Conformal invariants and nonlinear elliptic equations

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Abstract. We describe several "uniformizing" problems in conformal geometry, all of which can be formulated as problems of existence for solutions of certain elliptic partial differential equations. For the sake of exposition we divide the discussion according to the type of PDE, beginning with semilinear equations related to the scalar curvature, then higher order equations arising from the Q-curvature, and finally fully nonlinear equations.

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1. Introduction

In this article we outline several problems related to finding a canonical representative of a conformal equivalence class of Riemannian metrics. The earliest and best known result of this kind is the Uniformization Theorem for surfaces. Although originally conceived as a problem in complex function theory, in view of the modern development of the theory of Riemann surfaces it can be stated in purely geometric terms: Given a compact surface (M^2, g), there is a conformal metric $\tilde{g} = e^{2w} g$ with constant curvature.

In higher dimensions the Uniformization Theorem is no longer true, even locally. For example, in dimensions $n \geq 4$ the Weyl tensor $W(g)$ provides an obstruction to a metric being locally conformal to a flat metric. Despite this fact – or perhaps because of it – there are numerous ways to define what constitutes a canonical metric. In this article we will outline several "uniformizing" problems, each involving the study of a PDE of a different type: second order semilinear (in the case of the scalar curvature), higher order semilinear (for the Q-curvature), and fully nonlinear (when considering the σ_k-curvature). An underlying theme will be the connection between conformal invariants associated to the problem, spectral properties of the relevant differential operator, and the interplay of both with the topology of the manifold.

2. Semilinear examples

2.1. The modified scalar curvature. Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 3$, and let $R(g)$ denote its scalar curvature. If $\tilde{g} = u^{4/(n-2)} g$ is a

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conformal metric, then the scalar curvature $R(\tilde{g})$ of $\tilde{g}$ is given by

$$L_g u + R(\tilde{g}) u^{\frac{n+2}{n-2}} = 0, \tag{2.1}$$

where $L_g = \frac{4(n-1)}{(n-2)}\Delta_g - R(g)$ is the *conformal laplacian*. The *Yamabe problem* is to establish the existence of a conformal metric with constant scalar curvature (see [32]). Since the scalar curvature of a surface is twice the Gauss curvature, the Yamabe problem can be viewed as one possible generalization of the Uniformization Theorem.

An important property of the operator L_g is its *conformal covariance*: If $\tilde{g} = u^{4/(n-2)}g$, then

$$L_{\tilde{g}} v = u^{-\frac{n+2}{n-2}} L_g(uv). \tag{2.2}$$

Consequently, the sign of the principle eigenvalue $\lambda_1(-L_g)$ is a conformal invariant ([30]).

A generalization of the Yamabe problem was introduced by the author, for the purposes of studying a rigidity question for Einstein metrics [22]. Subsequently it has been used to estimate the Yamabe invariant of certain four-manifolds [23], the principal eigenvalue of the Dirac operator [28], and in the context of Seiberg–Witten theory [31].

Given a Riemannian manifold (M^n, g), let $\mathcal{G} \subset S^2T^*M^n$ denote the ray bundle consisting of metrics in the conformal class of g. Let $\delta_s : \mathcal{G} \to \mathcal{G}$ denote the dilations $\delta_s(g) = s^2 g$, with $s > 0$. Functions on $\mathcal{G}$ which are homogeneous of degree β with respect to δ_s are known as *conformal densities of weight* β.

Definition 2.1. Given a density ϕ of weight -2, we define the *modified scalar curvature* associated to ϕ by

$$\hat{R}(g) = R(g) - \phi. \tag{2.3}$$

For example, in some applications (M^4, g) is an oriented four-dimensional manifold and $\phi = 2\sqrt{6}|W^{\pm}(g)|$, the (anti-) self-dual part of the Weyl curvature tensor. For Kähler manifolds of positive scalar curvature the corresponding modified scalar curvature $\hat{R}(g) = R(g) - 2\sqrt{6}|W^+|^2 \equiv 0$. As this example illustrates, in general we do not assume the density is smooth, but at least Lipschitz continuous.

Since ϕ is a conformal density of weight -2 the operator $\hat{L}_g = \frac{4(n-1)}{(n-2)}\Delta_g - \hat{R}(g)$ enjoys the same invariance as the conformal laplacian given in (2.2). In particular, the sign of $\lambda_1(-\hat{L}_g)$ is a conformal invariant.

In analogy with the Yamabe problem, we can ask whether there exists a conformal metric whose modified scalar curvature is constant. Due to the conformal covariance of $\hat{L}$, the sign of this constant will agree with the sign of $\lambda_1(-\hat{L}_g)$. Since the choice of density obviously plays an important role in the question of existence, a completely general theory would seem unlikely. For particular cases, though, some existence results have been appeared (see [29]).

In fact, for many applications the relevant question is the *sign* of the modified scalar curvature, or equivalently, the sign of $\lambda_1(-\hat{L}_g)$. Typically, the density ϕ is chosen by examining the curvature term in the Weitzenböck formula for a harmonic section of some vector bundle; then the sign of $\lambda_1(-\hat{L}_g)$ can be thought of as an obstruction to the existence of non-trivial harmonic sections. Of course, Lichnerowicz used precisely this kind of argument with the Dirac operator on a spin manifold to prove obstructions to the existence of metrics with positive scalar curvature [33].

To illustrate this technique with one important example, consider a self-dual harmonic two-form $\omega \in H^2_+(M^4, \mathbb{R})$. In this case the Weitzenböck formula is given by

$$\frac{1}{2}\Delta_g|\omega|^2 = |\nabla\omega|^2 - 2W^+(\omega,\omega) + \frac{1}{3}R(g)|\omega|^2. \tag{2.4}$$

Another intriguing element in the study of the modified scalar curvature is the important role played by *refined Kato inequalities* (see [24]). In the case of self-dual harmonic two-forms this takes the form

$$|\nabla\omega|^2 \geq \frac{3}{2}|\nabla|\omega||^2 \tag{2.5}$$

([37]). Substituting into (2.4), and using the fact that $W^+\colon \Lambda^2_+ \to \Lambda^2_+$ is a trace-free endomorphism, we eventually arrive at

$$\Delta_g|\omega|^{2/3} \geq \frac{1}{6}\big[-2\sqrt{6}|W^+(g)| + R(g)\big]|\omega|^{2/3}. \tag{2.6}$$

Taking $\phi = 2\sqrt{6}|W^+|$, we conclude

$$\hat{L}_g|\omega|^{2/3} \geq 0 \tag{2.7}$$

which implies $\lambda_1(-\hat{L}_g) \leq 0$. Thus, there are cohomological obstructions to the existence of metrics with positive first eigenvalue.

On the other hand, when $\lambda_1(-\hat{L}_g) \leq 0$ one obtains L^p-estimates for the scalar curvature. For example, in four dimensions

$$\int R(g)^2\, dv(g) \leq \int \phi^2\, dv(g). \tag{2.8}$$

Since ϕ is a density of weight -2, the integral on the right-hand side of (2.8) is a conformal invariant. Thus, we have a connection between the spectral properties of the conformally covariant operator $\hat{L}_g$, and L^2-conformal invariants. We will encounter a similar phenomenon when considering higher order elliptic equations.

2.2. Higher order equations and the Q-curvature. In an unpublished preprint [35], the late Stephen Paneitz constructed a fourth order conformally covariant operator defined on a (pseudo)-Riemannian manifold (M^4, g) of dimension $n \geq 3$. In four dimensions, his operator is given by

$$P_g = (-\Delta_g)^2 - \delta_g\Big\{\Big[\frac{2}{3}R(g)g - 2\,\mathrm{Ric}(g)\Big]\circ\nabla\Big\}, \tag{2.9}$$

where $\delta_g \colon \Lambda^1(M^4) \to C^\infty(M^4)$ is the divergence. If $\tilde{g} = e^{2w}g$ is a conformal metric, then

$$P_{\tilde{g}} = e^{-4w}P_g. \tag{2.10}$$

In particular, the sign of $\lambda_1(P_g)$ and the kernel of P_g are both conformally invariant. Since $P_g(1) = 0$, we always have $\lambda_1(P_g) \leq 0$.

Branson [4] subsequently observed the connection between Paneitz's operator and what Branson called the *Q-curvature*, defined by

$$Q(g) = \frac{1}{12}\big(-\Delta_g R(g) + R(g)^2 - 3|\,\mathrm{Ric}(g)|^2\big). \tag{2.11}$$

In fact, if $\tilde{g} = e^{2w}g$, then the Q-curvature of $\tilde{g}$ is given by

$$P_g w + 2Q(g) = 2Q(\tilde{g})e^{4w}. \tag{2.12}$$

It follows that the integral of the Q-curvature is another conformal invariant:

$$\int Q(\tilde{g})\,dv(\tilde{g}) = \int Q(g)\,dv(g). \tag{2.13}$$

The Q-curvature and Paneitz operator have become important objects of study in the geometry of four-manifolds, and play a role in the such diverse topics as the Moser–Trudinger inequalities ([3], [5]), twistor theory ([14]), gauge choices for Maxwell's equations ([13]), and conformally compact AHE manifolds ([15], [16]). In addition, they naturally suggest another "uniformizing" problem, that of finding a conformal metric with constant Q-curvature.

Chang and Yang ([9]) studied this problem using the direct variational method, by attempting to prove the existence of minimizers of the non-convex functional

$$\begin{aligned} F[w] = \int wP_g w\,dv(g) + 4\int wQ(g)\,dv(g) \\ - \Big(\int Q(g)\,dv(g)\Big)\log\int e^{4w}\,dv(g). \end{aligned} \tag{2.14}$$

However, if the Paneitz operator has a negative eigenvalue and the conformal invariant (2.13) is positive, then Chang and Yang showed that $\inf F = -\infty$. Even when F is bounded below the compactness of a minimizing sequence is a delicate matter. Using a sharp form of Adam's inequality ([1]) Chang and Yang [9] were able to prove the existence of minimizer assuming the invariant (2.13) is less than its value on the standard sphere:

Theorem 2.2. *Let* (M^4, g) *be a closed four-manifold, and assume* (i) $P_g \geq 0$ *with* $\mathrm{Ker}\, P_g = \{\mathrm{const.}\}$, *and* (ii) $\int Q(g) dv(g) < 8\pi^2$. *Then there exists a minimizer* $w \in C^\infty$ *of* F, *which satisfies* (2.12) *with* $Q(\tilde{g}) = \mathrm{const.}$

Thus, the question of existence is reduced to studying the spectrum of the Paneitz operator and the conformal invariant (2.13). As we shall see, there is a connection between these problems.

First, it is not difficult to see that when g has positive scalar curvature, the invariant (2.13) is *always* dominated by its value on the sphere. (Somewhat surprisingly, equality can be characterized without resorting to the Positive Mass Theorem; see [21]). Thus, when (M^4, g) has positive scalar curvature the assumption (ii) in the Theorem of Chang-Yang is superfluous, except in the case of the sphere.

Turning to the first assumption of Theorem 2.2, we begin by noting the Dirichlet form associated to the Paneitz operator is given by

$$\int \psi P_g \psi \, dv(g) = \int \left[(\Delta_g \psi)^2 + \frac{2}{3} R(g) |\nabla \psi|^2 - 2\, \mathrm{Ric}(g)(\nabla \psi, \nabla \psi) \right] dv(g). \tag{2.15}$$

Using the Bochner formula we can rewrite this as

$$\begin{aligned} \int \psi P_g \psi \, dv(g) &= \int \left[\frac{4}{3} |\mathring{\nabla}^2 \psi|^2 \, dv(g) + \frac{2}{3} \big(R(g) g - \mathrm{Ric}(g) \big)(\nabla \psi, \nabla \psi) \right] dv(g) \\ &\geq \int \frac{4}{3} |\mathring{\nabla}^2 \psi|^2 \, dv(g) + \int \frac{2}{3} T(\nabla \psi, \nabla \psi) \, dv(g), \end{aligned} \tag{2.16}$$

where $\mathring{\nabla}^2 \psi = \nabla^2 \psi - \frac{1}{4} (\Delta_g \psi) g$ is the trace-free Hessian and $T = R(g) g - \mathrm{Ric}(g)$. Consequently, if the right-hand side of (2.16) is positive for all $\psi \in C^\infty$, then $\lambda_1(P_g) = 0$ and $\mathrm{Ker}\, P_g = \{\mathrm{const.}\}$. By conformal invariance it is enough to show that this property holds for some metric in the conformal class of g.

By using a kind of "regularized" version of the functional F (which was also studied by Chang and Yang), the author was able to construct a metric $\tilde{g} \in [g]$ for which the right-hand side of (2.16) is positive for all $\psi \in C^\infty$, provided the scalar curvature of g is positive (see [21]):

Theorem 2.3. *If* (M^4, g) *has positive scalar curvature and* $\int Q(g)\, dv(g) \geq 0$, *then* $P_g \geq 0$ *and* $\mathrm{Ker}\, P_g = \{\mathrm{const.}\}$. *In particular, there is a conformal metric* $\tilde{g}$ *with* $Q(\tilde{g}) = \mathrm{const.}$

The assumptions of Theorem 2.3 imply the first Betti number of M^4 vanishes; see [20]. On the other hand, in [14] Eastwood–Singer constructed metrics on $k(S^3 \times S^1)$ for all $k > 0$ with $P_g \geq 0$ and $\mathrm{Ker}\, P_g = \{\mathrm{const.}\}$. For this reason, it would be desirable to relax the assumption on the integral of $Q(g)$. A result of this kind appears in [25], which relied on solving a fully nonlinear equation of the type described in the next section to prove the positivity of the tensor field $T = R(g) g - \mathrm{Ric}(g)$ appearing in (2.16):

Theorem 2.4. *If (M^4, g) has positive Yamabe invariant $Y(g)$ and the Q-curvature satisfies $\int Q(g)\, dv(g) + \frac{1}{6}Y(g)^2 \geq 0$, then $P_g \geq 0$ and* $\operatorname{Ker} P_g = \{\text{const.}\}$. *In particular, there is a conformal metric $\tilde{g}$ with* $Q(\tilde{g}) = \text{const.}$

Since Theorem 2.4 allows the integral of Q to be negative, we are able to construct many new examples of conformal manifolds which admit a metric with constant Q-curvature (see Section 7 of [25].

There have been other approaches to the problem of finding metrics with constant Q-curvature. Malchiodi [34], Malchiodi–Djadli [11], and Druet–Robert [12] have studied existence and compactness of the solution space by a delicate blow-up analysis. In particular, the positivity of the total Q-curvature is not required. However, their work indicates that the assumption $\operatorname{Ker} P_g = \{\text{const.}\}$ is unlikely to be merely technical. Brendle [7] and Baird–Fardoun–Regbauoi [2] have used parabolic methods; they also assume $P_g \geq 0$ with trivial kernel.

The problem of finding metrics with constant Q-curvature originally appeared in the more general context of studying variational properties of the regularized determinant (see [9]). In contrast to the study of the Q-curvature, the existence theory has not developed very much beyond the results in [9] and [21]. The associated Euler–Lagrange equation includes terms which are nonlinear in the second derivatives of the solution, providing an important link to the material in the next section.

3. Fully nonlinear equations

Finally, we give a brief synopsis of a very active area which can be viewed as a fully nonlinear version of the Yamabe problem. For Riemannian manifolds of dimension $n \geq 3$ we define the *Weyl–Schouten* tensor by

$$A(g) = \frac{1}{(n-2)}\left(\operatorname{Ric}(g) - \frac{1}{2(n-1)}R(g)g\right). \tag{3.1}$$

In [39], J. Viaclovsky initiated the study of the fully nonlinear equations arising from the transformation of A under conformal deformations. More precisely, let $g_u = e^{-2u}g$ denote a conformal metric, and consider the equation

$$\sigma_k^{1/k}(g_u^{-1} \circ A_u) = f(x), \tag{3.2}$$

where $\sigma_k(\cdot)$ denotes the k-th elementary symmetric polynomial, applied to the eigenvalues of $g_u^{-1} \circ A_u$. Since A_u is related to A by the formula

$$A(g_u) = A(g) + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g, \tag{3.3}$$

equation (3.2) is equivalent to

$$\sigma_k^{1/k}(A(g) + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g) = f(x)e^{-2u}. \tag{3.4}$$

Note that when $k = 1$, $\sigma_1(g^{-1}A(g)) = \text{trace}(A) = \frac{1}{2(n-1)}R(g)$; therefore, (3.4) is equivalent to equation (2.1). When $k > 1$ the equation is fully nonlinear, but not necessarily elliptic. A sufficient condition for a solution $u \in C^2(M^n)$ to be elliptic is that the eigenvalues of $A = A(g)$ are in $\Gamma_k^+ = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \sigma_1(\lambda) > 0, \sigma_2(\lambda) > 0, \dots, \sigma_k(\lambda) > 0\}$ at each point of M^n. In this case we say that g is *admissible* (or k-admissible); likewise, if $-A(g) \in \Gamma_k^+$ then we say g is *negative admissible*.

The most straightforward question one can pose about equation (3.4) is existence: given an admissible metric $g \in \Gamma_k^+(M^n)$ and a positive function $f \in C^\infty(M^n)$, does there always exist a solution $u \in C^\infty(M^n)$ to (3.4)? When $f(x) = \text{const.} > 0$ this is referred to as the *σ_k-Yamabe problem*. An important distinction between the σ_k-Yamabe problem and the classical Yamabe problem is that the former is not in general variational.

The study of equation (3.4) in general and the σ_k-Yamabe problem in particular has seen an explosion of activity in recent years. We will highlight some results for the case $k > n/2$ as the theory is more developed; admissibility has a more geometric interpretation; and in contrast to the case $k \leq n/2$, solutions of (3.4) have a variational characterization.

First, Guan–Viaclovsky–Wang [19] showed that when g is k-admissible the Ricci curvature satisfies the sharp inequality

$$\text{Ric}(g) \geq \frac{(2k-n)}{2n(k-1)} R(g) g. \tag{3.5}$$

In particular, if $k > n/2$, the Ricci curvature is positive. Using the Newton–Maclaurin inequality and Bishop's volume comparison, one can quantify this in the following way: If $g_u = e^{-2u}g$ is an admissible solution of (3.4) with $f(x) \geq c_0 > 0$, then

$$\text{Vol}(g_u) = \int e^{-nu}\, dv(g) \leq C(k, n, c_0). \tag{3.6}$$

We define the k^{th}*-maximal volume* of the admissible metric g by

$$\Lambda_k(M^n, [g]) = \sup\{\text{Vol}(g_u) \mid A(g_u) \in \Gamma_k^+, \sigma_k(g_u^{-1}A(g_u)) \geq \sigma_k(S^n)\}, \tag{3.7}$$

where $\sigma_k(S^n)$ is the value attained by the round sphere. This definition suggests the following variational problem: given a k-admissible metric g, is there is conformal metric which attains the k^{th}-maximal volume? It is easy to see that any metric which does will satisfy (3.4) with $f(x) \equiv \sigma_k^{1/k}(S^n)$. A similar variational problem was formulated by Guan and Spruck in studying the curvature of hypersurfaces in Euclidean space [17].

In joint work with J. Viaclovsky we used this variational scheme to study the σ_k-Yamabe problem in three- and four-dimensions [26]. The dimension restriction is a result of the difficulty of proving sharp estimates for the maximal volume in high dimensions. In three dimensions we could prove such estimates thanks to Bray's

Football Theorem [6], and in four dimensions by using the Chern–Gauss–Bonnet formula. An approach somewhat similar in spirit was implicit in earlier work of Chang–Gursky–Yang [8] and Viaclovsky [40].

Another consequence of the volume bound (3.6) is the finiteness of the blow-up set for a sequence of solutions to (3.4). This follows from the ε-regularity result of Guan and Wang [18], which in turn is based on their local C^1- and C^2-estimates for solutions. In fact, using the estimates of Guan-Wang, it is possible to show that a divergent sequence $\{u_i\}$ of solutions to (3.4) will blow up at finitely many points, and converge uniformly to $-\infty$ off the singular set. By rescaling this sequence, one obtains a limiting viscosity solution $w \in C^{1,1}_{\mathrm{loc}}$ with $f(x) \geq 0$.

In recent work with Viaclovsky [27], we carried out a careful analysis of the tangent cone at infinity of the $C^{1,1}$-metric $g_\infty = e^{2w} g$. In particular, we showed the volume growth at infinity is Euclidean. Since the Ricci curvature is non-negative, this implies the metric is flat, and (M^n, g) is conformally the sphere.

Theorem 3.1. *Let (M^n, g) be closed n-dimensional Riemannian manifold, and assume*

(i) *g is k-admissible with $k > n/2$, and*

(ii) *(M^n, g) is not conformally equivalent to the round n-dimensional sphere.*

Then given any smooth positive function $f \in C^\infty(M^n)$ there exists a solution $u \in C^\infty(M^n)$ of (3.4)*, and the set of all such solutions is compact in the C^m-topology for any $m \geq 0$.*

In fact, our proof gives the existence of solutions to

$$F(A_u) = f(x)e^{-2u},$$

where $F\colon \Gamma \to \mathbb{R}$ is a symmetric function of the eigenvalues of A_u defined on a cone $\Gamma \subset \mathbb{R}^n$ which satisfies some explicit structural conditions. For more general equations we need to use the C^2-estimates developed by S. Chen [10]. Trudinger and Wang [38] have proved a similar existence result, along with a remarkable Harnack inequality for admissible metrics.

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