Asymptotic behavior of smooth solutions for partially dissipative hyperbolic systems and relaxation approximation

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Abstract. We study two problems related to hyperbolic systems with a dissipative source.

In the first part, we consider the asymptotic time behavior of global smooth solutions to general entropy dissipative hyperbolic systems of balance law in m space dimensions, under a coupling condition among hyperbolic and dissipative part known as the Shizuta–Kawashima condition. Under the assumption of small initial data, these solutions approach constant equilibrium state in the L^p-norm at a rate $O\big(t^{-\frac{m}{2}(1-\frac{1}{p})}\big)$, as $t \to \infty$, for $p \in [\min\{m, 2\}, \infty]$. The main tool is given by a detailed analysis of the Green function for the linearized problem. If the space dimension $m = 1$ or the system is rotational invariant, it is possible to give an explicit form to the main terms in the Green kernel.

In the second part, we consider the hyperbolic limit of special systems of balance laws: this means to study the limit of the solution to a system of balance laws under the rescaling $(t, x) \mapsto (t/\varepsilon, x/\varepsilon)$, as $\varepsilon \to 0$. For some special dissipative systems in one space dimension, it is possible to prove the existence of the limit and to identify it as a solution to a system of conservation laws.

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1. Asymptotic behavior of smooth solutions to balance laws

We consider the Cauchy problem for a general hyperbolic symmetrizable m-dimensional system of balance laws

$$u_t + \sum_{\alpha=1}^{m} (f_\alpha(u))_{x_\alpha} = g(u), \tag{1.1}$$

with the initial conditions

$$u(x, 0) = u_0(x), \tag{1.2}$$

where $u = (u_1, u_2) \in \Omega \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with $n_1 + n_2 = n$. We also assume that there are n_1 conservation laws in the system, namely that we can take

$$g(u) = \begin{pmatrix} 0 \\ q(u) \end{pmatrix} \quad \text{with } q(u) \in \mathbb{R}^{n_2}. \tag{1.3}$$

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According to the general theory of hyperbolic systems of balance laws [11], if the flux functions f_α and the source term g are smooth enough, it is well known that problem (1.1)–(1.2) has a unique local smooth solution, at least for some time interval $[0, T)$ with $T > 0$, if the initial data are also sufficiently smooth. In the general case, and even for very good initial data, smooth solutions may break down in finite time, due to the appearance of singularities, either discontinuities or blow-up in L^∞.

The easiest example is Burgers equation with an initial data strictly decreasing in some interval. With more generality, if we have a system of balance laws, it may happen that the source terms have no influence on the mechanism generating a singularity, or it may also help the solution to blow up.

Despite these general considerations, sometimes dissipative mechanisms due to the source term can prevent the formation of singularities, at least for some restricted classes of initial data, as observed for many models which arise to describe physical phenomena. A typical and well-known example is given by the compressible Euler equations with damping, see [20], [14] for the 1-dimensional case and [23] for an interesting 3-dimensional extension.

Recently, in [13], it was proposed a quite general framework of sufficient conditions which guarantee the global existence in time of smooth solutions. Actually, for the systems which are endowed with a strictly convex entropy function $\mathcal{E} = \mathcal{E}(u)$, a first natural assumption is the *entropy dissipation condition*, see [10], [19], [21], [24], namely for every $u, \bar{u} \in \Omega$, with $g(\bar{u}) = 0$,

$$\big(\nabla\mathcal{E}(u) - \nabla\mathcal{E}(\bar{u})\big) \cdot g(u) \leq 0,$$

where $\mathcal{E}'(u)$ is considered as a vector in $\mathbb{R}^n$ and "$\cdot$" is the scalar product in the same space.

Roughly speaking, the above condition means that the source is dissipative in some integral norm, typically L^2. Thus one expects that blow up in L^∞ could be prevented, or certainly it does not happen for space independent solutions.

Unfortunately, it is easy to see that this condition is too weak to prevent the formation of singularities: just consider the system

$$\begin{cases} u_t + uu_x = 0, \\ \qquad v_t = -v \end{cases} \tag{1.4}$$

with entropy u^2+v^2. The key point in this system is that the dissipative source $(0, -v)$ is not acting on the first equation, so that it cannot prevent the shock formation.

A quite natural supplementary condition can be imposed to entropy dissipative systems, following the classical approach by Shizuta and Kawashima [16], [22], and in the following called condition (SK), which in the present case reads

$$\mathrm{Ker}(Dg(\bar{u})) \cap \left\{\text{eigenspaces of } \textstyle\sum_{\alpha=1}^m Df_\alpha(\bar{u})\xi_\alpha\right\} = \{0\}, \tag{1.5}$$

for every $\xi \in \mathbb{R}^m \setminus \{0\}$ and every $\bar{u} \in \Omega$, with $g(\bar{u}) = 0$. It is possible to prove that this condition, which is satisfied in many interesting examples, is also sufficient to establish a general result of global existence for small perturbations of equilibrium constant states.

As an example, it is easy to see that (1.4) does not satisfy (SK) condition, while the system

$$\begin{cases} u_t + uu_x + v_x = 0, \\ v_t + u_x = -v \end{cases} \tag{1.6}$$

fulfills requirement (1.5).

We investigate the asymptotic behavior in time of the global solutions, always assuming the existence of a strictly convex entropy and the (SK) condition.

Our starting point is a careful and refined analysis of the behavior of the Green function for the linearized problem around an equilibrium state $\bar{u}$,

$$u_t + \sum_{\alpha=1}^{m} A_\alpha \partial_{x_\alpha} u = Bu, \quad A^\alpha = \nabla f^\alpha(u)|_{u=\bar{u}}, \; B = \nabla g(u)|_{u=\bar{u}} \tag{1.7}$$

The conditions on the existence of a dissipative strictly convex entropy and (SK) condition (and also (1.3) implies that

1. the matrices A_α are symmetric and

$$B = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}, \quad D \in \mathbb{R}^{n_2 \times n_2}, \tag{1.8}$$

   with D strictly negative definite;

2. no eigenvectors of $\sum_\alpha \xi_\alpha A_\alpha$ are in the null space of B for all $\xi \in \mathbb{R}^m$.

It is possible to show that the Green kernel $\Gamma(t)$ can be written as the sum of the kernels

$$\Gamma(t) = K(t) + \mathcal{K}(t). \tag{1.9}$$

The first term corresponds to a uniformly parabolic (pseudo) differential operator, while the first satisfies a uniform exponential decay in L^2:

$$\|D^\beta \mathcal{K}(t) w^0\|_{L^2} \le Ce^{-ct} \|D^\beta w^0\|_{L^2}. \tag{1.10}$$

It can be also shown that $\Gamma(t)$ has bounded support, so that both $K(t)$, $\mathcal{K}(t)$ have bounded support.

For the term $K(t)$ it is possible to give a more precise description, by using the two projectors Q_0, $Q_- = I - Q_0$: Q_0 is the projector on the conservative part of (1.7), i.e.

$$Q_0 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \quad Q_- \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ u_2 \end{pmatrix}.$$

Writing then

$$K(t) = \begin{bmatrix} K_{11}(t) & K_{12}(t) \\ K_{21}(t) & K_{22}(t) \end{bmatrix}, \tag{1.11}$$

it is possible to prove that K_{11} decays as a heat kernel, while the other three decays as a derivative of the heat kernel:

$$\begin{aligned} \|D^\beta K_{11}(t)w\|_{L^p} &\le C(|\beta|)\min\Big\{1, t^{-\frac{m}{2}(1-\frac{1}{p})-|\beta|/2}\Big\}\|w\|_{L^1}, \\ \|D^\beta K_{12}(t)w\|_{L^p}, \|D^\beta K_{21}(t)w\|_{L^p} &\le C(|\beta|)\min\Big\{1, t^{-\frac{m}{2}(1-\frac{1}{p})-1/2-|\beta|/2}\Big\}\|w\|_{L^1}, \\ \|D^\beta K_{22}(t)w\|_{L^p} &\le C(|\beta|)\min\Big\{1, t^{-\frac{m}{2}(1-\frac{1}{p})-1-|\beta|/2}\Big\}\|w\|_{L^1}. \end{aligned} \tag{1.12}$$

Using this Green kernel representation, one can show that solution to (1.1) satisfies the same decay estimates as $\Gamma(t)$. More precisely, one can prove that

$$\|u(t) - K(t)Q_0u(0)\|_{L^p} \le C\min\big\{1, t^{-\frac{m}{2}(1-\frac{1}{p})-1/2}\big\}\|u(0)\|_{L^1}.$$

Clearly to have more information on the asymptotic behavior of $u(t)$ one needs to know more information on $K(t)$. This can be done in two situations: if the space dimension is $m = 1$, or under the assumption of rotational invariance. In both cases the Fourier components of the differential operators can be inverted.

As an example, we can consider the linearized isentropic Euler equations with damping,

$$\begin{cases} \rho_t + \mathrm{div} v = 0, \\ v_t + \nabla\rho = -v. \end{cases} \tag{1.13}$$

One can check that the three conditions are satisfied. We can decompose the Green kernel Γ in three parts

$$\Gamma(t,x) = K(t,x) + R(t,x) + \mathcal{K}(t,x), \tag{1.14}$$

where $K(t,x)$ can be computed to be

$$K(t,x) = \begin{bmatrix} G(t,x) & (\nabla G(t,x))^T \\ \nabla G(t,x) & \nabla^2 G(t,x) \end{bmatrix} + R_1(t,x), \tag{1.15}$$

where $G(t,x)$ is the heat kernel for $u_t = \Delta u$, and the rest term $R_1(t,x)$ satisfies the bound

$$R_1(t,x) = \frac{e^{-c|x|^2/t}}{(1+t)^2}\begin{bmatrix} \mathcal{O}(1) & \mathcal{O}(1)(1+t)^{-1/2} \\ \mathcal{O}(1)(1+t)^{-1/2} & \mathcal{O}(1)(1+t)^{-1} \end{bmatrix}. \tag{1.16}$$

In particular the principal part of $\Gamma(t)$ is given by the heat kernel $G(t,x)$.

The rest part R_1 is exponentially decreasing and smooth, while $\mathcal{K}(t,x)$ can be computed to be

$$\mathcal{K}(t,x) = \begin{bmatrix} 0 & 0 \\ 0 & e^{-t}\mathcal{P} \end{bmatrix} + e^{-t}\begin{bmatrix} W_{00}(t,x) & W_{01}(t,x) \\ W_{10}(t,x) & W_{11}(t,x) \end{bmatrix} + R_2(t,x). \tag{1.17}$$

Here $\mathcal{P}: (L^2(\mathbb{R}^3))^3 \mapsto (L^2(\mathbb{R}^3))^3$ is the orthogonal projection of L^2 vector fields on the subspace of divergence free vector fields. $\mathcal{P}v$ is characterized by

$$\mathcal{P}v \in (L^2(\mathbb{R}^3))^3, \quad \mathrm{div}\mathcal{P}v = 0, \quad \mathrm{curl}(v - \mathcal{P}v) = 0,$$

and so we have that

$$v - \mathcal{P}v = \nabla\psi \quad \text{with } \Delta\psi = \mathrm{div}v.$$

This yields

$$\mathcal{P}v = v - \nabla(\Delta^{-1}\mathrm{div}v). \tag{1.18}$$

In fact, in Fourier coordinates, we have

$$\widehat{\mathcal{P}v}(\xi) = \hat{v}(\xi) - |\xi|^{-2}(\xi \cdot \hat{v}(\xi))\xi = \hat{v}(\xi) - |\xi|^{-2}\xi\xi^T \cdot \hat{v}(\xi). \tag{1.19}$$

The matrix valued function

$$W(t,x) = W_1(t,x) + W_2(t,x) = \begin{bmatrix} W_{00}(t,x) & W_{01}(t,x) \\ W_{10}(t,x) & W_{11}(t,x) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \delta(x)\mathcal{P} \end{bmatrix} \tag{1.20}$$

is the matrix valued Green function of the system

$$\begin{cases} \rho_t + \mathrm{div}v = 0, \\ v_t + \nabla\rho = 0, \end{cases}$$

and it can be written by means of the fundamental solution to the wave equation $u_{tt} = \Delta u$. In fact, W_{00} is the solution of $u_{tt} = \Delta u$ with initial data $u = \delta(x)$, $u_t = 0$, and

$$W_1 = \begin{bmatrix} W_{00} & \nabla^T\partial_t(-\Delta)^{-1}W_{00} \\ \nabla\partial_t(-\Delta)^{-1}W_{00} & -\nabla^2(-\Delta)^{-1}W_{00} \end{bmatrix}. \tag{1.21}$$

In particular one can check that W_2 corresponds to incompressible vector fields, while W_1 corresponds to curl free vector fields.

An interesting open question is whether the (SK) can be relaxed by considering it acting only on a subset of the eigenspaces of $\sum Df_\alpha\xi_\alpha$. As a trivial example, if f^α are linear functions, then no dissipativity condition is needed. In [27] it is possible to find a non trivial example of a system which does not satisfy (SK) condition but still the smooth solution exists for all $t \geq 0$.

2. Relaxation limit of balance laws

After having proved existence for small data of the system

$$u_t + \sum_{\alpha=1}^{m} (f_\alpha(u))_{x_\alpha} = \frac{g(u)}{\varepsilon}, \tag{2.1}$$

under the dissipativity and (SK) condition, one can ask the question of describing the hyperbolic limit as $\varepsilon \to 0$. At a formal level, by writing

$$g(u) = g(u_1, u_2) = 0 \quad \Longrightarrow \quad u_1 = q(u_2) \tag{2.2}$$

for the equilibrium manifold, at a formal level we obtain the symmetrizable hyperbolic system

$$u_{1,t} + \sum_{\alpha=1}^{m} \left(f_\alpha(u_1, q(u_1))\right)_{x_\alpha} = 0. \tag{2.3}$$

As an example, if (2.1) is a kinetic scheme, then this limit corresponds to the hydrodynamic limit, i.e. to the rescaling $(t, x) \mapsto (t/\varepsilon, x/\varepsilon)$. The most outstanding problem in this direction is the hydrodynamic limit of Boltzmann equation, at least in one space dimension. In this direction the major result is the convergence under the assumption of piecewise smooth solution to the limiting Euler system with non interacting shocks [26]. We remark that hyperbolic limit must be based on the proof of local existence for data which do not satisfy any integral estimates.

We want to underline the difference among the known results and the theory of hyperbolic system. The assumption of non interacting shocks is equivalent to saying that we do not need to control the non linear interaction among waves, which is one of the key aspects of hyperbolic systems.

The main tool for proving existence and stability of solutions to hyperbolic systems is the local decomposition of the solution into waves and the description of their interaction. As an example, for BV solutions of an $n \times n$ system, the derivative of the solution is decomposed in n scalar measures, each measure representing the waves of one of the characteristic families of u. The key point is that one can describe the evolution of these waves as they interact and find an interaction functional which bounds the total interaction.

We believe that this functional would be a key point in proving hyperbolic limit of balance laws. It is clear that understanding this functional implies the understanding of the wave structure of the solution.

In general it seems difficult to prove convergence of (2.1) as $\varepsilon \to 0$, under the assumptions considered in the previous part. We thus restrict to particular quasilinear systems of the form (BGK schemes with one moment) in one space dimension:

$$\partial_t F^\alpha + \alpha F^\alpha_x = \frac{1}{\varepsilon}\big(M^\alpha(u) - F^\alpha\big), \quad u = \sum_\alpha F^\alpha. \tag{2.4}$$

The functions $M^\alpha(u)$ are the Maxwellians. At a formal level, as $\varepsilon \to 0$ one obtains

$$F^\alpha = M^\alpha(u), \quad u_t + \mathcal{F}(u)_x = 0,$$

where the flux function $\mathcal{F}(u)$ is given by

$$\mathcal{F}(u) = \sum_\alpha \alpha M^\alpha(u).$$

The easiest example is the scheme introduced in [15],

$$\begin{cases} u_t + v_x = 0, \\ v_t + \Lambda^2 u_x = \dfrac{1}{\varepsilon}(\mathcal{F}(u) - v), \end{cases} \tag{2.5}$$

which can be put in the form (2.4) by diagonalizing the semilinear part.

By differentiating the second equation of (2.5) w.r.t. x and using the first one obtains the nonlinear wave equation

$$u_t + A(u)u_x = \frac{1}{\varepsilon}(u_{xx} - u_{tt}), \tag{2.6}$$

with $A(u) = D\mathcal{F}(u)$. The above equation is meaningful also in the case $A(u)$ is not a Jacobian matrix, so that one cannot write a conservative form like (2.5). For this particular system it is now proved the existence and stability of solutions with initial data of small BV norm [3].

In this last part we describe the structure of the waves of (2.6). Writing the system in the form

$$\begin{cases} F_t^- - F_x^- = (M^-(u) - F^-)/\varepsilon, \\ F_t^* + F_x^* = (M^+(u) - F^+)/\varepsilon, \end{cases} \qquad u = F^- + F^+,\ M^\pm(u) = \frac{u \pm \mathcal{F}(u)}{2} \tag{2.7}$$

with $F^-, F^+ \in \mathbb{R}^n$, and assuming the stability condition $|D\mathcal{F}(u)| < 1$, at a formal level one expects that as $\varepsilon \to 0$, the function u converges to a solution to

$$u_t + \mathcal{F}(u)_x = 0, \quad u \in \mathbb{R}^n. \tag{2.8}$$

Concerning the solution of (2.8), we know that its structure can be described as the nonlinear sum of n shock waves, corresponding to the characteristic speed of $D\mathcal{F}$,

$$u_x(t,x) = \sum_{i=1}^n v_i(t,x)\tilde{r}_i(t,x), \quad v_i \in \mathbb{R},\ \tilde{r}_i \in \mathbb{R}^n. \tag{2.9}$$

The vectors $\tilde{r}_i$ are in general not the eigenvectors of $D\mathcal{F}$, but are close to them for small data. Also the propagation speeds of the scalar v_i is close to the i-th eigenvalue. Their interaction is described by a Glimm type functional [4]. Thus, a possible way of thinking of the waves in (2.7) is to imagine that the solution (F^-, F^+) is the sum of n waves (the wave decomposition of u) and some remaining term v which is dissipating entropy.

It turns out that a more natural description is that the solution to (2.7) can be decomposed into $2n$ waves, n for each component F^-, F^+,

$$F_x^-(t,x) = \sum_{i=1}^n f_i^-(t,x)\tilde{r}_i^-(t,x), \quad F_x^+(t,x) = \sum_{i=1}^n f_i^+(t,x)\tilde{r}_i^+(t,x). \tag{2.10}$$

This in some sense is a different philosophy, because we are not describing the solution as the approximate solution to the limiting hyperbolic system plus a term which is dissipating: we give a full nonlinear structure to the solution to the kinetic scheme.

We remark that with more generality, the BGK system (2.4) is decomposed as the sum of n waves for each component F^α. It is an open question which is the right decomposition in non linear waves for the more complicated discrete models, for example the well-known Broadwell model in one space dimension.

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