing theorems in dynamics tend to take the following form: given a dynamical system of a specified class, the existence of some invariant set of one topological type implies the existence of invariant sets of other topological types. This forcing is often encoded by some sort of ordering on topological types of invariant sets.

1.1. Examples. Three canonical examples of forcing theorems frame our work.

Example 1 (*Morse Theory* [43]). The class of systems is that of nondegenerate gradient flows on an oriented manifold M. The invariant sets of interest are the fixed points, and the topological type associated to a fixed point is its *Morse index* – the dimension of its unstable manifold. A suitable chain complex generated by fixed points and graded by the Morse index yields a homology which is isomorphic to that

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of M, allowing one to deduce the existence and indices of additional critical points based on partial knowledge of the invariant sets and the homology of M.

Morse theory has blossomed into a powerful array of topological and dynamical theories. One significant extension is the theory of Conley [14] which associates to an 'isolated' invariant set of a finite dimensional dynamical system an index – the *Conley index* – which, like the Morse index, can be used to force the existence of certain invariant sets. Instead of being a number (the dimension of the unstable manifold), the Conley index is a homotopy class of spaces (roughly speaking, the homotopy type of the unstable set). See [44] and the references therein for a sampling of applications to differential equations.

Following on the heels of Conley's index theory is the extension of Floer to infinite-dimensional gradient-like dynamics. This, in turn, has led to an explosion of results in topology and geometry. The recent flurry of activity in contact homology and symplectic field theory [18] is a descendent of these foundational ideas.

Example 2 (*The Poincaré–Birkhoff Theorem* [5]). This theorem applies to orientation and area preserving homeomorphisms of the annulus whose boundaries are twisted in opposite directions. As with Morse theory, the forcing is in terms of a lower bound (two) on the number of fixed points. The Poincaré–Birkhoff Theorem is the first of many dynamical theorems to exploit the particular features of symplectic manifolds and maps which preserve this symplectic structure. The marriage of this type of theorem with the Morse-type forcing results is the *Arnold Conjecture*, for which Floer theory was first and most strikingly used.

There is a very powerful extension of the Poincaré–Birkhoff Theorem due to Franks [25] (Gambaudo and LeCalvez [39, App.] proved a slightly different version independently at about the same time). Franks' theorem states that if an area and orientation preserving diffeomorphism of the annulus has at least one periodic point, then it has infinitely many periodic orbits. See [26] for this and related results. Franks' Theorem is an excellent example of how a forcing theorem in dynamics often provides a sharp threshold for complicated dynamics: one simple invariant set implies the existence of infinitely many others. This principle finds its clearest exponent in the theorem of Sharkovsky.

Example 3 (*Sharkovsky's Theorem* [48]). For continuous maps of the compact interval to the reals, this theorem gives a total ordering $\lhd$ on the periods of periodic orbits. The theorem states that if a map has an orbit of minimal period P then it has periodic orbits of minimal period Q for all $P \lhd Q$. That the minimal element of $\lhd$ is three has led to the popular coinage *"period three implies chaos."*

The Sharkovsky theorem is remarkable in that there are no assumptions on the systems beyond dimension and continuity. Yet, the topological datum assigned to a periodic orbit is merely the period and nothing more sophisticated. In general, the resolution with which a forcing theorem can act depends on two factors: (1) how narrowly one constrains the class of dynamical systems; and (2) what type of topological data one assigns to the invariant sets.

1.2. Overview. This paper motivates and describes a forcing theory developed by R. Vandervorst in collaboration with J.-B. Van den Berg and the author. In this context, the class of dynamics is, roughly speaking, scalar parabolic lattice dynamics. The topological data which drives the forcing theory is a relative Conley index for invariant sets based on the theory of *braids*.

The resulting forcing theory shares features with all three of the above examples. The index we construct – the *homotopy braid index* – is a Conley–Morse index and leads to Morse-type inequalities. The discrete version of the forcing theory is similar in spirit to LeCalvez' work on twist maps for annuli [38], [39], which itself is an elegant descendent of the Poincaré–Birkhoff Theorem. As with the Sharkovsky Theorem, we obtain a (partial) order on invariant sets. This leads to very simple conditions on invariant sets which force an infinite collection of additional invariant sets.

1.3. Braids and braid types. The use of braids in forcing theorems is not without precedent. There are various types of topological forcing in dimensions two and three related to braids. In the two-dimensional discrete case, one considers the isotopy class of a map relative to some periodic orbit(s): these are related to braids.

One definition of a *topological braid* on n strands is a loop with basepoint in the configuration space of n distinct unlabeled points in the disc D^2. One usually visualizes a braid as an embedding of n intervals $\boldsymbol{u} = \{u^\alpha(t)\}_1^n$ into $D^2 \times [0, 1]$ such that each slice $D^2 \times \{t\}$ is a set of n points and the initial and final configurations the same: $\boldsymbol{u}(0) = \boldsymbol{u}(1)$. See Figure 1 [left]. Given a braid $\boldsymbol{u}$, its *braid class* $\{\boldsymbol{u}\}$ is the equivalence class of braids *isotopic* to $\boldsymbol{u}$, that is, homotopic to $\boldsymbol{u}$ through braids, fixing the endpoints.

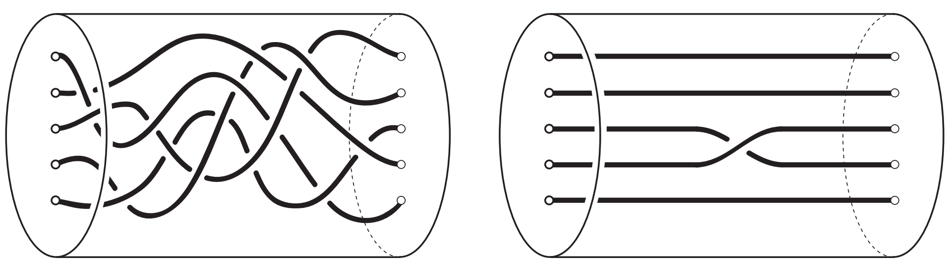

Figure 1. A braid on five strands, illustrated as a collection of embedded arcs in $D^2 \times [0, 1]$ [left]. A typical generator of the braid group has all strands 'straight' with a pair of incident strands crossing [right].

There is an obvious algebraic structure on n-strand braid classes by passing to the fundamental group B_n of the configuration space, the group operation being concatenation of the braids in $D^2 \times [0, 1]$. The standard presentation for B_n has $n - 1$ generators, where the i^{th} generator consists of n parallel strands (the identity braid) except that the i^{th} strand crosses over the $(i+1)^{\text{st}}$ strand as in Figure 3 [right]. See [6] for more details on the topology and algebra of braids.

There is a wonderful analogue of the Sharkovsky Theorem for forcing periodic orbits in surface homeomorphisms. In this setting, the period is not a sufficiently fine datum – one must use what Boyland [7] coined the *braid type* of a periodic orbit. Consider, for the sake of simplicity, an orientation preserving homeomorphism $f : D^2 \to D^2$ of the closed disc with a periodic orbit P of period n. The braid type $\text{BT}(P)$ is, roughly speaking, the isotopy class of f relative to P. Using the relationship between braid groups and mapping class groups [6], it is possible to formally identify $\text{BT}(P)$ with a conjugacy class in the braid group B_n modulo its center. This is best seen by suspending the disc map to a flow on $D^2 \times S^1$. When embedded in $\mathbb{R}^3$, the periodic orbit is a braid. The choice of how many meridional twists to employ in the embedding is the genesis of modding out by the center of B_n.

Boyland defined the following *forcing order* on braid types: one says that $\gamma \leq \beta$ if and only if for any homeomorphism $f : D^2 \to D^2$ with γ a braid type for some periodic orbit of f, then β must also be a braid type for some (perhaps distinct) periodic orbit of f as well. Boyland showed that this is a partial order on braid types [8], which, though weaker than the total order of the Sharkovsky theory, is nevertheless efficacious in forcing complicated dynamics.

Boyland's theory, when generalized to surfaces, entwines with the Nielsen–Thurston theory for surface homeomorphisms. This combination of braid types together with Nielsen–Thurston theory has matured to yield numerous strong results, not only in the dynamics of horseshoe and Hénon maps [11], [13], but also in problems of fluid mixing [9], [33].

1.4. Knots and links. In the case of flows in dimension three, embedding and isotopy data is likewise crucial. Since each periodic orbit is an embedded loop, it is a *knot*, and the set of periodic orbits forms a (perhaps infinite) *link*. The relationship between the link of periodic orbits and the dynamics of the flow is very subtle.

A forcing theory for flows in not straightforward. Roughly speaking, the counterexamples to the Seifert Conjecture constructed by K. Kuperberg [37] imply that there cannot be a forcing theorem for general smooth nonsingular 3-d flows – one can always insert a Kuperberg plug and destroy any isolated periodic orbit. At one extreme, Kuperberg's work implies that there exist smooth nonsingular flows on S^3 without any periodic orbits whatsoever. At the other extreme, it is possible to have a smooth, nonsingular, structurally stable flow on S^3 which displays all possible knots and links as periodic orbits [29]. These phenomena do not bode well for a forcing theory based on knots and links.

However, upon restriction to the correct subclass of flows, it is often possible to retain some vestige of forcing based on knot and link types. One principle which persists is that simple dynamics implicate simple types of knots. For example, in the class of nonsingular Morse–Smale flows on S^3, only certain knot types and link types can appear, a complete classification being given by Wada [54]. This result has a nearly dual counterpart in the class of integrable Hamiltonian dynamics on an invariant 3-sphere, as shown by Fomenko and Nguyen [24] and explained best by Casasayas et al. [12]. Other instantiations of this principle appear in smooth, integrable fluid flows on Riemannian 3-spheres [20] and in gradient fields on S^3 kinematically constrained by a plane field distribution [19].

A complementary principle also holds, that complex dynamics implicate complex knot types in a flow on a 3-sphere. The best example of this type of result is the theorem of Franks and Williams [27], which states that any C^2 flow with positive topological entropy has a link of periodic orbits which has an infinite number of distinct knot types represented. Other results on knotting and linking for suspensions of Smale horseshoes have been proved by Holmes and Williams [35] and used to force bifurcations in Hénon maps. These results all employ the relationship between knots, links, and *closed braids* – conjugacy classes of braids in the braid group which are visualized by identifying the left and right endpoints of a braid.

1.5. Toward higher dimensions. Forcing theorems based on knots, links, or braids in higher dimensional dynamics seem hopeless at first: these objects fall apart in dimension higher than three. One possibility is to try to work with embedding data associated to higher-dimensional invariant sets, say spheres or tori, which can be knotted and linked in the appropriate codimension. At present, there is some initial work on braiding of 2-d invariant tori in 4-d flows [50] which may lead to a forcing theory. There is a great deal now known about the peculiar constraints of embedding spheres and tori in symplectic manifolds, but as yet without much in the way of dynamical implications.

We now turn to a braid-theoretic forcing theory for certain types of PDEs, where the stationary equation allows us to import three-dimensional embedding constraints into an infinite-dimensional dynamical system.

2. Braids for parabolic dynamics

Our motivation for using braids to force dynamics comes from a very simple observation about parabolic partial differential equations.

2.1. Motivation: parabolic PDEs. Consider the scalar parabolic PDE

$$u_t = u_{xx} + f(x, u, u_x), \tag{1}$$

where f satisfies one's favorite technical assumptions to guarantee no finite-time blowups of solutions. For simplicity, we assume periodic boundary conditions ($x \in [0, 1]/0 \sim 1$). We view Equation (1) as an evolution equation on the curve $u(\cdot, t)$. As t increases, the graph of u evolves in the (x, u) plane. Thus, the PDE induces a flow on a certain infinite-dimensional space of curves. It is a result of Fiedler and Mallet-Paret [21] that a type of Poincaré–Bendixson Theorem holds for these types of equations: the only bounded invariant sets are stationary solutions, time-periodic solutions, and connecting orbits.

We augment the types of solutions under consideration as follows. First, we allow multiple graphs to evolve by the product flow. That is, if $u^1 = u^1(t)\colon [0,1] \to \mathbb{R}$ and $u^2 = u^2(t)\colon [0,1] \to \mathbb{R}$ are solutions to Equation (1), then we consider the union $\boldsymbol{u} = (u^1, u^2)$ as a solution to the product flow. These two strands evolve together, independently, as a pair of graphs on the (x, u) plane. In general, we can consider an n-tuple $\boldsymbol{u} = (u^k)_1^n$ of *strands* which evolve under the dynamics.

Second, we allow for strands of multiple spatial period. That is, we allow for a collection $\boldsymbol{u} = (u^k)_1^n$ of strands of the form $u^k\colon [0,1] \to \mathbb{R}$ with the endpoints equivalent as sets: $\{u^k(0)\}_1^n = \{u_k(1)\}$. Even though the endpoints do not match strandwise, the union of the endpoints of the strands do match, and thus the entire collection evolves under the PDE so as to respect the spatial periodicity. One can think of such a collection of strands as a single-strand curve on the n-fold cover $[0, n]/0 \sim n$ of the spatial variable x.

It is a well-known fact (going back to Sturm, but revived and extended considerably by Matano [41], Brunovsky and Fiedler [10], Angenent [1], and others) that there is a *comparison principle* for Equation (1). Specifically, let $u^1(t)$ and $u^2(t)$ be solutions to Equation (1). Then the number of intersections of the graphs of $u^1(t)$ and $u^2(t)$ is a weak Lyapunov function for the dynamics: it is non-increasing in t. Furthermore, at those particular times t for which the graphs of $u^1(t)$ and $u^2(t)$ are tangent, the number of intersections decreases strictly in t, even in the case where the tangencies are of arbitrarily high order [1]. These facts are all at heart an application of classical maximum principle arguments which have a topological interpretation: *parabolic dynamics separates tangencies monotonically.*

This monotonicity is easily seen. Assume that u^1 and u^2 are solutions to Equation (1) which have a simple tangency where $u^1(x,t) = u^2(x,t)$. Then the evolution of the difference between u^1 and u^2 is given by

$$\frac{\partial}{\partial t}\left(u^1(x,t) - u^2(x,t)\right) = \frac{\partial^2}{\partial x^2}\left(u^1(x,t) - u^2(x,t)\right). \tag{2}$$

Since the nonlinear terms cancel, the evolution is governed purely on the basis of the curvature of the graphs.

Using this comparison principle (also known as *lap number* or *zero crossing* techniques), numerous authors have analyzed the dynamics of Equation (1) in varying degrees of generality. We note in particular the paper of Fiedler and Mallet-Paret [21], in which the comparison principle is used to show that the dynamics of Equation (1) is often Morse–Smale, and also the paper of Fiedler and Rocha [22], in which the global attractor for the dynamics is roughly classified.

2.2. Idea: dynamics on spaces of braids. A typical collection of strands is illustrated in Figure 2 [left], in which one notices a resemblance to the planar projection of a braid. By lifting this collection of strands in the (x, u) plane to the 1-jet extension of the strands in (x, u, u_x) space, we obtain a *Legendrian braid* tangent to the contact structure $\{dy - z\,dx = 0\}$. Such a braid is *closed*, due to the periodicity of the strands. Being Legendrian, the braid is *positive* – in the standard generators for the braid group, only positive powers of generators are permitted.

There is a globalization of the comparison principle using braids. For a motivating example, consider again a pair of evolving curves $u^1(t)$ and $u^2(t)$ in the (x, u) plane. If we lift these curves to the three-dimensional (x, u, u_x) space, we no longer have intersecting curves, unless t is such that the planar graphs of u^1 and u^2 intersect tangentially. The graphs of u^1 and u^2 in the (x, u, u_x) space are instead a closed braid on two strands. What was the intersection number of their projections is now the *linking number* of the pair of strands.

We see therefore that the comparison principle takes on a linking number interpretation (a fact utilized in a discrete setting by LeCalvez [38]). After lifting solutions u^1 and u^2 to the (x, u, u_x) space, the comparison principle says that the linking number is a nonincreasing function of time which decreases strictly at those times at which the curves are tangent. This two-strand example is merely motivation for adopting a braid-theoretic perspective on multiple strands, as in Figure 2.

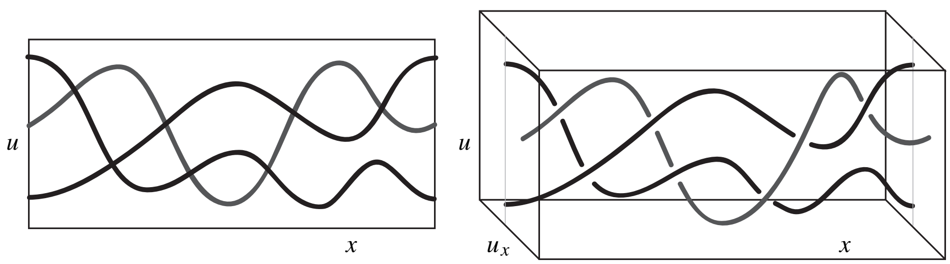

Figure 2. Curves in the (x, u) plane lift to a braid in (x, u, u_x).

The key observation is that the comparison principle passes from a local statement ("linking number decreases at a tangency") to a global statement ("algebraic length in the braid group decreases at a tangency"). A related globalization of the comparison principle for geodesic flows on Riemannian surfaces appears in the recent work of Angenent [2].

2.3. Goal: forcing. Our goal is to produce a forcing theory for the dynamics of Equation (1) and more general parabolic systems. For simplicity, we focus on forcing stationary solutions, though periodic and connecting orbits are likewise accessible. Say that one has found a *skeleton* of stationary strands $\{v^1, v^2, \ldots, v^m\}$ for a particular representative of Equation (1). How many and which types of other stationary curves are forced to be present? Since the skeleton of known fixed curves $\boldsymbol{v} = \{v^1\}_{i=1}^m$ lifts to a braid, the problem is naturally couched in braid-theoretic terms: given a braid $\boldsymbol{v}$ fixed by a particular uniform parabolic PDE, which other classes of braids $\boldsymbol{u}$ are forced to exist as stationary curves?

The spirit of our forcing theory is as follows:

1. Given a braid of stationary solutions $\boldsymbol{v}$, construct the configuration space of all n-strand braids $\boldsymbol{u}$ which have $\boldsymbol{v}$ as a sub-braid.

2. Use the braid-theoretic comparison principle to give a Morse-type decompose of this configuration space into dynamically isolated braid classes.

3. Define the *homotopy braid index* – a Conley index for relative braid classes which depends only on the topology of the braids, and not on the analytical details of the dynamics.

4. Prove Morse-type inequalities for forcing stationary and/or time-periodic solutions.

To execute this requires a significant generalization to spatially discretized systems, which in turn generalizes the results far beyond parabolic PDEs.

3. Spaces of braids for parabolic dynamics

3.1. Braids, topological and discrete. The motivation of §2.1 leads one to consider spaces of braids obtained from curves in the (x, u) plane. Consider the space of all such n-strand braids $\boldsymbol{u}$ which are both closed and positive. For the sake of intuition, one should think of these topological braids as smooth braids lifted from the 1-jet extension of graphs in the plane. In reality, one completes this space to include non-smooth braids as well. These configuration spaces of braids are infinite dimensional. By projecting to finite dimensional approximations, we avoid a great deal of analytic and topological difficulties. We briefly outline the "finite dimensional" braid theory needed.

The class of *discretized braids* are best visualized as piecewise-linear braid diagrams, as in Figure 3 [left]. A discretized braid, $\boldsymbol{u}$, on n strands of period p, is determined by *np anchor points*: $\boldsymbol{u} = \{u_i^\alpha\}_{i=0,\ldots,p}^{\alpha=1,\ldots,n}$. Superscripts $\alpha = 1, \ldots, n$ refer to strand numbers, and subscripts $i = 0, \ldots, p$ refer to spatial discretizations. One connects the anchor point u_i^α to u_{i-1}^α and u_{i+1}^α via straight lines. Since "height" is determined by slope, all crossings in the braid diagram are of the same sign (as in Figure 3 [left] but not in Figure 1 [left]). Since we employ periodic boundary conditions on the x variable, all of the braids are closed: left and right hand endpoints of strands are abstractly identified and the endpoints are free to move. This necessitates a periodicity convention for the subscript. For a single-strand component u^α, we have that $u_{i+p}^\alpha = u_i^\alpha$ for all i. For multi-strand components, one cycles between the strands according to the permutation of strands. Denote by $\mathcal{D}_p^n$ the space of all n-strand period p discretized braids: $\mathcal{D}_p^n$ is homeomorphic to $\mathbb{R}^{np}$.

For topological braids, a *singular braid* is one for which one or more strands intersect. For braids which are lifts of graphs, the only possible intersection is that which occurs when two strands are tangent in the projection. For a discretized braid $\boldsymbol{u}$,

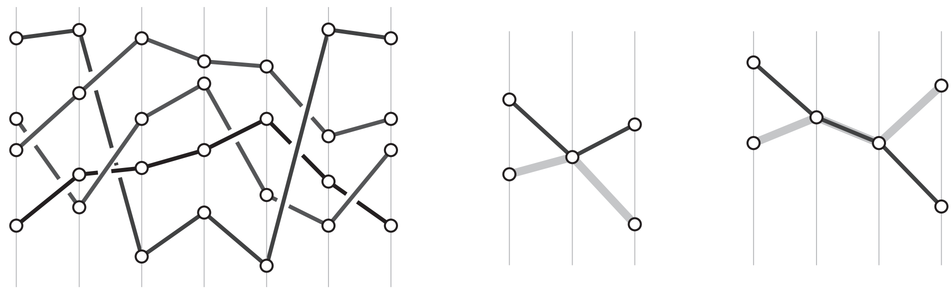

Figure 3. A discretized braid in $\mathcal{D}_6^4$ with three components (note: left and right hand sides are identified) [left]; Two types of singular discretized braids: a simple tangency, and a high-order contact [right].

the singular braids are defined to be those braids at which anchor points on two different strands coincide in a topologically non-transverse fashion with respect to immediate neighbors. Denote by Σ the singular braids,

$$\Sigma = \{\boldsymbol{u} : u_i^\alpha = u_i^\beta \text{ for some } i \text{ and } \alpha \neq \beta, \text{ and } (u_{i-1}^\alpha - u_{i-1}^\beta)(u_{i+1}^\alpha - u_{i+1}^\beta) \geq 0\}. \tag{3}$$

The set Σ is a discriminant that carves $\mathcal{D}_p^n$ into components: these are the *discretized braid classes*, denoted $[\boldsymbol{u}]$. Within Σ, there is a subspace of *collapsed* braids, $\Sigma^- \subset \Sigma$, consisting of those braids for which distinct components of the braid (or a single component with multiple period) collapse to yield a braid on fewer strands. More specifically,

$$\Sigma^- = \{\boldsymbol{u} \in \Sigma \ : \ u_i^\alpha = u_i^\beta \text{ for all } i \in \mathbb{Z} \text{ and some } \alpha \neq \beta\}, \tag{4}$$

under the convention of subscript periodicity mod p as regulated by the braid.

3.2. Parabolic dynamics on braids. A parabolic PDE of the form in Equation (1) gives rise to a flow on the space of topological braids. There is likewise a broad class of flows on spaces of discretized braids which are best described as parabolic. These come from nearest-neighbor lattice dynamics.

Discretizing Equation (1) in the standard way would yield a family of nearest neighbor equations of the form $\frac{d}{dt}u_i = f_i(u_{i-1}, u_i, u_{i+1})$ in which uniform parabolicity would manifest itself in terms of the derivatives of f_i with respect to the first and third variables. Instead of explicitly discretizing the PDE itself, we use the broadest possible category of nearest neighbor equations for which a comparison principle holds: these are related to the *monotone systems* of, e.g., [49], [34], [21] and others.

A *parabolic relation* of period p is a sequence of maps $\mathcal{R} = \{\mathcal{R}_i : \mathbb{R}^3 \to \mathbb{R}\}$, such that $\partial_1 \mathcal{R}_i > 0$ and $\partial_3 \mathcal{R}_i \geq 0$ for every i. These include discretizations of uniform parabolic PDE's, as well as a variety of other discrete systems [40], [42], including monotone twist maps [38]. The small amount of degeneracy permitted ($\partial_3 \mathcal{R}_i = 0$) does not prevent the manifestation of a comparison principle. Given a discretized braid $\boldsymbol{u} = \{u_i^\alpha\}$ and a parabolic relation $\mathcal{R}$, one evolves the braid according to the equation

$$\frac{d}{dt}(u_i^\alpha) = \mathcal{R}_i(u_{i-1}^\alpha, u_i^\alpha, u_{i+1}^\alpha). \tag{5}$$

Any parabolic relation $\mathcal{R}$ therefore induces a flow on $\mathcal{D}_p^n$. Fixed points of this flow correspond to stationary braids $\boldsymbol{u}$ satisfying $\mathcal{R}_i(u_i^\alpha) = 0$ for all i and α. It will be useful at certain points to work with parabolic relations which induce a gradient flow on $\mathcal{D}_p^n$. One calls $\mathcal{R}$ *exact* if there exist generating functions S_i such that

$$\mathcal{R}_i(u_{i-1}, u_i, u_{i+1}) = \partial_2 S_{i-1}(u_{i-1}, u_i) + \partial_1 S_i(u_i, u_{i+1}), \tag{6}$$

for all i. In the exact case, the flow of Equation (5) is given by the gradient of $\sum_i S_i$.

All parabolic relations, exact or non-exact, possess a discrete braid-theoretic comparison principle.

Lemma 4 (Comparison principle for braids [32]). *Let $\mathcal{R}$ be any parabolic relation and $\boldsymbol{u} \in \Sigma - \Sigma^-$ any non-collapsed singular braid. Then the flowline $\boldsymbol{u}(t)$ of $\mathcal{R}$ passing through $\boldsymbol{u} = \boldsymbol{u}(0)$ leaves a neighborhood of Σ in forward and backward time so as to strictly decrease the algebraic length of $\boldsymbol{u}(t)$ in the braid group as t increases through zero.*

Lemma 4 implies that the flow of parabolic dynamics is gradient-like on the (non-collapsed portions of) boundaries of braid classes. This suggests a Morse-theoretic approach. For example, if the flow points in to a given braid class everywhere along the boundary, then the braid class should serve as a 'sink' for the dynamics and thus be assigned a Morse index of zero. At least some invariant set would have to lie within this braid class, even if the dynamics is not gradient everywhere. For more complicated behaviors on the boundary of a braid class, Conley's version of Morse theory is the appropriate tool, with the notion of a Morse index generalizing to the Conley index, a homotopy class of spaces.

4. The homotopy braid index

One significant problem with this idea is the prevalence of collapsed braids, which are invariant under the flow and foil the straightforward application of Morse theory. Clearly, *any* braid class $[\boldsymbol{u}]$ borders the set of collapsed braids Σ^- somewhere. One need simply collapse all the strands together as an extreme degeneracy.

4.1. Relative braids. We are therefore naturally confronted with the need for a forcing theory. Given that a particular parabolic relation possesses a stationary braid $\boldsymbol{v}$, does it force some other braid $\boldsymbol{u}$ to also be stationary with respect to the dynamics? This necessitates understanding how the strands of $\boldsymbol{u}$ braid relative to those of $\boldsymbol{v}$.

Given a discrete braid $\boldsymbol{v} \in \mathcal{D}_p^m$, consider the set of nonsingular braids

$$\left\{ \boldsymbol{u} \in \mathcal{D}_p^n \ : \ \boldsymbol{u} \cup \boldsymbol{v} \in \mathcal{D}_p^{n+m} - \Sigma_p^{n+m} \right\},$$

the path components of which define the *relative braid classes* $[\boldsymbol{u} \text{ REL } \boldsymbol{v}]$. Not only are tangencies between strands of $\boldsymbol{u}$ illegal, so are tangencies with the strands of $\boldsymbol{v}$. In this setting, the braid $\boldsymbol{v}$ is called the *skeleton*. Elements within $[\boldsymbol{u} \text{ REL } \boldsymbol{v}]$ are equivalent as discrete braids fixing all strands of $\boldsymbol{v}$.

In this context, it is possible to define a Conley index for certain discrete relative braid classes. To do so, it must be shown that the braid classes $[\boldsymbol{u} \text{ REL } \boldsymbol{v}]$ are *isolated* in the sense that no flowlines within $[\boldsymbol{u} \text{ REL } \boldsymbol{v}]$ are tangent to the boundary of this set. It follows from Lemma 4 that $[\boldsymbol{u} \text{ REL } \boldsymbol{v}]$ is isolated for the flow of Equation (5) assuming that the braid class avoids the collapsed braids Σ^-. We therefore declare a braid class $[\boldsymbol{u} \text{ REL } \boldsymbol{v}]$ to be *proper* if no free strands of $\boldsymbol{u}$ can "collapse" onto $\boldsymbol{v}$ or onto each other: see Figure 4. Furthermore, to ensure compactness, it is convenient to assume that the braid class $[\boldsymbol{u} \text{ REL } \boldsymbol{v}]$ is *bounded* – free strands cannot wander off to $\pm\infty$.

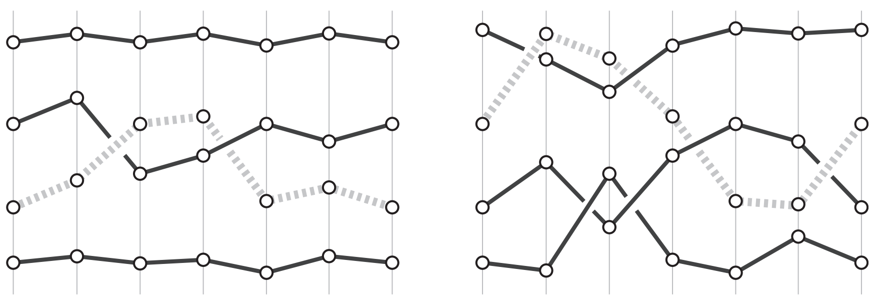

Figure 4. A bounded but improper braid class [left]. A proper, but unbounded braid class. Solid strands form the skeleton; dashed strands are free [right].

4.2. The index: discrete version. The *homotopy braid index* of a proper, bounded, discrete relative braid class $[\boldsymbol{u} \text{ REL } \boldsymbol{v}]$ is defined as follows. Choose any parabolic relation $\mathcal{R}$ which fixes $\boldsymbol{v}$ (such an $\mathcal{R}$ exists). Define $\mathcal{E}$ to be the *exit set*: those braids on the boundary of the braid class $[\boldsymbol{u} \text{ REL } \boldsymbol{v}]$ along which evolution under the flow of $\mathcal{R}$ exits the braid class. The homotopy braid index is defined to be the pointed homotopy class

$$h([\boldsymbol{u} \text{ REL } \boldsymbol{v}]) = \left(\overline{[\boldsymbol{u} \text{ REL } \boldsymbol{v}]}/\mathcal{E}, \{\mathcal{E}\} \right). \tag{7}$$

This is simply the Conley index of the closure of $[\boldsymbol{u} \text{ REL } \boldsymbol{v}]$ in $\mathcal{D}_p^n$ under the flow of $\mathcal{R}$. Lemma 4, combined with the basic stability properties of the Conley index yields the following:

Lemma 5. *The index* $h([\boldsymbol{u} \text{ REL } \boldsymbol{v}])$ *is well-defined and independent of the choice of* $\mathcal{R}$ (*so long as it is parabolic and fixes* $\boldsymbol{v}$) *as well as the choice of* $\boldsymbol{v}$ *within its braid class* $[\boldsymbol{v}]$.

Thanks to the comparison principle for braids, the computation of the index h does not require a choice of $\mathcal{R}$. One can identify the exit set $\mathcal{E}$ purely on the basis of which singular braids will decrease algebraic length under parabolic evolution. This is the basis for an algorithm to compute the homological index $H_*(h[\boldsymbol{u} \text{ REL } \boldsymbol{v}])$ numerically [17].

Example 6. Consider the proper period-2 braid illustrated in Figure 5 [left]. There is exactly one free strand with two anchor points (via periodicity). The anchor point in the middle, u_1, is free to move vertically between the fixed points on the skeleton. At the endpoints, one has a singular braid in Σ which is on the exit set. The end anchor point, u_0 $(= u_2)$ can freely move vertically in between the two fixed points on the skeleton. The singular boundaries are not on the exit set since pushing u_0 across the skeleton increases the number of crossings.

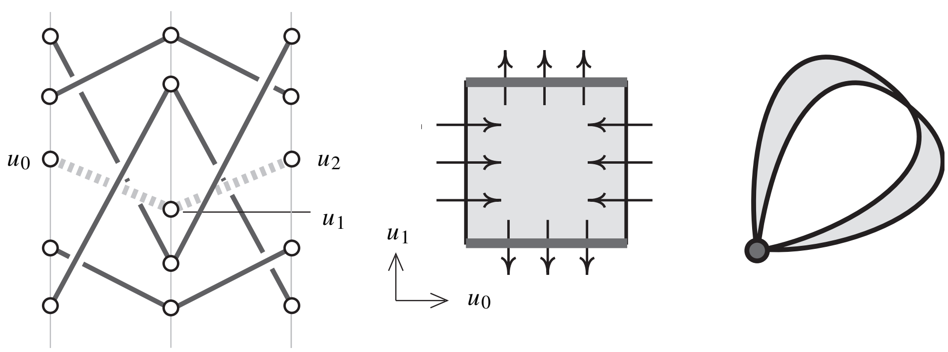

Figure 5. The braid of Example 6 [left] and the associated configuration space with parabolic flow [middle]. Collapsing out the exit set leads to a space [right] which has the homotopy type of a circle.

Since the points u_0 and u_1 can be moved independently, the braid class is the product of two intervals. The exit set consists of those points on the boundary for which u_1 is a boundary point. Thus, the homotopy braid index is S^1, as seen in Figure 5 [right].

Example 7. Consider the proper relative braid presented in Figure 6 [left]. Since there is one free strand of period three, the configuration space is determined by the vector of positions (u_0, u_1, u_2) of the anchor points. This example differs greatly from the previous example. For instance, the point u_0 (as represented in the figure) may pass through the nearest strand of the skeleton above and below without changing the braid class. The points u_1 and u_2 may not pass through any strands of the skeleton without changing the braid class unless u_0 has already passed through. In this case, either u_1 or u_2 (depending on whether the upper or lower strand is crossed) becomes free.

The skeleton induces a cubical partition of $\mathbb{R}^3$ by planes of singular braids. The relative braid class is the collection of cubes in $\mathbb{R}^3$ illustrated in Figure 6 [right]: it is homeomorphic to $D^2 \times S^1$. In this case, the exit set is the entire boundary and the quotient space is homotopic to the wedge-sum $S^2 \vee S^3$, the space defined by abstractly gluing a point of S^2 to a point of S^3.

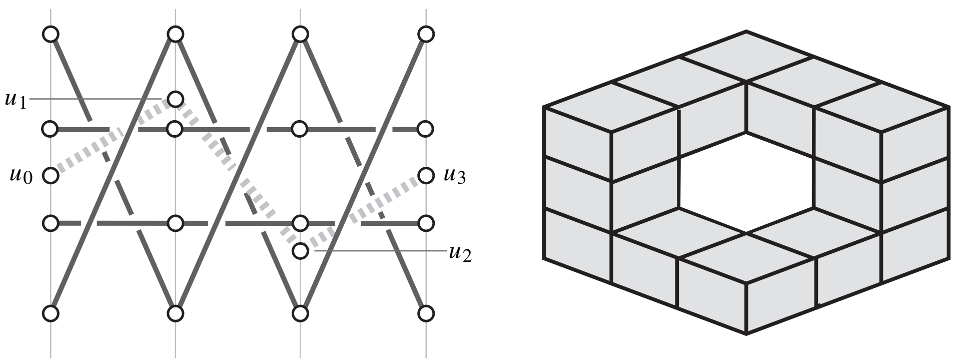

Figure 6. The braid of Example 7 and the associated relative braid class.

Example 8. The braid pair of Figure 7 [right] has index $h \simeq S^4 \vee S^5$ (as computed in [32, Lem. 50]); the pair on the left has trivial index, even though the linking numbers and periods of all strands are identical. This exemplifies the extra information carried by the braiding data.

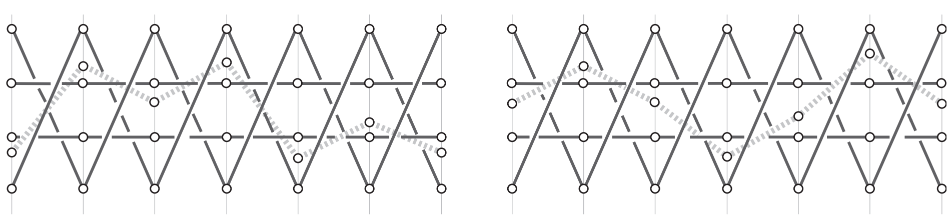

Figure 7. Discretized braid pairs with trivial [left] and nontrivial [right] homotopy index.

4.3. The index: topological version. As defined, the homotopy braid index h is a function of discretized braid classes. For *topological* braids, one could hope that any discretization yields the same discrete index. It does, modulo two technicalities.

The first is simple. Given a topological relative braid pair $\boldsymbol{u}$ REL $\boldsymbol{v}$ and a discretization period p, consider the discrete braid pair whose anchor points are defined in the obvious way using $x_i = i/p$ as the spatial discretization points. Only for p sufficiently large will this discrete braid pair be isotopic as a topological braid to the pair $\boldsymbol{u}$ REL $\boldsymbol{v}$. Thus, one must choose p so that the correct braid class is obtained by discretization.

The second technicality is more subtle. Even if the discretized braid is topologically isotopic to the original, it is possible to "fracture" the homotopy type of the topological braid class via discretization. Consider the discrete braids of Figure 8: these braid pairs are equivalent as topological closed braids, but *not* as discrete closed braids. There is simply not enough freedom to maneuver.

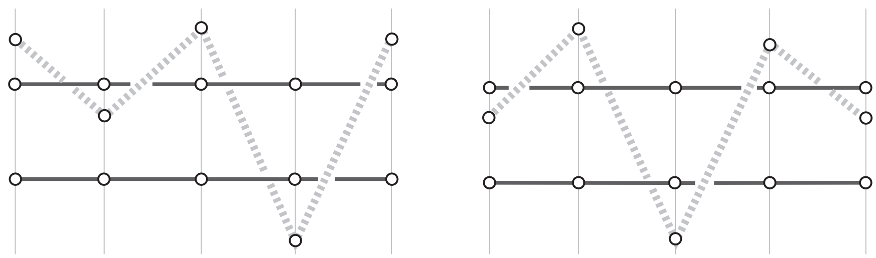

Figure 8. An example of two discretized braids which are of the same topological braid class but define disjoint discretized braid classes in $\mathcal{D}^1_4$ REL $\boldsymbol{v}$.

To overcome this difficulty, we define a modification of the homotopy braid index as follows. Given a fixed period p and a discrete proper relative braid class $\beta = [\boldsymbol{u} \text{ REL } \boldsymbol{v}] \in \mathcal{D}^n_p$, let $\mathcal{S}(\beta)$ denote the set of all braid classes in $\mathcal{D}^n_p$ REL $\boldsymbol{v}$ which are isotopic as *topological* braids to a representative of β. Define the index $\boldsymbol{H}$ to be

$$\boldsymbol{H}(\beta) = \bigvee_{\beta_i \in \mathcal{S}(\beta)} h(\beta_i). \tag{8}$$

This is a wedge sum of the indices of all discrete period-p representatives of the given topological braid class. The wedge sum is well-defined since each h is a pointed homotopy class.

This index $\boldsymbol{H}$ is an invariant of topological braid classes. Consider the following *stabilization operator*, $\mathbb{E}\colon \mathcal{D}^n_p \to \mathcal{D}^n_{p+1}$, which appends a trivial period-1 braid to the end of a discrete braid:

$$(\mathbb{E}\boldsymbol{u})^\alpha_i = \begin{cases} u^\alpha_i, & i = 0, \dots, p, \\ u^\alpha_p, & i = p+1. \end{cases} \tag{9}$$

The most important result about the index is the following invariance theorem:

Theorem 9 (Stabilization [32]). *For $\boldsymbol{u}$ REL $\boldsymbol{v}$ any bounded proper discretized braid pair, the topological homotopy braid index is invariant under the extension operator:*

$$\boldsymbol{H}(\mathbb{E}\boldsymbol{u} \text{ REL } \mathbb{E}\boldsymbol{v}) = \boldsymbol{H}(\boldsymbol{u} \text{ REL } \boldsymbol{v}). \tag{10}$$

The proof of this theorem involves, surprisingly enough, a dynamical argument, utilizing a singular perturbation of a particular parabolic relation adapted to $\mathbb{E}$. This is a very convenient way to prove homotopy equivalence, given the robustness of the Conley index with respect to singular perturbations [15]. This theorem allows for a proof of topological invariance.

Theorem 10 (Invariance [32]). *Given $\boldsymbol{u}$ REL $\boldsymbol{v} \in \mathcal{D}^n_p$ REL $\boldsymbol{v}$ and $\tilde{\boldsymbol{u}}$ REL $\tilde{\boldsymbol{v}} \in \mathcal{D}^n_{\tilde{p}}$ REL $\tilde{\boldsymbol{v}}$ which are topologically isotopic as bounded proper braid pairs, then*

$$\boldsymbol{H}(\boldsymbol{u} \text{ REL } \boldsymbol{v}) = \boldsymbol{H}(\tilde{\boldsymbol{u}} \text{ REL } \tilde{\boldsymbol{v}}). \tag{11}$$

The key ingredients in this proof are the Stabilization Theorem combined with a braid-theoretic argument that the moduli space of discretized braids converges to that of topological braids under sufficiently many applications of $\mathbb{E}$ – the length of the braid in the word metric suffices.

5. Forcing theorems: parabolic lattice dynamics

The dynamical consequences of the index are forcing results. A simple example: given any parabolic relation $\mathcal{R}$ which has as stationary solutions the skeleton of Figure 7 [right], then, since adding the dashed strand from that figure yields a nontrivial braid index, there must be some invariant set for $\mathcal{R}$ within this braid class. At this point, one uses Morse theory ideas: if $\mathcal{R}$ is exact, then there must be a stationary solution of the form of the grey strand. If the flow is not a gradient flow, then finer information can still detect stationary solutions.

More specifically, let h be the homotopy braid index of a proper bounded discrete braid class $[\boldsymbol{u}$ REL $\boldsymbol{v}]$. Let $P_\tau(h)$ denote the *Poincaré polynomial* of the index – the polynomial in $\mathbb{Z}[\tau]$ whose coefficients are the Betti numbers of the homology of the index, $H_*(h;\mathbb{R})$. The following results are consequences of degenerate Morse theory (cf. [16]).

Theorem 11 ([32]). *Given a parabolic relation $\mathcal{R}$ which fixes $\boldsymbol{v}$ and $h = h([\boldsymbol{u}$ REL $\boldsymbol{v}])$, the following hold:*

1. *The number of stationary braids in this braid class is bounded below by the Euler characteristic $\chi(h) = P_{-1}(h)$.*

2. *If $\mathcal{R}$ is exact, then the number of stationary braids in this braid class is bounded below by the number of nonzero monomials of $P_\tau(h)$.*

Stronger results are available if it is known that the parabolic relation is nondegenerate. By iterating the process of adding free strands and computing a nontrivial index, one can go quite far. The following forcing theorem (for exact $\mathcal{R}$) is very general, requiring only that the parabolic relation is exact (yielding a gradient flow) and *dissipative*, meaning that $\mathcal{R}_i \to -\infty$ as $|u_i| \to +\infty$.

Theorem 12 ([32]). *Let $\mathcal{R}$ be a parabolic relation which is both exact and dissipative. If $\mathcal{R}$ fixes a discretized braid $\mathbf{v}$ which is not a trivial braid class, then there exist an infinite number of distinct braid classes which arise as stationary solutions of $\mathcal{R}$.*

This theorem is very much in the spirit of "period-three implies chaos." The dissipative boundary condition at infinity can be replaced with a coercive condition (infinity is attracting) or with mixtures thereof with only minor adjustments to the theorem statements [32].

6. Forcing theorems: second-order Lagrangians

This forcing theory gives an elegant approach to a class of fourth-order equations arising from a Lagrangian. Consider a *second order Lagrangian*, $L(u, u_x, u_{xx})$, such as is found in the Swift–Hohenberg equation:

$$L = \frac{1}{2}(u_{xx})^2 - (u_x)^2 + \frac{1-\alpha}{2}u^2 + \frac{u^4}{4}. \tag{12}$$

Assume the standard convexity assumption that $\partial^2_{u_{xx}} L \geq \delta > 0$. The Euler–Lagrange equations yield a fourth-order ODE. The objective is to find bounded functions $u : \mathbb{R} \to \mathbb{R}$ which are stationary for the action integral $J[u] = \int L(u, u_x, u_{xx})\, dx$. Due to the translation invariance $x \mapsto x + c$, the solutions of the Euler–Lagrange equation satisfy the energy constraint

$$\left(\frac{\partial L}{\partial u_x} - \frac{d}{dx}\frac{\partial L}{\partial u_{xx}}\right)u_x + \frac{\partial L}{\partial u_{xx}}u_{xx} - L(u, u_x, u_{xx}) = E = \text{ constant}, \tag{13}$$

where E is the energy of a solution. To find bounded solutions for given values of E, we employ the variational principle $\delta_{u,T} \int_0^T (L(u, u_x, u_{xx}) + E)\, dx = 0$, which forces solutions to have energy E.

The Lagrangian problem can be reformulated as a two degree-of-freedom Hamiltonian system. In that context, bounded periodic solutions are closed characteristics of the corresponding energy manifold $M^3 \subset \mathbb{R}^4$. Unlike the case of first-order Lagrangian systems, the energy hypersurface is not of contact type in general [4], and is never compact. The recent stunning results in contact homology [18] are inapplicable.

6.1. The twist condition. The homotopy braid index provides a very effective means of forcing periodic orbits. By restricting to systems which satisfy a mild variational hypothesis, one can employ a "broken geodesics" construction which yields a restricted form of parabolic relation.

Closed characteristics at a fixed energy level E are concatenations of monotone laps between alternating minima and maxima $(u_i)_{i\in\mathbb{Z}}$, which form a periodic sequence with even period. The problem of finding closed characteristics can, in most cases, be formulated as a finite dimensional variational problem on the extrema (u_i), as realized by Vandervorst, in his definition of the *twist condition*. The twist condition is a weaker version of the hypothesis that assumes that the monotone laps between extrema are unique and is valid for large classes of Lagrangians L, including Equation (12). The following result of [52] is the motivation and basis for the applications of the homotopy braid index to second-order Lagrangians.

Lemma 13. *Extremal points* $\{u_i\}$ *for bounded solutions of second order Lagrangian twist systems are solutions of an exact parabolic relation with the constraints that* (i) $(-1)^i u_i < (-1)^i u_{i+1}$*; and* (ii) *the relation blows up along any sequence satisfying* $u_i = u_{i+1}$.

6.2. A general result. It is necessary to retool the homotopy braid index to the setting of Lemma 13 and show that the index properties with respect to this restricted class of parabolic relations are invariant. Upon so doing, one extracts very general forcing theorems, a simple example of which is the following:

Theorem 14 ([32]). *Let* $L(u, u_x, u_{xx})$ *be a Lagrangian which is dissipative* (*infinity is repelling*) *and twist. Then, at any regular energy level, the existence of a single periodic orbit which traces out a self-intersecting curve in the* (u, u_x) *plane implies the existence of infinitely many other periodic orbits at this energy level.*

Additional results give lower bounds on the multiplicity of solutions in a given braid class based on the Poincaré polynomial and apply to singular energy levels, as well as to non-dissipative systems [32].

7. Forcing theorems: parabolic PDEs

The homotopy braid index, being inspired by parabolic PDEs, is efficacious in this context also, thanks to Theorem 10. By performing a spatial discretization of the dynamics of Equation (1), it is possible to reduce the dynamics of the PDE to those of a parabolic relation on a finite-dimensional space of discretized braids.

On account of the robustness of the homotopy index with respect to the dynamics, there is very little one needs to assume about the nonlinearity in Equation (1). The first, crucial, hypothesis is a growth condition on the u_x term of f. For simplicity, let us call Equation (1) *subquadratic* if there exist constants $C > 0$ and $0 < \gamma < 2$, such that $|f(x, u, v)| \leq C(1 + |v|^\gamma)$, uniformly in both $x \in S^1$ and on compact intervals in u. This is necessary for regularity and control of derivatives of solution curves, cf. [3]. This condition is sharp: one can find examples of f with quadratic growth in u_x for which solutions have singularities in u_x. Since our topological data are drawn from graphs of u, the bounds on u imply bounds on u_x and u_{xx}.

A second gradient hypothesis will sometimes be assumed. One says Equation (1) is *exact* if

$$u_{xx} + f(x, u, u_x) = a(x, u, u_x)\left[\frac{d}{dx}\partial_{u_x} L - \partial_u L\right], \tag{14}$$

for a strictly positive and bounded function $a = a(x, u, u_x)$ and some Lagrangian L satisfying $a(x, u, u_x) \cdot \partial^2_{u_x} L(x, u, u_x) = 1$.

In this case, one has a gradient system whose stationary solutions are critical points of the action $\int L(x, u, u_x)\, dx$ over loops of integer period in x. This condition holds for a wide variety of systems. In general, systems with Neumann or Dirichlet boundary conditions admit a gradient-like structure which precludes the existence of nonstationary time-periodic solutions. It was shown by Zelenyak [55] that this gradient-like condition holds for many nonlinear boundary conditions which are a mixture of Dirichlet and Neumann.

7.1. Stationary solutions. Assume for the following theorems that $\{\boldsymbol{u} \text{ REL } \boldsymbol{v}\}$ is a topological braid class which is both bounded and proper. Assume further that $\boldsymbol{v}$ is stationary for Equation (1). We state our existence and multiplicity results in terms of the Poincaré polynomial $P_\tau(\boldsymbol{H})$ of the topological (as opposed to the discrete) braid index $\boldsymbol{H} = \boldsymbol{H}\{\boldsymbol{u} \text{ REL } \boldsymbol{v}\}$, computed via a discretization of the topological braid.

Theorem 15 ([31]). *Let Equation* (1) *be subquadratic with* $\boldsymbol{v}$ *a stationary braid, and* $\boldsymbol{H} = \boldsymbol{H}(\{\boldsymbol{u} \text{ REL } \boldsymbol{v}\})$.

1. *There exists a stationary solution in this braid class if the Euler characteristic of the index,* $\chi(\boldsymbol{H}) = P_{-1}(\boldsymbol{H})$*, is nonvanishing.*

2. *If Equation* (1) *is furthermore exact, then there exists a stationary solution in this braid class if* $P_\tau(\boldsymbol{H}) \neq 0$.

Additional results are available concerning multiplicity of solutions, alternate boundary conditions, and non-uniformly parabolic equations: see [31]. A version of Theorem 12 on infinite numbers of braids being forced by a single nontrivial stationary braid persists in this context. The result is simplest to state if the PDE is *dissipative*; that is, $u\, f(x, u, 0) \to -\infty$ as $|u| \to +\infty$ uniformly in $x \in S^1$. This is a fairly benign restriction.

Theorem 16 ([31]). *Let Equation* (1) *be subquadratic, exact, and dissipative. If* $\boldsymbol{v}$ *is a nontrivially braided stationary skeleton, then there are infinitely many braid classes represented as stationary solutions. Moreover, the number of single-free-strand braid classes is bounded from below by* $\lceil \iota/2 \rceil - 1$*, where* ι *is the maximal number of intersections between two strands of* $\boldsymbol{v}$.

7.2. Examples. The following family of spatially inhomogeneous Allen–Cahn equations was studied by Nakashima [45], [46]:

$$\varepsilon^2 u_t = \varepsilon^2 u_{xx} + g(x)u(1-u^2), \tag{15}$$

where $g\colon S^1 \to (0,1)$ is not a constant. This equation has stationary solutions $u = 0, \pm 1$ and is exact with Lagrangian

$$L = \frac{1}{2}\varepsilon^2 u_x^2 - \frac{1}{4}g(x)u^2(2-u^2).$$

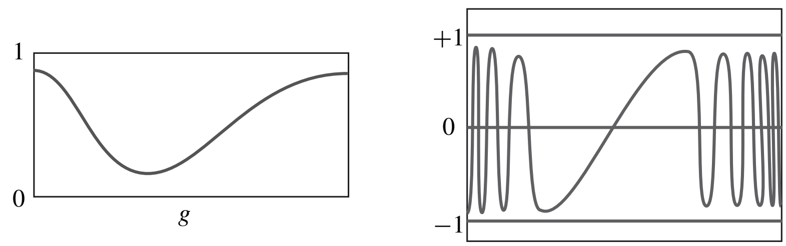

Figure 9. Given a function $g\colon S^1 \to (0,1)$ and ε small, there exists a skeleton of stationary curves for Equation (15) which forms a nontrivial braid. This forces infinitely many other stationary braids.

According to [45], for any $N > 0$, there exists an $\varepsilon_N > 0$ so that for all $0 < \varepsilon < \varepsilon_N$, there exist at least two stationary solutions which intersect $u = 0$ exactly N times. (The cited works impose Neumann boundary conditions: it is a simple generalization to periodic boundary conditions.) Via Theorem 16 we have that for any such g and any small ε, this equation admits an infinite collection of stationary periodic curves; furthermore, there is a lower bound of N on the number of 1-periodic solutions.

As a second explicit example, consider the equation

$$u_t = u_{xx} - \frac{5}{8}\sin 2x\, u_x + \frac{\cos x}{\cos x + \frac{3}{\sqrt{5}}}u(u^2-1), \tag{16}$$

with $x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$. This gives an exact system with Lagrangian

$$L = e^{-\frac{5}{16}\cos 2x}\left(\frac{1}{2}u_x^2 - \frac{\cos x}{\cos x + \frac{3}{\sqrt{5}}}\frac{(u^2-1)^2}{4}\right), \tag{17}$$

and weight $a(x,u,u_x) = e^{\frac{5}{16}\cos 2x}$ (cf. Equation (14)).

One checks easily that there are stationary solutions ± 1 and $\pm\frac{1}{2}\big(\sqrt{5}\cos x + 1\big)$, as in Figure 10 [left]. These curves comprise a skeleton $\boldsymbol{v}$ which can be discretized to yield the skeleton of Example 6. This skeleton forces a stationary solution of the braid

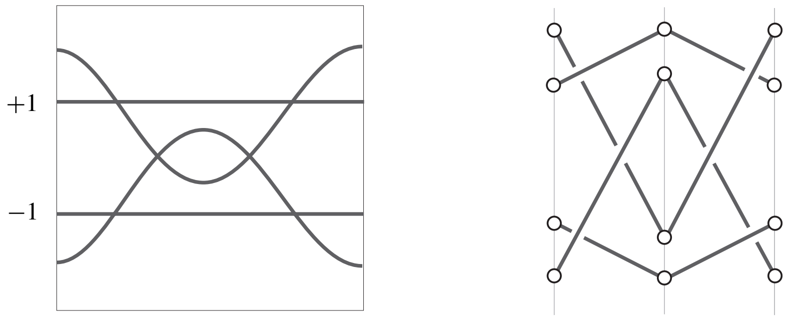

Figure 10. This collection of stationary solutions for Equation (16) [left] discretizes to the braid skeleton of Example 6.

class indicated in Figure 5 [left]: of course, this is detecting the obvious stationary solution $u = 0$. Note, however, that since $\boldsymbol{H} \simeq S^1$, this solution is unstable.

What is more interesting is the fact that one can take periodic extensions of the skeleton and add free strands in a manner which makes the relative braid spatially non-periodic. Let $\boldsymbol{v}^n$ be the n-fold periodic extension of $\boldsymbol{v}$ on $[0, n]/0 \sim n$ and consider a single free strand that weaves through $\boldsymbol{v}^n$ as in Figure 11. The homotopy index of such a braid is a sphere whose dimension is a function of the linking number of the free strand with the skeletal strands. The appropriate Morse inequalities imply that for each $n > 0$ there exist at least $3^n - 2$ distinct stationary solutions. This information can be used to prove that the time-2π map of the stationary equation has positive entropy, see e.g. [47], [53].

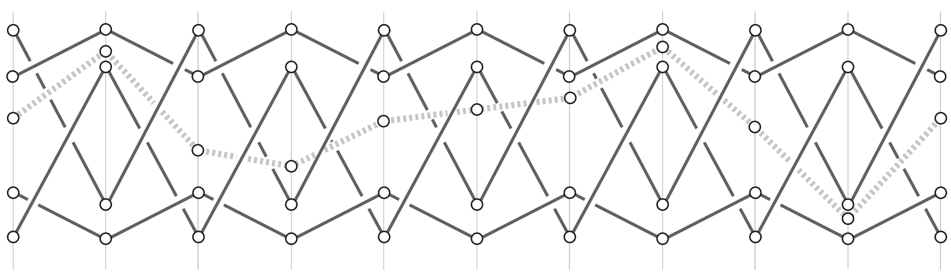

Figure 11. Taking a lift of the spatial domain allows one to weave free strands through the lifted skeleton. These project to multiply-periodic solutions downstairs. The braid pictured has index $\boldsymbol{H} \simeq S^2$.

7.3. Time-periodic solutions. A fundamental class of time-periodic solutions to Equation (1) are the so-called *rotating waves*. For an equation which is autonomous in x, one makes the rotating wave hypothesis that $u(t, x) = U(x - ct)$, where c is the unknown wave speed. Stationary solutions for the resulting equation on $U(\xi)$ yield rotating waves. In [3] it was proved that time-periodic solutions are necessarily rotating waves for an equation autonomous in x. However, in the non-autonomous case, the rotating wave assumption is highly restrictive.

The homotopy braid index presents a very general technique for finding time-periodic solutions without the rotating wave hypothesis.

Theorem 17 ([31]). *Let $\{\boldsymbol{u} \text{ REL } \boldsymbol{v}\}$ be a bounded proper topological braid class with $\boldsymbol{u}$ a single-component braid, $\boldsymbol{v}$ an arbitrary stationary braid, and $P_\tau(\boldsymbol{H}) \neq 0$. If the braid class is not stationary for Equation* (1) *– the equation does not contain stationary braids in this braid class – then there exists a time-periodic solution in this braid class.*

It was shown in [3] that a singularly perturbed van der Pol equation,

$$u_t = \varepsilon u_{xx} + u(1 - \delta^2 u^2) + u_x u^2, \tag{18}$$

possesses an arbitrarily large number of rotating waves for $\varepsilon \ll 1$ sufficiently small and fixed $0 < \delta$. The homotopy braid index methods extend these results dramatically.

Theorem 18 ([31]). *Consider the equation*

$$u_t = u_{xx} + ub(u) + u_x c(x, u, u_x), \tag{19}$$

where c has sub-linear growth in u_x at infinity. Moreover, b and c satisfy the following hypotheses:

1. $b(0) > 0$*, and b has at least one positive and one negative root;*
2. $c(x, 0, 0) = 0$*, and $c > 0$ on $\{uu_x \neq 0\}$.*

Then this equation possesses time-periodic solutions spanning an infinite collection of braid classes.

All of the periodic solutions implied are dynamically unstable. In the most general case (those systems with x-dependence), the periodic solutions are not rigid rotating waves and thus would seem to be very difficult to detect.

8. What does this index mean?

The most important fact about the homotopy braid index $\boldsymbol{H}$ is that it is an invariant of topological braid pairs. Though it is not realistic to think that this is of interest in knot theory as a means of distinguishing braid pairs, the homotopy braid index nevertheless entwines both topological and dynamical content.

Thinking in terms of braid classes gives finer information than relying merely on intersection numbers. With the braid-theoretic approach, various analytic conditions on a PDE or lattice system (dispersive, coercive, etc.) can be 'modeled' by an auxiliary braid when computing the index. Likewise, spatial boundary conditions (Neumann, Dirichlet, periodic, etc.) can be viewed as restrictions on braids (fixed, closed, etc.). Any such restrictions which yield topologically equivalent braids have the same dynamical implications with respect to forcing. One may replace complicated analytic constraints with braids.

The precise topological content to the homotopy braid index is not transparent. A few steps toward unmasking the meaning of the index are as follows.

8.1. Duality. One special feature of working with discretized braids in a fixed period is a natural duality made possible by the fact that the index pair used to compute the homotopy braid index can be chosen to be a manifold pair.

The *duality operator* on discretized braids of even period is the map $\mathbb{D}\colon \mathcal{D}^n_{2p} \to \mathcal{D}^n_{2p}$ given by

$$(\mathbb{D}\boldsymbol{u})^\alpha_i = (-1)^i u^\alpha_i. \tag{20}$$

Clearly $\mathbb{D}$ induces a map on relative braid diagrams by defining $\mathbb{D}(\boldsymbol{u} \text{ REL } \boldsymbol{v})$ to be $\mathbb{D}\boldsymbol{u} \text{ REL } \mathbb{D}\boldsymbol{v}$. The topological action of $\mathbb{D}$ is to insert a half-twist at each spatial segment of the braid. This has the effect of linking unlinked strands, and, since $\mathbb{D}$ is an involution, linked strands are unlinked by $\mathbb{D}$, as in Figure 12.

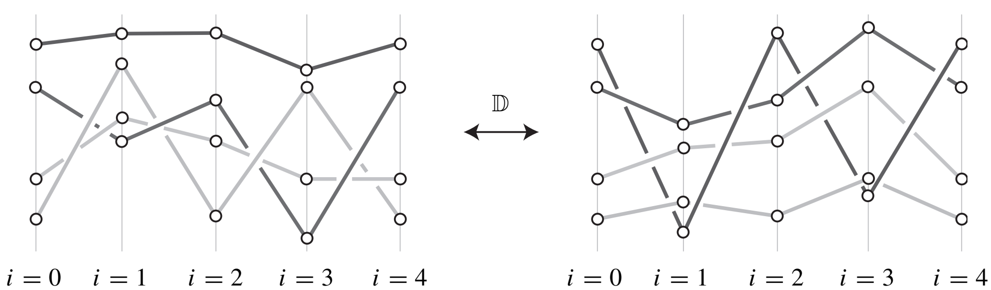

Figure 12. The topological action of $\mathbb{D}$.

For the two duality theorems to follow, we assume that all braids considered have even periods and that all of the braid classes and their duals are proper, so that the homotopy index is well-defined. In this case, the duality map $\mathbb{D}$ respects braid classes: if $[\boldsymbol{u}] = [\boldsymbol{u}']$ then $[\mathbb{D}(\boldsymbol{u})] = [\mathbb{D}(\boldsymbol{u}')]$. Bounded braid classes are taken to bounded braid classes by $\mathbb{D}$.

The effect of $\mathbb{D}$ on the index pair is to reverse the direction of the parabolic flow. This is the key to proving the following:

Theorem 19 (Duality [32][1]). *For $[\boldsymbol{u} \text{ REL } \boldsymbol{v}]$ having period $2p$ and n free strands,*

$$H_q(\boldsymbol{H}(\mathbb{D}(\boldsymbol{u} \text{ REL } \boldsymbol{v})); \mathbb{R}) \cong H_{2np-q}(\boldsymbol{H}(\boldsymbol{u} \text{ REL } \boldsymbol{v}); \mathbb{R}). \tag{21}$$

This duality operator is very useful in computing the homology of the braid index: see the computations in [32].

8.2. Twists. The duality operator yields a result on the behavior of the index under appending a full twist.

Theorem 20 (Shift [32]). *Appending a full twist to a braid shifts the homology of the index up by dimension equal to twice the number of free strands.*

We include a sketch of the proof (a more careful version of which would deal with some boundedness issues). Assume that $[\boldsymbol{u} \text{ REL } \boldsymbol{v}]$ is a braid of period $2p$ with n free strands. A period two full-twist braid can be realized as the dual of the trivial braid of period two. Thus, the effect of adding a full twist to a braid can be realized by the operator $\mathbb{DEED}$. By combining Theorems 9 and 19, we obtain:

$$\begin{aligned} H_q(\boldsymbol{H}(\mathbb{DEED}[\boldsymbol{u} \text{ REL } \boldsymbol{v}])) &\cong H_{2np+2n-q}(\boldsymbol{H}(\mathbb{DEE}[\boldsymbol{u} \text{ REL } \boldsymbol{v}])) \\ &\cong H_{2np+2n-q}(\boldsymbol{H}(\mathbb{D}[\boldsymbol{u} \text{ REL } \boldsymbol{v}])) \\ &\cong H_{q-2n}(\boldsymbol{H}([\boldsymbol{u} \text{ REL } \boldsymbol{v}])). \end{aligned} \tag{22}$$

A homotopy version of Equation (22) should be achievable by following a similar procedure as in the proof of Theorem 9. We suspect one obtains an iterated suspension of the homotopy index, as opposed to a shift in homology.

9. Toward arbitrary braids

Given the motivation from PDEs and the comparison principle, the types of braids considered in this paper are positive braids. One naturally wonders whether an extension to arbitrary braids – those with mixed crossing types – is possible. Unfortunately, passing to discretized braids is no longer simple, as anchor points alone cannot capture crossing information for arbitrary braids.

One way to define a formal index for general braid pairs is to use Garside's Theorem [6], slightly modified. Garside's Theorem states that any braid can be placed into a unique normal form of a positive braid times a (minimal) number of negative half-twists. Clearly, one can define a modified Garside normal form that gives a unique decomposition into a positive braid and a (minimal) number of negative full twists. By applying Theorem 20, one can define a homological braid index (with negative grading permitted) by shifting the braid index of the positive normal form down by

[1] The theorem in the reference has a slight error in the statement. There, it was implicitly assumed that the braid has one free strand. The present statement is correct for arbitrary numbers of strands.

the appropriate amount. A homotopy theoretic version could be defined in terms of spectra via suspensions. This, then, yields a formal index for arbitrary (proper) braid pairs.

The real question is what dynamical meaning this generalized index entails. The passage from positive braids to arbitrary braids is akin to the passage from a Lagrangian to a Hamiltonian settings, and such an extended index appears to be a relative Floer homology for (multiply) periodic solutions to time-periodic Hamiltonian systems.

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