Nonlinear Schrödinger equations in inhomogeneous media: wellposedness and illposedness of the Cauchy problem

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Abstract. We survey recent wellposedness and illposedness results for the Cauchy problem for nonlinear Schrödinger equations in inhomogeneous media such as Riemannian manifolds or domains of the Euclidean space, trying to emphasize the influence of the geometry. The main tools are multilinear Strichartz estimates for the Schrödinger group.

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1. Introduction

The nonlinear Schrödinger equation arises in several areas of Physics (see the book [55] for an introduction), such as Optics or Quantum Mechanics, where it is related to Bose–Einstein condensation or Superfluidity. From the mathematical point of view, this equation has been studied on the Euclidean space since the seventies. However, it is quite relevant, in the above applications, to consider this equation on inhomogeneous media. In Optics, for instance, this naturally corresponds to a variable optical index; more specifically, spatial inhomogeneity has been recently used in the modelization of broad-area semiconductor lasers (see [38]). One of the main mathematical questions is then to evaluate the impact of the inhomogeneity on the dynamics of the equation, in particular regarding the wellposedness theory of the Cauchy problem. The goal of this paper is to survey recent mathematical contributions in this direction.

Let us precise what we mean by inhomogeneous medium in this context. Our physical space M is either the space $\mathbb{R}^d$ or a compact manifold, endowed in both cases with a second order differential operator P, which is elliptic, positive and selfadjoint with respect to some Lebesgue density μ and satisfies $P(1) = 0$.[1]

In coordinates, this means that P and μ are given by

$$Pu = -\frac{1}{\rho}\nabla.(A\nabla u), \quad d\mu = \rho(x)\,dx, \tag{1.1}$$

[1] Notice that this latter condition prevents potential terms in P. We impose this condition here for the sake of concision, though potential terms may of course be quite relevant too.

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where ρ is a smooth positive function, and where A is a smooth function valued in positive definite matrices. If $M = \mathbb{R}^d$, we shall impose the following additional conditions

$$0 < c \le \rho(x), \quad cI \le A(x), \quad |\partial^\alpha \rho(x)| + |\partial^\alpha A(x)| \le C_\alpha, \quad \alpha \in \mathbb{N}^d, \tag{1.2}$$

in order to avoid degeneracy at infinity. An example of such an operator is of course minus the Laplace operator associated to a Riemannian metric g on M which, in the case $M = \mathbb{R}^d$, satisfies moreover

$$cI \le g(x), \quad |\partial^\alpha g(x)| \le C_\alpha, \quad \alpha \in \mathbb{N}^d.$$

In this setting, the nonlinear Schrödinger equation (NLS) reads

$$i\frac{\partial u}{\partial t} - Pu = F(u), \tag{1.3}$$

where the unknown complex function u depends on $t \in \mathbb{R}$ and on $x \in M$, and the Cauchy problem consists in imposing the initial value of u at $t = 0$. Here the nonlinearity F is a smooth function on $\mathbb{C}$, which we assume to satisfy the following normalization and growth conditions:

$$F(0) = 0, \quad |D^k F(z)| \le C_k(1 + |z|)^{1+\alpha-k}, \quad k = 0, 1, 2, \dots. \tag{1.4}$$

In many situations, we require additional conditions on the structure of F. The most common one imposes that F derives from a potential function

$$F(z) = \frac{\partial V}{\partial \bar{z}}, \quad V : \mathbb{C} \to \mathbb{R}. \tag{1.5}$$

In this case (1.3) is a Hamiltonian system with the following Hamiltonian functional:

$$H(u) = \int_M (Pu\,\bar{u} + V(u))\, d\mu, \tag{1.6}$$

and consequently it formally enjoys the conservation law

$$H(u(t)) = H(u(0)). \tag{1.7}$$

Furthermore, if we assume the following gauge-invariance condition

$$V(\mathrm{e}^{i\theta} z) = V(z), \quad \theta \in \mathbb{R}, \tag{1.8}$$

i.e. $V(z) = G(|z|^2)$, $F(z) = G'(|z|^2)z$, we also have the L^2 conservation law

$$\|u(t)\|_{L^2(M,\mu)} = \|u(0)\|_{L^2(M,\mu)}. \tag{1.9}$$

A typical example of nonlinearity F satisfying (1.4), (1.5) and (1.8) is

$$F_{\alpha,\pm}(z) = \pm(1 + |z|^2)^{\alpha/2} z. \tag{1.10}$$

Finally, let us indicate that we shall sometimes discuss another kind of inhomogeneous NLS, namely when $-P$ is the Laplace operator with Dirichlet or Neumann boundary condition on a smooth domain of the Euclidean space. However, the theory is much less complete in this context.

This paper is organized as follows. After defining three different notions of wellposedness for the Cauchy problem for (1.3) on the scale of Sobolev spaces in Section 2, we make some general observations based on scaling considerations in Section 3. We begin Section 4 by recalling the role of Strichartz estimates in the analysis of (1.3) on the Euclidean space. We insist that this part is by no means an exhaustive review of the NLS theory on the Euclidean space. In particular, we did not discuss the recent contributions on scattering theory and on blow up. Then we really start the study of the influence of the geometry by observing that losses of derivatives may appear in Strichartz inequalities in the case of inhomogeneous media. In Section 5, we revisit the wellposedness problems by introducing multilinear Strichartz estimates, which originate in the works of Bourgain for Schrödinger and of Klainerman–Machedon for the wave equations. Finally, Section 6 is devoted to discussing in details the case of simple Riemannian compact manifolds, such as tori and spheres.

2. Some notions of wellposedness

We start with defining precisely the notions of wellposedness we are going to use throughout this paper. Indeed, since our evolution problem is nonlinear, several notions are available. We shall define these notions for the nonlinear Schrödinger equation (1.3) but it is clear that these notions are quite general and can be applied to other evolution equations.

Definition 2.1. We shall say that the Cauchy problem for equation (1.3) is (locally) *well-posed* on $H^s(M)$ if, for every bounded subset B of $H^s(M)$, there exists $T > 0$ and a Banach space X_T continuously contained in $C([-T, T], H^s(M))$ such that:

i) For every Cauchy data $u_0 \in B$, (1.3) has a unique solution $u \in X_T$ such that $u(0) = u_0$.

ii) If $u_0 \in H^\sigma(M)$ for $\sigma > s$, then $u \in C([-T, T], H^\sigma(M))$.

iii) The map
$$u_0 \in B \mapsto u \in X_T$$
is continuous.

Moreover, we shall say that the Cauchy problem for equation (1.3) is *globally well-posed* on $H^s(M)$ if properties i), ii), iii) above hold for every time $T > 0$.

Notice that in some cases local wellposedness can be combined with the conservation laws (1.9) and (1.7) to provide global wellposedness. Specifically, assume for instance that (1.3) is well-posed on $L^2(M) = H^0(M)$ and that F is gauge-invariant.

Since the L^2 conservation law holds for every solution in $C([-T,T], H^s(M))$ with s large enough, it results from requirements ii) and iii) that this conservation law holds on $[-T,T]$ as soon as $u_0 \in L^2$. Combining this observation with requirement i), we conclude that (local) wellposedness on L^2 implies global wellposedness on L^2. Similarly, one can show that local wellposedness on H^1 implies global wellposedness on H^1, under the assumption that a bound on $\|f\|_{L^2}$ and on $H(f)$ is equivalent to a bound on $\|f\|_{H^1}$, as it is the case, for instance, if F is gauge invariant and derives from a nonnegative potential with $(d-2)\alpha \le 4$.

Definition 2.2. We shall say that the Cauchy problem for equation (1.3) is (locally) *uniformly well-posed* on $H^s(M)$ if it is well-posed on $H^s(M)$ and if, with the notation of Definition 2.1 the map $u_0 \in B \mapsto u \in X_T$ is *uniformly* continuous.

One defines similarly *global uniform wellposedness*. Compared to Definition 2.1, uniform wellposedness can be understood as an additional requirement of high frequency stability for small uniform time. Let us mention that, in all the positive results of this paper, uniform continuity will come from Lipschitz continuity. As we shall see in the next sections, uniform wellposedness is rather natural for *semilinear* equations such as (1.3), but it may be violated for other natural evolution equations. For instance, it can be shown (see e.g. [59]) that the Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

is well-posed on $H^s(\mathbb{R},\mathbb{R})$ for $s > 3/2$, but is not uniformly well-posed. This is related to the *quasilinear* hyperbolic feature of Burgers' equation. A more subtle example is the modified Korteweg–de Vries equation on the one-dimensional torus,

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + \left(u^2 - \int_{\mathbb{T}} u^2\,dx\right)\frac{\partial u}{\partial x} = 0,$$

which is uniformly well-posed on $H^s(\mathbb{T},\mathbb{R})$ for $s > 1/2$ (see [10]), but is not for $s \in]3/8, 1/2[$, though it is well-posed for s in this interval. (see [56] and [40]).

Definition 2.3. We shall say that the Cauchy problem for equation (1.3) is (locally) *regularly well-posed* on $H^s(M)$ if it is well-posed on $H^s(M)$ and if, with the notation of Definition 2.1, the map $u_0 \in \mathring{B} \mapsto u \in X_T$ is *smooth*.

One defines similarly *global regular wellposedness*. As we shall see in the next section, regular wellposedness is quite a stringent notion if F is not a polynomial. On the other hand, for polynomial gauge-invariant nonlinearities, it will lead us to multilinear Strichartz estimates (see Section 5 below), which turn out to be the key estimates in this theory.

3. General observations

Since the free group e^{-itP} acts on $H^s(M)$, the following result is an elementary consequence of the classical nonlinear estimates in Sobolev spaces H^s for $s > d/2$, combined with the Duhamel formulation of the Cauchy problem for (1.3),

$$u(t) = \mathrm{e}^{-itP}u_0 - i\int_0^t \mathrm{e}^{-i(t-t')P}(F(u(t')))\,dt'. \tag{3.1}$$

Proposition 3.1. *If $s > d/2$, the Cauchy problem for* (1.3) *is regularly well-posed on $H^s(M)$.*

The above proposition has the following partial converse.

Proposition 3.2. *Assume $F(0) = 0$ and $D^k F(0) \neq 0$ for some $k \geq 2$ and that the Cauchy problem for* (1.3) *is regularly well-posed on $H^s(M)$. Then $s \geq \frac{d}{2} - \frac{2}{k-1}$.*

Corollary. *If $F(0) = 0$ and F is real analytic and is not polynomial, then the Cauchy problem for* (1.3) *is not regularly well-posed on $H^s(M)$ for $s < \frac{d}{2}$.*

Proposition 3.2 relies on a very simple idea: if we solve (1.3) with the following bounded data in H^s,

$$u(0,x) = f_N(x) = N^{d/2-s}\varphi(Nx),$$

where φ is a suitable cutoff function – so that the above expression makes sense on a manifold, choosing local coordinates –, and N is a large parameter, then, because P is a differential operator of order 2, on times t such $|t| \ll N^{-2}$, the term Pu can be neglected at the first order. For instance, for such times,

$$\mathrm{e}^{-itP} f_N - f_N \to 0$$

in H^s as N tends to infinity. Then one uses this remark to compute successively

$$v_j(t) = \frac{d^j}{d\delta^j}(u^\delta(t))_{|\delta=0}, \quad j \geq 1,\ |t| \ll N^{-2},$$

where u^δ denotes the solution to (1.3) such that $u^\delta(0) = \delta f_N$. The continuity of the differential of order k of the map $u_0 \in H^s \mapsto u(t) \in H^s$ implies

$$\|v_k(t)\|_{H^s} \leq C\|f_N\|_{H^s}^k$$

which, for $t = N^{-2-\varepsilon}$, yields the claimed condition on s.

Combined with more delicate estimates, this idea can also be used to disprove other kinds of wellposedness, as in the following adaptation to inhomogeneous media of a recent result by Christ, Colliander and Tao.

Theorem 3.3 (Christ–Colliander–Tao [27], Burq–Gérard–Tzvetkov [17]). *If $\alpha > 0$, the Cauchy problem for* (1.3) *with $F = F_{\alpha,\pm}$ given by* (1.10) *is not uniformly well-posed on $H^s(M)$ if $s < 0$, and not well-posed on $H^s(M)$ if $0 < s < \frac{d}{2} - \frac{2}{\alpha}$.*

The main point of the proof is to establish that, for sufficiently small times – but not too small – , the solution of the above equation with data $\kappa_N f_N$, where κ_N is a small coefficient to be adjusted, is approximated by the solution v_N of the ordinary differential equation

$$i\partial_t v_N = \pm(1 + |v_N|^2)^{\alpha/2} v_N$$

with the same Cauchy data. Of course v_N can be computed explicitly,

$$v_N(t,x) = \mathrm{e}^{\mp it(1+\kappa_N^2|f_N(x)|^2)^{\alpha/2}} \kappa_N f_N(x),$$

and one checks that the above oscillating term induces instability in the first case, while in second case it produces norm inflation, namely the H^s norm of the solution can become unbounded for a sequence of times tending to 0, though the Cauchy data tend to 0 in H^s.

4. The role of Strichartz inequalities

In this section we first recall the basic role played by Strichartz inequalities in the analysis of equation (1.3) on Euclidean spaces, quoting some important results in this context, without pretending to be exhaustive. Then we discuss extensions of these inequalities to different geometries.

4.1. The Euclidean case. In this subsection we assume that $M = \mathbb{R}^d$ and that $-P$ is the Laplace operator. In this case, the solution of the linear Schrödinger equation is explicit,

$$\mathrm{e}^{it\Delta} u_0(x) = \frac{1}{(4i\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{i|x-y|^2/4t} u_0(y)\, dy, \tag{4.1}$$

and this implies the following dispersion estimate:

$$\|\mathrm{e}^{it\Delta} u_0\|_{L^\infty(\mathbb{R}^d)} \le \frac{1}{(4\pi|t|)^{d/2}} \|u_0\|_{L^1(\mathbb{R}^d)}. \tag{4.2}$$

By a classical functional-analytic tool (known as the TT^* trick), this estimate implies important inequalities for the solution of the linear Schrödinger equation with L^2 data. In order to state these inequalities we shall say that a pair $(p,q) \in [1,\infty] \times [1,\infty]$ is d-admissible if

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}, \quad p \ge 2,\ (p,q) \ne (2,\infty). \tag{4.3}$$

Moreover, if $r \in [1, \infty]$, we denote by $\bar{r}$ the conjugate exponent of r, characterized by

$$\frac{1}{\bar{r}} + \frac{1}{r} = 1.$$

Proposition 4.1 (Strichartz [54], Ginibre–Velo [34], Yajima [61], Keel–Tao [42]). *Let (p_1, q_1), (p_2, q_2) be d-admissible pairs. There exists $C > 0$ such that, for every $u_0 \in L^2(\mathbb{R}^d)$, for every $T > 0$ and $f \in L^{\bar{p}_1}([0, T], L^{\bar{q}_1}(\mathbb{R}^d))$, the solution u of*

$$i\partial_t u + \Delta u = f, \quad u(0) = u_0,$$

satisfies $u \in L^{p_2}([0, T], L^{q_2}(\mathbb{R}^d))$ with the inequality

$$\|u\|_{L^{p_2}([0,T],L^{q_2}(\mathbb{R}^d))} \leq C\,(\|u_0\|_{L^2(\mathbb{R}^d)} + \|f\|_{L^{\bar{p}_1}([0,T],L^{\bar{q}_1}(\mathbb{R}^d))}). \tag{4.4}$$

These inequalities were first proved in [54] in the particular cases $p_1 = q_1$ and $p_2 = q_2$, then in [34] in the cases $p_1 = p_2 > 2$, then in [61] for $p_1, p_2 > 2$ arbitrary, and finally, for the endpoint case $p = 2$, in [42], where an abstract presentation of the TT^* trick is also available. Notice that these estimates are optimal: indeed, the first condition in (4.3) is related to the scale invariance of the Schrödinger equation, the second condition comes from general properties of translation invariant operators on L^p (see *e.g.* Theorem 1.1 of Hörmander [39]), while the special forbidden case $p = 2, q = \infty$, which only arises for $d = 2$, has been checked by Montgomery-Smith [46]. A variant of the proof of the necessity of the scaling condition consists in testing the above estimates for $f = 0$ and

$$u_0(x) = \varphi(Nx),$$

where $\varphi \in C_0^\infty(\mathbb{R}^d)$ and N is a large parameter. As we already observed in the previous section, for $t \ll N^{-2}$ the solution $u(t, x)$ stays essentially constant. This is made more precise by the ansatz

$$u(t, x) \sim \psi(N^2 t, Nx)$$

where $\psi(s, .)$ belongs to the Schwartz class, uniformly as s stays bounded, and $\psi(0, x) = \varphi(x)$. Consequently, if (p, q) is an admissible pair, then the $L_t^p(L_x^q)$ norm of u on the thin slab $[0, N^{-2}] \times \mathbb{R}^d$ is equivalent to $N^{-2/p-d/q} = N^{-d/2}$, which is the magnitude of the L^2 norm of u_0. In other words, the main contribution in the $L_t^p(L_x^q)$ norm of u already lies in the thin slab $[0, N^{-2}] \times \mathbb{R}^d$: this is a striking illustration of the dispersive character of the Schrödinger equation. Moreover, this remark can be carried out to inhomogeneous media, showing that *the above Strichartz inequalities cannot be improved in any inhomogeneous media.* On the other hand, as we shall see in the sequel, they may be dramatically altered.

We close this subsection by pointing that Strichartz inequalities on the Euclidean space essentially provide reverse statements of the necessary conditions to wellposedness of Proposition 3.2 and Theorem 3.3.

Theorem 4.2 (Ginibre–Velo [33], [34], Kato [41], Cazenave–Weissler [24], Tsutsumi [58], Yajima [61]). *Let F satisfy conditions* (1.4) *and let $s \geq 0$ satisfy*

$$s > \frac{d}{2} - \frac{2}{\alpha}.$$

Then equation (1.3) *is uniformly well-posed on $H^s(\mathbb{R}^d)$ if $d \leq 6$. Moreover, if F is a polynomial of degree $1+\alpha$, then* (1.3) *is regularly well-posed on $H^s(\mathbb{R}^d)$ for every d.*

The unexpected condition $d \leq 6$ relies on lack of good estimates for large derivatives of $F(u)$ if F is non-polynomial. This limitation may not be optimal, but so far it cannot be dropped, for instance for the proof of propagation of high regularity.

In the polynomial case, such a condition is not necessary, and we observe that, under the additional conditions (1.5) and (1.8), thresholds of the three wellposedness properties coincide, for every d, with $\max(0, d/2 - 2/\alpha)$. Furthermore, let us mention that some regular wellposedness results were proved on H^s for $s < 0$, in the one-dimensional case, for quadratic nonlinearities which do not satisfy the gauge invariance condition (see Kenig–Ponce–Vega [43]).

The critical cases $s_c = d/2 - 2/\alpha \geq 0$ are not covered by the above theorem. Using the same Strichartz inequalities, it is possible to extend the wellposedness results of this theorem on a global time interval *for data which lie in a small neighborhood of* 0 *in H^s* (see [24]). For large data however, this question is still the object of intensive work, particularly for Hamiltonian gauge-invariant nonlinearities and $s_c = 0$ or 1. For instance, if $d = 2$ and $F(u) = -|u|^2 u$, $s_c = 0$ and the existence of blow up solutions (see Zakharov [62]) together with a scaling argument yields a family of solutions with bounded data in $L^2(\mathbb{R}^2)$ and which blow up at arbitrarily small times. On the other hand, if $F(u) = |u|^2 u$, the question of global existence of a solution with L^2 solution and the related question of (regular) wellposedness are widely open problems. Another important example of such critical problems is $d = 3$, $F(u) = |u|^4 u$, for which $s_c = 1$. In this case, Colliander–Keel–Staffilani–Takaoka–Tao [29] have recently proved global (regular) wellposedness.

4.2. Operators on the real line. The first kind of nonhomogeneous medium to which it is natural to generalize the above Strichartz inequalities – and hence the nonlinear results of Theorem 4.2 – is of course the real line, with an operator P satisfying the assumptions (1.1), (1.2) of the introduction. In this case, the extension is rather straightforward for *local in time* Strichartz estimates. Indeed, if

$$P = -\frac{1}{\rho}\frac{d}{dx} a \frac{d}{dx},$$

the global change of variable $dy = (\rho(x)/a(x))^{1/2}\, dx$ and the conjugation $\lambda^{-1} P \lambda$ with the function $\lambda(y) = (a(x)\rho(x))^{-1/4}$ lead to the operator

$$\tilde{P} = -\frac{d^2}{dy^2} + V(y), \quad V = -\frac{\lambda''}{\lambda} + 2\left(\frac{\lambda'}{\lambda}\right)^2.$$

Since V is a bounded function, the Strichartz inequalities for $\tilde{P}$ on a *finite time* interval $[0, T]$ with $C = C(T)$ are a straightforward consequence of the Euclidean ones, considering the term Vu as a source term in the right hand side. We refer to [49] for a slightly different proof.

Though we shall not pursue in this direction, let us indicate that the question of singular coefficients ρ, a has been addressed quite recently. In [2], Banica observed that, if $\rho = 1$ and a is a piecewise constant function with a finite number of discontinuities, the dispersion estimate – and hence Strichartz inequalities – is valid, while it fails for an infinite number of discontinuities. Burq and Planchon [21] generalized this observation by proving that *global* Strichartz estimates hold as soon as a has bounded variation. Notice that the BV regularity seems to be a relevant threshold, since, for every $s < 1$, $W^{s,1}$ functions a are constructed in the appendix of [21] such that every kind of Strichartz inequality fails (see also an earlier result by Castro and Zuazua [22], which shows the same phenomenon for Hölder continuous functions a of any exponent $\alpha < 1$).

4.3. Strichartz inequalities with loss of derivative. At this stage a natural question is of course to extend Strichartz inequalities (4.4) to variable coefficients in several space dimensions, as we did on the line. The following example shows that the situation is much more complicated. Our starting point is the following identity for the ground state of the harmonic oscillator,

$$(-h^2\partial_s^2 + s^2 - h)\big(\mathrm{e}^{-s^2/2h}\big) = 0.$$

Setting $s = r - 1$ and, for every positive integer n,

$$\psi_n(r, \theta) = \mathrm{e}^{-(r-1)^2/2h_n + in\theta}$$

where $h_n > 0$ is such that $h_n^{-2} = h_n^{-1} + n^2$, we infer

$$(-\partial_r^2 - \partial_\theta^2 - h_n^{-2}(1 - (r-1)^2))\psi_n = 0. \tag{4.5}$$

Notice that ψ_n is the expression in polar coordinates of a smooth function on the complement of the origin in the plane, and that the operator

$$P_0 = -\frac{1}{1 - (r-1)^2}(\partial_r^2 + \partial_\theta^2)$$

is a positive elliptic operator of order 2 with smooth real coefficients, on the complement Ω of the origin in the disc of radius 2 endowed with the density $d\mu = (1 - (r-1)^2)\, dr\, d\theta = (2 - |x|)dx$. Since $P_0(1) = 0$, it follows that P_0 can be written $P_0 = -\frac{1}{\rho_0}\nabla.(A_0\nabla)$, where ρ_0, A_0 are smooth functions on Ω valued in positive numbers and definite positive matrices respectively. Notice that (4.5) reads $P_0\psi_n = h_n^{-2}\psi_n$, and that, as n tends to ∞, ψ_n is exponentially concentrating on the circle of radius 1. By cutting off ψ_n near this circle, we obtain a sequence of functions $\tilde{\psi}_n$ and a differential operator P on $\mathbb{R}^2$ satisfying the assumptions of the introduction, such that

$$P\tilde{\psi}_n = h_n^{-2}\tilde{\psi}_n + r_n \tag{4.6}$$

where, for every $q \geq 1$ and for every $s \geq 0$,

$$\|\tilde{\psi}_n\|_{L^q} \sim n^{-1/2q}, \quad \|\tilde{\psi}_n\|_{H^s} \sim n^{s-1/4}, \quad \|r_n\|_{H^s} \leq C_s \mathrm{e}^{-\delta n}$$

for some $\delta > 0$. This sequence of functions is called a quasimode for the operator P. The geometric interpretation is that the circle of radius 1 is a (sufficiently stable) geodesic curve for the Riemannian metric defined by the principal symbol of P (see e.g. Ralston [47]). By adding a suitable remainder term $w_n(t)$, we can write

$$u_n(t) := \mathrm{e}^{-itP}\tilde{\psi}_n = \mathrm{e}^{-ith_n^{-2}}\tilde{\psi}_n + w_n(t)$$

with $\|u_n\|_{L^p([0,T],L^q(\mathbb{R}^2))} \sim T^{1/p} n^{-1/2q}$ for every $p, q \geq 1$ and every $T > 0$. By using u_n as a test function we conclude that, for every $q > 2$, for every $p \geq 1$, the estimate

$$\|u\|_{L^p([0,T],L^q)} \leq C\|u(0)\|_{L^2}$$

fails. More precisely, in view of the behavior of u_n it is even impossible to replace the L^2 norm in the right hand side by the H^s norm if

$$s < \frac{1}{4} - \frac{1}{2q}. \tag{4.7}$$

In other words, *in multidimensional heterogeneous media, losses of derivatives in Strichartz inequalities cannot be avoided.* The next question is of course to estimate precisely this loss of derivatives in terms of p, q and the geometry of the medium. There is no need to say that this is a very difficult open problem. However, it is possible to give a general upper bound, which is valid for every geometry and already gives interesting applications to nonlinear problems.

Theorem 4.3 (Staffilani–Tataru [53], Burq–Gérard–Tzvetkov [12]). *If (p, q) is a d-admissible pair, the solution u of the equation*

$$i\partial_t u - Pu = f, \quad u(0) = u_0, \tag{4.8}$$

satisfies the inequality

$$\|u\|_{L^p([0,T],L^q(M))} \leq C_T \left(\|u_0\|_{H^{1/p}(M)} + \|f\|_{L^1([0,T],H^{1/p}(M))}\right). \tag{4.9}$$

Proof (*sketch*). Firstly, by Duhamel's formula and Minkowski's inequality, (4.9) is reduced to the case $f = 0$. Then, by Littlewood–Paley's analysis, we can assume that u_0 is spectrally supported in a dyadic interval, namely $\varphi(N^{-2}P)u_0 = u_0$ for some $\varphi \in C_0^\infty(\mathbb{R})$, where N is a large dyadic integer. The advantage of this spectral localization is that we can describe rather explicitly, by a standard semiclassical WKB analysis, the solution

$$u(t) = \mathrm{e}^{-itP}u_0$$

on a time interval of order $N^{-1} = h$. By a stationary phase argument, this implies the following dispersion estimate:

$$\|u(t)\|_{L^\infty(M)} \le \frac{C}{|t|^{d/2}}\|u_0\|_{L^1(M)}, \quad |t| \lesssim \frac{1}{N}. \tag{4.10}$$

From this dispersion estimate, the TT^* trick leads to the following semi-classical Strichartz inequalities

$$\|u\|_{L^p([0,N^{-1}],L^q(M))} \le C\|u_0\|_{L^2(M)}, \quad u_0 = \varphi(N^{-2}P)u_0, \tag{4.11}$$

where (p, q) stands for any d-admissible pair. The last step of the proof consists in iterating the estimates (4.11) on N intervals of length N^{-1} covering the interval [0, 1]. This yields a factor $N^{1/p}$ in the right hand side, and this completes he proof since

$$N^{1/p}\|u_0\|_{L^2(M)} \simeq \|u_0\|_{H^{1/p}(M)}. \qquad \square$$

Remark 4.4. Notice that when $d = 2$, the loss $\frac{1}{p} = \frac{1}{2} - \frac{1}{q}$ is twice as big as the threshold (4.7) derived from our counterexample in the beginning of this subsection. Indeed, the last step of the above proof may seem quite rough, since the decomposition of [0, 1] into N intervals of length N^{-1} does not take into account the geometric features of M and P. However, it is interesting to notice that there are geometries where some inequalities (4.9) are optimal. Indeed, if M is compact and $d \ge 3$, inequality (4.9) with $p = 2$ applied to $f = 0$ and to the special Cauchy data $u_0 = \psi_\lambda$, where ψ_λ is an eigenfunction of P associated to a large eigenvalue λ^2, provides the estimate

$$\|\psi_\lambda\|_{L^q(M)} \le C\,\lambda^{1/2}\|\psi_\lambda\|_{L^2(M)}, \quad q = \frac{2d}{d-2}. \tag{4.12}$$

Estimate (4.12) is one of the estimates obtained by Sogge [51] for the L^r norms of the eigenfunctions of elliptic operators on compact manifolds, and it is known that this estimate is optimal if M is the sphere $\mathbb{S}^d$ and P is the standard Laplace operator, for spherical harmonics ψ_λ which are functions of the distance to a fixed point (see [50]). A similar phenomenon occurs for $d = 2$ with $q = \infty$, except that our inequalities need an extra ε-derivative, due to the forbidden case $p = 2$, $q = \infty$.

Using inequalities (4.9), we obtain wellposedness results for nonlinear Schrödinger equations (1.3). For simplicity, we only state the case of polynomial nonlinearities.

Corollary 4.5 (Burq–Gérard–Tzvetkov[12]). *Assume $d \ge 2$ and suppose that F is a polynomial in u, $\bar{u}$ of degree $1 + \alpha \ge 2$. Then the Cauchy problem for* (1.3) *is regularly well-posed on $H^s(M)$ for*

$$s > \frac{d}{2} - \frac{1}{\max(\alpha, 2)}.$$

Moreover, if $d = 3$ and $F(u) = |u|^2 u$, then the Cauchy problem for (1.3) *is globally well-posed on $H^1(M)$, and it is globally regularly well-posed on $H^s(M)$ if $s > 1$.*

If F is both gauge invariant and Hamiltonian with a nonnegative potential V, we can combine Corollary 4.5 with the conservation laws (1.9) and (1.7) to deduce global regular wellposedness on $H^1(M)$ if $d = 2$. In the special case of a cubic NLS in three space dimensions, observe that regular wellposedness is only known for $s > 1$; the uniform or regular wellposedness is still open in general on the energy space $H^1(M)$. This is in strong contrast with the case of the Euclidean case, where the critical nonlinearity is quintic. However, in Section 6 we shall improve Corollary 4.5 for several specific three-dimensional geometries.

We conclude this subsection by quoting a recent result concerning boundary value problems. In this case the WKB analysis is much more problematic, due to glancing rays. However it is possible to reduce the analysis, by a reflection argument, to the case of a boundaryless manifold endowed with a Lipschitz continuous Riemannian metric. Combining the method of proof of Theorem 4.3 with earlier smoothing ideas due to Bahouri and Chemin [3] (see also Tataru [57]) in the context of nonlinear wave equations, it is possible to obtain the following result, which, in the particular case of a plane domain, provides the first global wellposedness result for super-cubic nonlinearities (for the cubic case, earlier wellposedness results were due to Brezis–Gallouet [11] and Vladimirov [60], by different arguments).

Theorem 4.6 (Anton [1]). *Assume that M is a compact manifold and that P is given by* (1.1) *where ρ and A are Lipschitz continuous. Then, for every d-admissible pair (p, q), the solution of* (4.8) *satisfies*

$$\|u\|_{L^p([0,T],L^q(M))} \leq C_{s,T}\left(\|u_0\|_{H^s(M)} + \|f\|_{L^1([0,T],H^s(M))}\right), \quad s > \frac{3}{2p}. \tag{4.13}$$

In particular, the estimate (4.13) *still holds if M is replaced by a smooth bounded open set in $\mathbb{R}^d$ if $-P$ is the Laplace operator Δ_D (resp. Δ_N) with Dirichlet (resp. Neumann) boundary conditions and if the space $H^s(M)$ is replaced by the domain of the power $s/2$ of P. Consequently, if $d = 2$ and F satisfies* (1.4)*, is gauge invariant and Hamiltonian with a nonnegative potential V, the equation* (1.3) *with Dirichlet (resp. Neumann) boundary condition has a unique global solution $u \in C(\mathbb{R}, H_0^1(\Omega))$ (resp. $u \in C(\mathbb{R}, H^1(\Omega))$) if $u_0 \in H_0^1(\Omega)$ (resp. $u_0 \in H^1(\Omega)$), and the map $u_0 \mapsto u$ is Lipschitz continuous.*

4.4. Non-trapping metrics. Though we are rather interested in new phenomena induced by the heterogeneity of the medium, we cannot conclude this section devoted to Strichartz inequalities without quoting a series of results giving sufficient conditions on the geometry of the operator P on $M = \mathbb{R}^d$ in order that Euclidean inequalities (4.4) hold. All these conditions concern the Laplace operator with a non-trapping metric, namely a Riemannian metric on $\mathbb{R}^d$ such that no geodesic curve stays in a compact set during an arbitrarily long time: notice that this prevents counterexamples like the one in the previous subsection. First of all, Staffilani–Tataru [53] proved (4.4) on finite time intervals if the non-trapping metric is the perturbation of the Euclidean metric by a C^2 compactly supported function. Then Robbiano–Zuily [48] generalized this result to short range perturbations of the Euclidean metric by a very precise parametrix construction. Similar results were obtained by Hassell–Tao–Wunsch on asymptotically conic manifolds, using different methods. Finally, Bouclet–Tzvetkov [5] recently tackled the case of long range perturbations of the Euclidean metric. Notice that the proofs in [53] and [5] rely on the local smoothing effect for non-trapping metrics (see Doi [30]) which appears to be the complementary property of estimates (4.9) to obtain Strichartz inequalitites (4.4) without loss on finite time intervals. Finally, we refer to [15] for applications of this smoothing effect for non-trapping exterior domains to boundary problems for nonlinear Schrödinger equations.

5. Multilinear Strichartz estimates

In Subsection 4.3, we observed that, in several space dimensions, the geometry of the medium may induce losses of derivatives in Strichartz inequalities, and that some of these losses are optimal in specific geometries such as the sphere. The wellposedness results deduced from these Strichartz inequalities with loss are altered with respect to the Euclidean case – compare Theorem 4.2 and Corollary 4.5. However, so far we did not give evidence of this alteration. Indeed, as we shall see in the sequel, the whole range of Strichartz inequalities is not necessary to give optimal wellposedness results for NLSs. In order to understand this, we begin with revisiting the question of regular wellposedness for the cubic NLS.

5.1. A criterion for regular wellposedness of the cubic NLS. Given a dyadic integer N, let us say that a function u on M is *spectrally localized at frequency N* if

$$\mathbf{1}_{[N,2N]}(\sqrt{1+P})(u) = u. \tag{5.1}$$

We start with the important notion of bilinear Strichartz estimate, which originates in the works of Bourgain [6] and of Klainerman–Machedon [45] in the context of null forms for the wave equation.

Definition 5.1. Let $s \geq 0$. We shall say that *the Schrödinger group for P satisfies a bilinear Strichartz estimate of order s on M* if there exists a constant C such that, for all dyadic integers N, L, for all functions u_0, v_0 on M spectrally localized at frequencies N, L respectively, the functions

$$u(t) = \mathrm{e}^{-itP}(u_0), \quad v(t) = \mathrm{e}^{-itP}(v_0)$$

satisfy the inequality

$$\|uv\|_{L^2([0,1]\times M)} \leq C \min(N, L)^s \|u_0\|_{L^2(M)} \|v_0\|_{L^2(M)}. \tag{5.2}$$

Notice that by setting $v_0 = u_0$, $L = N$ and by using the Littlewood–Paley inequality, one easily shows that a bilinear Strichartz estimate of order s implies a Strichartz-type estimate of the space-time L^4 norm of a solution to the linear Schrödinger equation in terms of the $H^{s/2}$ norm of the Cauchy data. However, if $s > 0$, a bilinear Strichartz estimate says more, since the price to pay for estimating the L^2 norm of a product of such solutions only involves the lowest frequency of these solutions. The importance of bilinear Strichartz estimates in the wellposedness theory for NLSs clearly appears in the following theorem, which is a slight reformulation of some results from [16].

Theorem 5.2 (Burq–Gérard–Tzvetkov [16]). *Assume $F(u) = \pm|u|^2 u$ and $s \geq 0$.*

i) *If the Cauchy problem for* (1.3) *is regularly well-posed on $H^s(M)$, then the Schrödinger group for P satisfies a bilinear Strichartz estimate of order s on M.*

ii) *If the Schrödinger group for P satisfies a bilinear Strichartz estimate of order s on M, then, for every $\sigma > s$, the Cauchy problem for* (1.3) *is regularly well-posed on $H^\sigma(M)$.*

In other words, the existence of a bilinear Strichartz estimate of order s is *almost* a criterion for regular wellposedness for cubic NLSs on $H^s(M)$. Notice that the strict inequality $\sigma > s$ in ii) cannot be extended to an equality in general. Indeed, for the Euclidean Laplace operator on $\mathbb{R}^2$, the Strichartz inequality (4.4) for the admissible pair (4, 4) combined with the Hölder inequality implies a bilinear estimate of order 0; however, we already observed at the end of Subsection 4.1 that the focusing cubic equation with $F(u) = -|u|^2 u$ is not well-posed on $L^2(\mathbb{R}^2)$.

Another comment on the above statement is that it provides counterexamples to regular wellposedness. Indeed, in the beginning of Subsection 4.3 we constructed an example of an operator P on $\mathbb{R}^2$ such that the estimate

$$\|u\|_{L^4([0,1]\times\mathbb{R}^2)} \leq C\|u_0\|_{H^s(\mathbb{R}^2)}$$

fails if $s < 1/8$. Consequently, the Schrödinger group for this operator does not enjoy bilinear Strichartz estimates of order $s < 1/4$, and thus *the cubic NLS for this operator is not regularly well-posed on $H^s(\mathbb{R}^2)$ for $s < 1/4$*. This is in strong contrast with the Euclidean case, where we know that regular wellposedness holds for every $s > 0$ (see Theorem 4.2).

Proof (*sketch*). The proofs of parts i) and ii) of Theorem 5.2 are of unequal length and difficulty. As a matter of fact, using regular wellposedness of (1.3) for smooth data stated in Proposition 3.1 and the propagation of regularity contained in the definition of regular wellposedness, it is easy to check that the third differential at 0 of the map $\Phi_1 : u_0 \mapsto u(1)$ is given by the following polarized form of the first iteration of the Duhamel equation (3.1):

$$D^3\Phi_1(0)(u_0, u_0, v_0) = -2i \int_0^1 \mathrm{e}^{-i(1-t)P}(2|u(t)|^2 v(t) + u(t)^2\overline{v(t)})\, dt.$$

Here we assumed $N \le L$ without loss of generality. We now compute the scalar product of both members of the above identity with $\mathrm{e}^{-iP}v_0$, and we use the assumed continuity of the trilinear map $D^3\Phi_1(0)$ from $(H^s)^3$ to H^s. This yields

$$\|uv\|^2_{L^2([0,1]\times M)} \le C\|u_0\|^2_{H^s}\,\|v_0\|_{H^s}\,\|v_0\|_{H^{-s}}.$$

Using that

$$\|f\|_{H^{\pm s}} \simeq N^{\pm s}\|f\|_{L^2}$$

if f is spectrally supported at frequency N, we infer the bilinear Strichartz estimate (5.2), and hence part i) is proved.

Let us come to part ii). The main idea, which in this context is due to Bourgain [6], is to introduce the scale of Hilbert spaces

$$X^{s,b}(P,\mathbb{R}\times M) = \{v \in \mathcal{S}'(\mathbb{R}\times M) : (1+|i\partial_t - P|^2)^{b/2}\,(1+P)^{s/2}v \in L^2(\mathbb{R}\times M)\}$$

for $s, b \in \mathbb{R}$. We refer to [35] for a pedagogical introduction to this strategy. Denoting by $X^{s,b}_T(P)$ the space of restrictions of elements of $X^{s,b}(P,\mathbb{R}\times M)$ to $]-T,T[\times M$, it is easy to observe that

$$X^{s,b}_T(P) \subset C([-T,T],H^s(M)) \quad \text{for all } b > \tfrac{1}{2},$$

and that the solution of the linear Schrödinger equation with datum in H^s lies in $X^{s,b}_T(P)$ for every b. Moreover, the Duhamel term in the integral equation (3.1) can be handled by means of these spaces as

$$\left\|\int_0^t \mathrm{e}^{-i(t-t')P} f(t')\,dt'\right\|_{X^{s,b}_T(P)} \le C\,T^{1-b-b'}\|f\|_{X^{s,-b'}_T(P)}$$

if $0 < T \le 1$, $0 < b' < \frac{1}{2} < b$, $b + b' < 1$. The crux of the proof is then to observe that a bilinear Strichartz estimate of order s implies the following estimates, for $\sigma' \ge \sigma > s$ and suitable b, b' as above,

$$\|v_1\bar{v}_2v_3\|_{X^{\sigma,-b'}(P)} \le C\,\|v_1\|_{X^{\sigma,b}(P)}\,\|v_2\|_{X^{\sigma,b}(P)}\|v_3\|_{X^{\sigma,b}(P)},$$
$$\||v|^2v\|_{X^{\sigma',-b'}(P)} \le C\,\|v\|^2_{X^{\sigma,b}(P)}\,\|v\|_{X^{\sigma',b}(P)},$$

which allow the use a fixed point argument in $X^{\sigma,b}_T(P)$ in the resolution of the integral equation (3.1). □

Remark 5.3. Combining the Strichartz inequalities of Theorem 4.3 with the Sobolev inequalities, one easily shows that, if $d \ge 2$ and if u_0 spectrally localized at frequency N, the solution of the linear Schrödinger equation satisfies

$$\|u\|_{L^2([0,1],L^\infty(M))} \le C_sN^s\|u_0\|_{L^2(M)} \quad \text{for all } s > \tfrac{d-1}{2}.$$

Using Hölder's inequality and the conservation of the L^2 norm by the Schrödinger group, we infer, in the context of Definition 5.1,

$$\begin{aligned}\|uv\|_{L^2([0,1]\times M)} &\le \|u\|_{L^2([0,1],L^\infty(M))}\,\|v\|_{L^\infty([0,1],L^2(M))}\\ &\le C_s N^s\,\|u_0\|_{L^2(M)}\|v_0\|_{L^2(M)},\end{aligned}$$

namely a bilinear Strichartz estimate of order $s > (d-1)/2$. Applying Theorem 5.2, we conclude that a cubic NLS is regularly well-posed on $H^s(M)$ for every $s > (d-1)/2$ if $d \ge 2$, which is consistent with Corollary 4.5. However, we shall see in the next section that this bilinear Strichartz estimate is far from optimal in several specific cases, therefore the threshold of regular wellposedness will be improved through Theorem 5.2.

5.2. Generalization to subcubic nonlinearities. Bilinear Strichartz estimates can also be used to prove uniform wellposedness for (1.3) when F is not polynomial. In particular, combining the method of proof of part ii) in Theorem 5.2 with paradifferential expansions, it is possible to prove the following result.

Theorem 5.4. *Assume that the Schrödinger group for P satisfies a bilinear Strichartz estimate of order s on M and that F satisfies* (1.4) *with $\alpha \le 2$. Then the Cauchy problem for* (1.3) *is uniformly well-posed on $H^\sigma(M)$ for every $\sigma > s$.*

Compared to Theorem 4.2 it may seem surprising that the regularity threshold of uniform wellposedness for (1.3) does not depend on α. However we shall see such an example in Section 6, in the case of the two-dimensional sphere.

5.3. Higher order nonlinearities. By mimicking Definition 5.1 it is easy to define the notion of k-linear estimate for $k \ge 3$.

Definition 5.5. Let k be an integer ≥ 3 and $s_1, \dots, s_{k-1} \ge 0$. We shall say that *the Schrödinger group for P satisfies a k-linear Strichartz estimate of order* $(s_1, \dots, s_{k-1})$ *on M* if there exists a constant C such that, for all dyadic integers $N_1 \le \cdots \le N_k$, for all functions $u_{1,0}, \dots, u_{k,0}$ on M spectrally localized at frequencies $N_1, \dots, N_k$ respectively, the functions

$$u_j(t) = \mathrm{e}^{-itP}(u_{j,0}), \quad j = 1, \dots, k,$$

satisfy the inequality

$$\|u_1 \dots u_k\|_{L^2([0,1]\times M)} \le C\, N_1^{s_1} \dots N_{k-1}^{s_{k-1}} \|u_{1,0}\|_{L^2(M)} \dots |u_{k,0}\|_{L^2(M)}. \tag{5.3}$$

Remark 5.6. By an iterated use of Hölder's inequality, we can always assume $s_1 \ge \cdots \ge s_{k-1}$.

Next we have the equivalent of Theorem 5.2.

Theorem 5.7. *Let $s \geq 0$, m be an integer ≥ 2 and $F(u) = \pm|u|^{2m}u$. Then*

i) *If* (1.3) *is regularly well-posed on $H^s(M)$, the Schrödinger group for P satisfies an $(m+1)$-linear Strichartz estimate of order $(s,\dots,s)$.*

ii) *If the Schrödinger group for P satisfies an $(m+1)$-linear Strichartz estimate of order $(s,\dots,s)$, then* (1.3) *is regularly well-posed on $H^\sigma(M)$ for every $\sigma > s$.*

Moreover, the use of different exponents in the list $s_1,\dots,s_k$ can help to tackle non-polynomial nonlinearities. Let us give an example for nonlinearities which are intermediate between cubic and quintic, which is essentially borrowed from [17].

Theorem 5.8. *Assume the Schrödinger group for P satisfies a trilinear Strichartz estimate of order (s_1, s_2) with $s_1 > s_2 \geq 0$ and M is compact. Let F satisfy* (1.4) *with $2 < \alpha < 4$. Then* (1.3) *is uniformly well-posed on $H^s(M)$ for every*

$$s > \left(1 - \frac{2}{\alpha}\right)s_1 + \frac{2}{\alpha}s_2.$$

5.4. Multilinear estimates for spectral projectors. If M is compact, a k-linear Strichartz estimate for the Schrödinger implies a k-linear estimate for eigenfunctions of P of the following kind:

$$\|\varphi_1 \dots \varphi_k\|_{L^2(M)} \leq C\,\lambda_1^{s_1} \dots \lambda_{k-1}^{s_{k-1}} \|\varphi_1\|_{L^2(M)} \dots \|\varphi_k\|_{L^2(M)},$$

φ_j an eigenfunction of P associated to the eigenvalue λ_j^2 and $1 \leq \lambda_1 \leq \dots \leq \lambda_k$. These estimates can be seen as k-linear versions of Sogge's estimates [51], [52]. A first step is therefore to decide for which orders such k-linear estimates hold. The following result gives a fairly general answer to this question. For the sake of generality, we deal, as in [51], [52], with spectral projectors $\Pi_\lambda = \mathbf{1}_{\lambda \leq \sqrt{P} \leq \lambda+1}$ on clusters of bounded length for the square root of P. Notice that this is a much more stringent spectral localization than the dyadic one which we introduced in the beginning of this section. Under this form our result makes sense in the case $M = \mathbb{R}^d$ too.

Theorem 5.9 (Burq–Gérard–Tzvetkov [17]). *We have the bilinear estimates*

$$\|\Pi_{\lambda_1} f_1\, \Pi_{\lambda_2} f_2\|_{L^2(M)} \leq C\,\|f_1\|_{L^2(M)} \|f_2\|_{L^2(M)} \begin{cases} \lambda_1^{1/4} & \text{if } d = 2, \\ (\lambda_1\ \log(\lambda_1))^{1/2} & \text{if } d = 3, \\ \lambda_1^{(d-2)/2} & \text{if } d \geq 4 \end{cases}$$

if $2 \leq \lambda_1 \leq \lambda_2$. In the special case $d = 2$, we have the trilinear estimate

$$\|\Pi_{\lambda_1} f_1\, \Pi_{\lambda_2} f_2\, \Pi_{\lambda_3} f_3\|_{L^2(M)} \leq C(\lambda_1\lambda_2)^{1/4} \|f_1\|_{L^2(M)}\, \|f_2\|_{L^2(M)}\, \|f_3\|_{L^2(M)}$$

if $1 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3$.

Remark 5.10. The logarithmic factor in the right hand side of the bilinear estimate in three space dimensions may be just technical. Apart from this, the linear estimates deduced from the above bilinear and trilinear estimates by making all the frequencies λ_j equal and all the functions f_j equal, are exactly the L^4 and L^6 estimates among the L^p estimates proved by Sogge in [51], which are known to be optimal. Moreover, all the exponents in the multilinear estimates of Theorem 5.9 are optimal in the particular case of the sphere.

Finally, we did not state the other multilinear estimates for those spectral projectors, since they are essentially straightforward consequences of the above ones and of the L^∞ estimate due to Sogge,

$$\|\Pi_\lambda f\|_{L^\infty(M)} \le C\lambda^{\frac{d-1}{2}}\|f\|_{L^2(M)}, \quad \lambda \ge 1. \tag{5.4}$$

Proof (*sketch*). There are several proofs of this result (see [16] for the bilinear estimate in the two-dimensional case, and [17] or [18] for the general case). Here we follow [18]. We set $u = \Pi_\lambda f$, $h = \lambda^{-1}$, and we observe that the spectral localization given by Π_λ can be formulated as a semiclassical PDE:

$$h^2 Pu - u = h\, r, \quad \|r\|_{L^2(M)} \le C\|u\|_{L^2(M)}. \tag{5.5}$$

The main observation is then that this equation can be seen microlocally as an evolution equation with respect to one of the spatial coordinates, say x_1. For this equation it is possible to perform exactly the same analysis as we did in the proof of Theorem 4.3. This yields semiclassical versions of Strichartz inequalities (4.9) for $(d-1)$-admissible pairs, and the result follows by using Hölder's inequality as in Remark 5.3. □

6. The case of some simple compact manifolds

In this section we investigate the Cauchy problem for (1.3) on tori, spheres and balls, where $-P$ is the standard Laplace operator, trying to improve the results of Section 4. The basic tools for positive results of wellposedness are borrowed from Section 5. Several illposedness results are also obtained by explicit constructions. Moreover, we point some open problems in this context.

6.1. Tori. The first compact manifolds on which the global Cauchy problem for (1.3) has been studied are the tori

$$\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d.$$

The main reference is the fundamental work of Bourgain [6], [7] (see also [8] and Ginibre [35]). We start with the one-dimensional case.

Theorem 6.1 (Bourgain [6]). *If $F(u) = \pm|u|^2u$, then* (1.3) *is globally regularly well-posed on $L^2(\mathbb{T})$.*

If $F(u) = \pm|u|^4 u$, *then* (1.3) *is locally regularly well-posed on* $H^\varepsilon(\mathbb{T})$, *but not on* $L^2(\mathbb{T})$.

If we compare with Theorem 4.2, we see that the quintic nonlinearity is critical for L^2 wellposedness on the line. However, on the line a quintic NLS is regularly well-posed on small neighborhoods of 0 in $L^2(\mathbb{R})$, while it is not the case on the circle.

The proof of Theorem 6.1 essentially combines Theorem 5.7 with the explicit spectral representation for the solution of the linear Schrödinger equation,

$$u(t,x) = \mathrm{e}^{it\partial_x^2} u_0\,(x) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{-itn^2} \mathrm{e}^{inx} \hat{u}_0(n),$$

which yields the needed bilinear and trilinear estimates by a direct calculation based on the Parseval formula in both time and space variables, following an old idea due to Zygmund [63]. The lack of regular wellposedness of a quintic NLS on L^2 results from the first part of Theorem 5.7 and an explicit example of a sequence of data u_0 such that $\|u\|_{L^6(\mathbb{T}^2)}\,\|u_0\|^{-1}_{L^2(\mathbb{T})}$ is not bounded.

Let us say a word about data in $H^s(\mathbb{T})$ for $s < 0$. In this case, there are several results of illposedness for (1.3). The most elementary one (see [13]) is the lack of uniform wellposedness which can be deduced from the explicit solutions

$$u(t,x) = \kappa n^{|s|} \mathrm{e}^{-it(n^2 + G'(\kappa^2 n^{2|s|}))} \mathrm{e}^{inx}$$

if $F(z) = G'(|z|^2)z$. Notice that, if G' behaves like a power at infinity, a small variation of κ around 1 induces a large variation of the phase shift of $u(t,x)$ for $t > 0$ and n large, whence some lack of stability. Notice that a similar argument had been used earlier by Birnir–Kenig–Ponce–Svanstedt–Vega [4] and by Kenig–Ponce–Vega [44] on the line. Furthermore, by a careful study of the interaction between high and low frequencies, Christ, Colliander and Tao prove in [28] that, for the cubic case, the flow map fails to be continuous from any ball of H^s to the space of distributions. As for the quintic equation, the flow map fails to be uniformly continuous from any ball of H^s endowed with the topology of C^∞, to the space of distributions. More recently, Christ [25] constructed non trivial weak solutions in $C_t(H^s)$ of a modified version of a cubic NLS, with zero Cauchy datum.

Let us come to the multidimensional case. Using the same method as for Theorem 6.1, it is possible to prove multilinear Strichartz estimates on $\mathbb{T}^d$ for $d \geq 2$. The interesting point is that these estimates are the same as on $\mathbb{R}^d$, except the fact that they are local in time, and that a loss ε may alter the orders. For instance, the Schrödinger group on $\mathbb{T}^d$ enjoys a bilinear Strichartz estimate of order $(d-2)/2 + \varepsilon$ for every $\varepsilon > 0$. In view of Theorems 5.2, 5.7 and 5.8, this implies in particular the following results.

Theorem 6.2 (Bourgain [6]). *If* $F(u) = \pm|u|^2 u$, *then* (1.3) *is regularly well-posed on* $H^\varepsilon(\mathbb{T}^2)$ *for every* $\varepsilon > 0$.

If F satisfies (1.4) *for some $\alpha \in [2,4]$, then* (1.3) *is uniformly wellposed on $H^s(\mathbb{T}^3)$ for every $s > 3/2 - 2/\alpha$. In particular, if moreover $\alpha < 4$, F is gauge invariant and Hamiltonian with a nonnegative potential,* (1.3) *is globally uniformly well-posed on $H^1(\mathbb{T}^3)$.*

In the case $d = 4$, the regularity $s = 1$ is critical for the cubic NLS. However, by means of logarithmic estimates based on a careful study of exponential sums, global wellposedness in $H^s(\mathbb{T}^4)$, $s > 1$, can be obtained for (1.3) with nonlinearities with quadratic growth such as $F = F_{1,+}$ (see Bourgain [7]).

Let us conclude this subsection by quoting two open problems. The first one concerns the quintic defocusing problem, namely $F(u) = |u|^4u$, on $\mathbb{T}^3$. According to Theorem 6.2, it is regularly wellposed on $H^s(\mathbb{T}^3)$ for every $s > 1$, but nothing is known about $s = 1$, even for small data. This would yield global regular wellposedness in view of conservation laws.

The second open problem is the generalization of the above results to tori of the type

$$\mathbb{T}^d(\theta_1, \dots, \theta_d) = \mathbb{R}/\theta_1\mathbb{Z} \times \cdots \times \mathbb{R}/\theta_d\mathbb{Z},$$

where the θ_j's are positive numbers, possibly irrationally independent. The possibly chaotic behavior of the spectrum

$$\lambda = \omega_1^2 n_1^2 + \cdots + \omega_d^2 n_d^2, \quad \omega_j = \frac{2\pi}{\theta_j},\ n_j \in \mathbb{Z},$$

makes the multilinear Strichartz estimates particularly delicate to obtain. For instance, if $d = 2$, the optimal order of the bilinear Strichartz estimate – and thus the threshold of regularity wellposedness of cubic NLSs – is not known. However, if $d = 3$, it is possible to prove that the local $L^p_t(L^4_x)$ norm of the solution of the linear Schrödinger equation scales as on the Euclidean space, for any $p > 16/3$ (see Bourgain [9]). This implies a trilinear estimate of order $\left(\frac{5}{4} - \varepsilon, \frac{3}{4} + \varepsilon\right)$ for every $\varepsilon > 0$, and, by Theorem 5.8, global uniform wellposedness of the defocusing subquintic NLS on $H^1(\mathbb{T}^3)$. Moreover, by more refined counting arguments, it is possible to reduce the order of the bilinear Strichartz estimate from $\frac{3}{4} + \varepsilon$ to $\frac{2}{3} + \varepsilon$ ([9]).

6.2. Spheres. The case of multidimensional spheres is of course very natural, since we observed in Subsection 4.3 that the loss in endpoint Strichartz inequalities is optimal on them. Therefore we could expect that the wellposedness results for (1.3) are the worst ones on spheres. The two-dimensional case is particularly interesting in this respect. The following theorem is a slight generalization of results in [13] and [16].

Theorem 6.3 (Burq–Gérard–Tzvetkov [13], [16]). *The Cauchy problem for the cubic NLS, i.e.* (1.3) *with $F(u) = \pm|u|^2u$, is regularly well-posed on $H^s(\mathbb{S}^2)$ for every $s > 1/4$ and* not *uniformly well-posed on $H^s(\mathbb{S}^2)$ for every $s < 1/4$.*

If $F = F_{\alpha,\pm}$ *(see* (1.10)) *with* $\alpha \in]0, 2]$*, it is uniformly well-posed on* $H^s(\mathbb{S}^2)$ *for every* $s > 1/4$*, and* not *uniformly well-posed for* $s < 1/4$*.*

The Cauchy problem for the quintic NLS, i.e. (1.3) *with* $F(u) = \pm|u|^4 u$ *is uniformly well-posed on* $H^s(\mathbb{S}^2)$ *for every* $s > 1/2$*.*

Remark 6.4. The first striking fact contained in the above result is that, unlike cubic NLSs on the Euclidean plane or on the (square) torus, a cubic NLS on the sphere has a threshold of regular (or uniform) wellposedness which is $1/4$ and not 0. Notice that we already met this exponent $1/4$ in Subsection 5.1, in connection with the counterexample of Subsection 4.3. In fact, the geometric phenomenon is the same here, namely concentration on a closed geodesic, but specific information about the sphere allow to get optimal results. A very natural open problem is of course to decide whether the *wellposedness* threshold is also $1/4$, or if it is smaller (for instance 0), as we quoted in Section 2 about the modified KdV equation.

Another important open question is raised by the comparison of the above result with Corollary 4.5, which, for a cubic NLS on general surfaces, needs $s > 1/2$ for regular wellposedness. In fact, we ignore if there is a compact surface where a cubic NLS is not regularly well-posed on H^s for some $s \in]1/4, 1/2]$.

A third observation is that, unlike positive thresholds on the Euclidean space, the one on the sphere is not always changing with the parameter α. Indeed, for $0 < \alpha < 2$, it is frozen at $1/4$.

Finally, for the quintic NLS, the last statement of Theorem 6.3, combined with Theorem 3.3 shows that the three wellposedness thresholds coincide with $1/2$, which is the Euclidean one. This suggests a general mechanism which we already met in the context of tori, namely that *the* $L^p_t(L^q_x)$ *estimates of the solutions of the linear Schrödinger equation seem to become as good as the Euclidean ones if* p, q *are large enough, so that, for* α *large enough, the wellposedness thresholds become identical to the Euclidean ones.* We shall check this phenomenon for higher dimensional spheres as well. However, we do not have any argument for proving it on a general compact manifold.

Proof (*sketch*). The positive results on uniform and regular wellposedness are consequences of Theorems 5.2, 5.4 and 5.7 and of multilinear Strichartz estimates for the Schrödinger group on $\mathbb{S}^2$. As in the previous subsection, we use the exact spectral representation of solutions to the linear Schrödinger equation,

$$u(t, x) = \sum_{n \sim N} \mathrm{e}^{-itn(n+1)} H_n(x),$$

where H_n are spherical harmonics of degree n, and the condition $n \sim N$ corresponds to the spectral localization at frequency N. Using Parseval formula in the time variable and multilinear spectral estimates given by Theorem 5.9, we obtain a bilinear Strichartz estimate of order $1/4+\varepsilon$ and a trilinear estimate of order $(3/4+\varepsilon, 1/4+\varepsilon)$ for every $\varepsilon > 0$. The latter is even better than what we need. In particular, using Theorem 5.8, we infer that (1.3), with $F = F_{\alpha,\pm}$ (see (1.10)) and $\alpha \in]2, 4[$, is uniformly well-posed on $H^s(\mathbb{S}^2)$ for every $s > 3/4 - 1/\alpha$. However we do not know if this threshold is optimal.

Here an important role is played by the localization of the spectrum around squares of the integers, so that this proof can only be generalized to Zoll manifolds (see [12] and [16] for details).

We come now to the illposedness result. We observe that, as in the counterexample given in Subsection 4.3, the following sequence of spherical harmonics

$$\psi_n(x) = (x_1 + ix_2)^n, \quad x_1^2 + x_2^2 + x_3^2 = 1$$

is concentrating exponentially on the closed geodesic $x_1^2 + x_2^2 = 1$ and satisfies

$$\|\psi_n\|_{L^q} \simeq n^{\frac{1}{4} - \frac{1}{2q}} \|\psi_n\|_{L^2}, \quad q \geq 2.$$

Moreover, ψ_n has the remarkable property that it is the ground state of the Laplace operator on the space V_n of functions f satisfying the symmetry property

$$f(R_\theta(x)) = \mathrm{e}^{in\theta} f(x),$$

where R_θ is the rotation of angle θ around the x_3 axis. The idea is to construct stationary solutions to (1.3) by minimizing the energy $H(f)$ on the L^2 sphere of V_n for small radii $\delta_n = \kappa_n n^{-s}$, for different values of the parameter $\kappa_n \sim 1$. It turns out that the minimizers f_n are very close to the line directed by ψ_n, and that the nonlinear eigenvalue ω_n can be precisely estimated as n goes to ∞, creating for the solution

$$u_n(t, x) = \mathrm{e}^{-it\omega_n} f_n(x)$$

the same kind of instability that we already observed in the case of the one-dimensional torus. We refer to [31] for details, or to [13] for a slightly different approach. □

Finally, we observe that the above methods can be applied to higher-dimensional spheres. We gather the most striking facts in the following theorem.

Theorem 6.5 (Burq–Gérard–Tzvetkov). *The Cauchy problem for a cubic NLS is regularly well-posed on $H^s(\mathbb{S}^3)$ for $s > 1/2$, and not uniformly well-posed for $s < 1/2$* ([17], [13]).

If $F = F_{\alpha,\pm}$ (see (1.10)*), it is uniformly well-posed on $H^s(\mathbb{S}^3)$ if $s > s(\alpha)$, and not uniformly well-posed for $s < s(\alpha)$, with $s(\alpha) = 1/2$ if $\alpha \leq 2$, and $s(\alpha) = 3/2 - 2\alpha$ if $\alpha \in [2, 4]$. In particularly it is globally uniformly well-posed on $H^1(\mathbb{S}^3)$ if $\alpha < 4$* ([17], [13]).

The Cauchy problem for a cubic NLS is regularly well-posed on $H^s(\mathbb{S}^4)$ if $s > 1$, but not for $s = 1$ ([16], [12]).

The Cauchy problem for (1.3) *with $F = F_{\alpha,\pm}$ is not uniformly well-posed on $H^1(\mathbb{S}^6)$ for every $\alpha \in]0, 1]$* ([13]).

In the case $d = 3$, as in the torus case, the question of (regular) wellposedness of the quintic NLS on small data in H^1 remains open. See however some partial results in this direction in [19]. As for global wellposedness of the subquintic case, it is also known on $\mathbb{S}^2 \times \mathbb{T}$, but is completely open for an arbitrary three–manifold.

In the case $d = 4$, global wellposedness for some smoothed variants of the cubic NLS can be found in [32].

Finally, let us emphasize that the illposedness result on $\mathbb{S}^6$ is in strong contrast with the case $d = 6$ in Theorem 4.2.

6.3. Balls. If $-P$ is the Laplace operator on the ball $\mathbb{B}^d$ of the d-dimensional Euclidean space with Dirichlet or Neumann boundary conditions, it is possible to take advantage of stronger concentration phenomena of the eigenfunctions at the boundary to produce illposedness for higher regularity.

Theorem 6.6 (Burq–Gérard–Tzvetkov [14], [20]). *Let $-P$ be the Laplace operator on $\mathbb{B}^d$ with Dirichlet* (*resp. Neumann*) *boundary conditions.*

If $d = 2$, the Cauchy problem for the cubic NLS is not uniformly well-posed on the domain of $P^{s/2}$ for $s \in [0, 1/3[$.

The Cauchy problem for (1.3) *with $F = F_{\alpha,\pm}$* (*see* (1.10)) *is not uniformly well-posed on $H_0^1(\mathbb{B}^5)$* (*resp. $H^1(\mathbb{B}^5)$*) *for every $\alpha \in]0, 1]$.*

Finally, let us mention that, contrarily to the case of spheres, global wellposedness for subquintic (1.3) with boundary conditions on $\mathbb{B}^3$ remains an open problem.

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