Asymptotic solutions for large time of Hamilton–Jacobi equations

Hitoshi Ishii*

Abstract. In this article we discuss some recent results on the large-time behavior of solutions of Hamilton–Jacobi equations as well as some ideas and observations behind them and historical remarks concerning them.

Mathematics Subject Classification (2000). Primary 35B40; Secondary 35F25, 37J99, 49L25.

Keywords. Large-time behavior, asymptotic solutions, Hamilton–Jacobi equations, Aubry sets, weak KAM theory, viscosity solutions.

1. Introduction

Hamilton–Jacobi equations play important roles in classical mechanics, geometric optics, optimal control, differential games, etc. We are here interested in global solutions of Hamilton–Jacobi equations. A well-known classical method of finding solutions of Hamilton–Jacobi equations is that of characteristics and its applications have serious difficulties in practice because of developments of shocks in solutions. At the beginning of 1980s M. G. Crandall and P.-L. Lions [10], [11] introduced the notion of viscosity solution in the study of Hamilton–Jacobi equations. It is a notion of generalized solutions for partial differential equations and it is based on the maximum principle while, in this regard, distributions theory is based on integration by parts. This notion has been successfully employed to study fully nonlinear partial differential equations (PDE for short), especially Hamilton–Jacobi equations and fully nonlinear elliptic or parabolic PDE. Important basic features of viscosity solutions are: they enjoy nice properties such as (1) stability under uniform convergence or under the processes of pointwise supremum or infimum, and (2) existence and uniqueness of solutions, under mild assumptions, of boundary value problems or the Cauchy problem for fully nonlinear PDE. See [2], [3], [24], [9] for general overviews and developments of the theory of viscosity solutions.

We recall the definition of viscosity solutions of $F(x, u(x), Du(x)) = 0$ in Ω. Let $u \in C(\Omega, \mathbb{R})$. It is called a viscosity subsolution (resp., supersolution) of $F[u] = 0$ in Ω if whenever $\varphi \in C^1(\Omega)$ and $u - \varphi$ attains a maximum (resp., minimum) at $y \in \Omega$, then $F(y, u(y), D\varphi(y)) \le 0$ (resp., $F(y, u(y), D\varphi(y)) \ge 0$). Then, $u \in C(\Omega)$

*The author was supported in part by Grant-in-Aids for Scientific Research, No. 15340051, JSPS.

Proceedings of the International Congress
of Mathematicians, Madrid, Spain, 2006

is called a viscosity solution of $F[u] = 0$ in Ω if it is both a viscosity sub- and supersolution of $F[u] = 0$ in Ω. We will be here focused on viscosity solutions, subsolutions, or supersolutions and will suppress the term "viscosity" in what follows as far as there is no danger of confusion.

In this article we consider the stationary Hamilton–Jacobi equation

$$H(x, Du) = 0 \quad \text{in } \Omega, \tag{1}$$

where Ω is an open subset of $\mathbb{R}^n$, and the Cauchy problem

$$u_t + H(x, Du) = 0 \quad \text{in } \Omega \times (0, \infty), \tag{2}$$

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega. \tag{3}$$

Here H is a continuous function on $\overline{\Omega} \times \mathbb{R}^n$ and u represents the real-valued unknown function on Ω in the case of (1) or on $\overline{\Omega} \times [0, \infty)$ in the case of (2) and (3), respectively. We write frequently $H[u]$ for $H(x, Du(x))$ for notational simplicity. We will be concerned also with the additive eigenvalue problem

$$H[v] = c \quad \text{in } \Omega. \tag{4}$$

Here the unknown is a pair (c, v) of a constant $c \in \mathbb{R}$ and a function v on Ω for which v satisfies (4).

The purpose of this article is to review some of recent results concerning the large-time behavior of solutions of (2). An interesting feature of the investigations towards such results is the interaction with the developments of weak KAM theory, and this review will touch upon weak KAM theory. For overviews and developments of weak KAM theory, we refer to [17], [14].

In Section 2 we discuss projected Aubry sets and representation formulas for solutions of (1). In Section 3 the main result concerning the large-time behavior of solutions of (2) are explained. In Section 4 we outline the proof of the main result.

2. Projected Aubry sets and representation of solutions

Weak KAM theory introduced by A. Fathi in [15], [17] has changed the viewpoint of uniqueness questions regarding (1).

To begin with, we recall that classical uniqueness or comparison results in viscosity solutions theory applied to the following simple PDE

$$\lambda u + H(x, Du) = 0 \quad \text{in } \Omega, \tag{5}$$

where λ is a positive constant, states:

Theorem 2.1. *Let $u,\ v \in C(\overline{\Omega})$ be a subsolution and a supersolution of* (5)*, respectively. Assume that Ω is bounded, that either u or v is locally Lipschitz in Ω, and that $u \le v$ on $\partial\Omega$. Then $u \le v$ in Ω.*

See [2], [3], [8] for general comparison results for Hamilton–Jacobi equations. The above theorem guarantees uniqueness of locally Lipschitz continuous solutions of the Dirichlet problem for (5). We will be concerned mostly with viscosity sub-, super-, or solutions of (1) which are locally Lipschitz continuous, and thus the assumption concerning local Lipschitz continuity of solutions in the above theorem is not any real restriction.

Another way of stating the above theorem is as follows.

Theorem 2.2. *Let $u,\ v \in C(\overline{\Omega})$ be solutions, respectively, of $H[u] \le -\varepsilon$ in Ω and of $H[v] \ge 0$ in Ω, where ε is a positive constant. Assume that Ω is bounded, that either u or v is locally Lipschitz in Ω, and that $u \le v$ on $\partial\Omega$. Then $u \le v$ in Ω.*

Let $\Omega = \operatorname{int} B(0,r)$, where $B(0,r)$ denotes the closed ball of radius $r > 0$ with center at the origin and int A denotes the interior of $A \subset \mathbb{R}^n$. The eikonal equation $|Du| = |x|$ in Ω has two solutions $u_\pm(x) := \pm(1-|x|^2)/2$, which in addition satisfies the boundary condition, $u(x) = 0$ on $\partial\Omega$. Indeed, the solutions of the Dirichlet problem, $|Du| = |x|$ in Ω and $u = 0$ on $\partial\Omega$, are given by the family of functions $u_a(x) := \min\{u_+(x), a + u_-(x)\}$ parametrized by $a \in [0,\ 1]$. This example tells us that the Dirichlet problem, $|Du| = |x|$ in Ω and $u = 0$ on $\partial\Omega$, has many solutions and that the condition, $\lambda > 0$, in Theorem 2.1 is sharp. Thus uniqueness of solution of the Dirichlet problem does not hold in general for (1).

We *assume* henceforth the following two assumptions, the convexity and coercivity of the Hamiltonian H:

$$\text{for each } x \in \Omega \text{ the function } p \mapsto H(x,p) \text{ is convex in } \mathbb{R}^n, \tag{6}$$

$$\lim_{r\to\infty} \inf\{H(x,p) \mid x \in \Omega,\ p \in \mathbb{R}^n \setminus B(0,r)\} = \infty. \tag{7}$$

We set $L(x,\xi) = \sup_{p\in\mathbb{R}^n}(\xi \cdot p - H(x,p))$ for $(x,\xi) \in \Omega\times\mathbb{R}^n$, where $\xi \cdot p$ denotes the Euclidean inner product of $\xi,\ p \in \mathbb{R}^n$. We call the function $L\colon \Omega \times \mathbb{R}^n \to (-\infty,\infty]$ the *Lagrangian*.

We define the function d_H on $\Omega \times \Omega$ by

$$d_H(x,y) = \sup\{v(x) \mid H[v] \le 0,\ v(y) = 0\}.$$

Classical results in viscosity solutions theory assure that the function d_H has the properties:

$$H[d(\,\cdot\,,y)] \le 0 \quad \text{in } \Omega, \tag{8}$$

$$H[d(\,\cdot\,,y)] \ge 0 \quad \text{in } \Omega \setminus \{y\}. \tag{9}$$

Following [18], we define the (projected) *Aubry set* $\mathcal{A}$ for the Lagrangian L (or for the Hamiltonian H) as the subset of Ω given by

$$\mathcal{A} = \{y \in \Omega \mid H[d(\,\cdot\,,y)] \ge 0 \text{ in } \Omega\}. \tag{10}$$

In view of (8), (9), and (10), it is easily seen that $y \in \Omega \setminus \mathcal{A}$ if and only if $H(y, p) < 0$ for some $p \in D_1^- d_H(y, y)$, where $D_1^- d_H(x, y)$ denotes the subdifferential of $d_H(\cdot, y)$ at x. Similarly, we may state that $y \in \Omega \setminus \mathcal{A}$ if and only if there are functions $\sigma \in C(\Omega)$ and $\psi \in C^{0+1}(\Omega)$ such that $\sigma \geq 0$ in Ω, $\sigma(y) > 0$, and $H[\psi] \leq -\sigma$ in Ω.

We now assume for simplicity of presentation that Ω is an n-dimensional torus $\mathbb{T}^n$. The following theorem is an improved version of classical results such as Theorems 2.1 or 2.2

Theorem 2.3. *Let $u, v \in C(\Omega)$ be a subsolution and a supersolution of $H = 0$ in Ω, respectively. Assume that $u \leq v$ on $\mathcal{A}$. Then $u \leq v$ in Ω.*

This theorem can be found in [17, Chap. 8]. A key observation for the proof of the above theorem is that for each compact $K \subset \Omega \setminus \mathcal{A}$ there exist a function $\psi_K \in C^{0+1}(\Omega)$ and a constant $\varepsilon_K > 0$ such that $H[\psi_K] \leq -\varepsilon_K$ in a neighborhood V_K of K.

Indeed, with such K, ψ_K, ε_K, and V_K, we see that for any $\lambda \in (0, 1)$, the function $u_\lambda := (1 - \lambda)u + \lambda\psi_K$ is a subsolution of $H[u_\lambda] \leq -\lambda\varepsilon_K$ in V_K, and hence from Theorem 2.2 that for all $x \in V_K$,

$$(1 - \lambda)u(x) + \lambda\psi_K(x) \leq v(x) + \sup_{\Omega\setminus K}[(1 - \lambda)u + \lambda\psi_K - v],$$

which implies that for all $x \in \Omega \setminus \mathcal{A}$,

$$u(x) \leq v(x) + \sup_{\mathcal{A}}(u - v).$$

In their study of semicontinuous viscosity solutions, E. N. Barron and R. Jensen [4], [5] have observed that under the convexity assumption (6), a function $u \in C(\Omega)$ is a viscosity subsolution of (1) if and only if $H(x, p) \leq 0$ for all $p \in D^- u(x)$ and $x \in \Omega$, where $D^- u(x)$ denotes the subdifferential of u at x. A consequence of this observation is that the pointwise infimum of a uniformly bounded family of solutions of (1) yields a solution of (1).

Theorem 2.4. *If $u \in C(\Omega)$ is a viscosity solution of $H[u] = 0$ in Ω, then*

$$u(x) = \inf\{d_H(x, y) + u(y) \mid y \in \mathcal{A}\} \quad \textit{for all } x \in \Omega.$$

We remark that if $\mathcal{A} = \emptyset$, then there exists no solution u of $H[u] = 0$ in Ω. The above theorem is a weaker version of [19, Theorem 10.4] which is formulated with the Mather set in place of the Aubry set $\mathcal{A}$ and with quasi-convex Hamiltonian H. In the case where $\mathcal{A}$ is a finite set, a corresponding result for the Dirichlet problem for bounded domains has already been obtained in [24].

Another remark here is on the representation of d_H as the value function of an optimal control problem associated with the Hamiltonian H. Let $I \subset \mathbb{R}$ be an interval and $\gamma : I \to \Omega$. We say that γ is a *curve* if it is absolutely continuous on any compact subinterval of I. For $(x, t) \in \Omega \times (0, \infty)$, let $\mathcal{C}(x, t)$ denote the space of curves γ on $[0, t]$ such that $\gamma(t) = x$.

Theorem 2.5. *Let $x, y \in \Omega$. Then*

$$d_H(x, y) = \inf\left\{ \int_0^t L(\gamma(s), \dot\gamma(s))\mathrm{d}s \;\middle|\; t > 0,\ \gamma \in \mathcal{C}(x, t),\ \gamma(0) = y \right\}, \quad (11)$$

where $\dot\gamma$ denotes the derivative of γ.

Now we turn to the case when Ω is an open bounded subset of $\mathbb{R}^n$ with regular boundary. We consider the Dirichlet problem

$$H[u] = 0 \quad \text{in } \Omega, \quad (12)$$

$$u|_{\partial\Omega} = g, \quad (13)$$

where g is a given continuous function on $\partial\Omega$. For the Dirichlet problem, we have to modify the definition of the Aubry set and for this we set $\mathcal{A}_D = \mathcal{A} \cup \partial\Omega$, where $\mathcal{A}$ is defined as before. Let $g \in C(\mathcal{A}_D)$ and assume that the following compatibility condition, for the solvability of the Dirichlet problem (12) and (13), holds:

$$g(x) - g(y) \le d_H(x, y) \quad \text{for all } x, y \in \mathcal{A}_D. \quad (14)$$

Theorem 2.6. *Under assumption* (14)*, the function $u \in C(\overline{\Omega})$, defined by*

$$u(x) = \inf\{d_H(x, y) + g(y) \mid y \in \mathcal{A}_D\} \quad \textit{for all } x \in \overline{\Omega},$$

is a solution of (12) *and* (13)*. Moreover it is a unique solution of* (12) *satisfying $u|_{\mathcal{A}_D} = g$.*

3. Asymptotic solutions

The following result is concerned with the unbounded domain $\Omega = \mathbb{R}^n$ and we need a further restriction on H. Indeed, we assume that there exist functions ϕ_i and σ_i, with $i = 0, 1$, such that

$$H[\phi_i] \le -\sigma_i \quad \text{in } \mathbb{R}^n,$$

$$\lim_{|x|\to\infty} \sigma_i(x) = \infty,$$

$$\lim_{|x|\to\infty} (\phi_0 - \phi_1)(x) = \infty.$$

Also, we need the following hypothesis:

$$\text{for each } x \in \Omega, \text{ the function } H(x, \cdot) \text{ is strictly convex in } \mathbb{R}^n. \quad (15)$$

We introduce the spaces Φ_0 and Ψ_0 of functions on $\mathbb{R}^n$ and on $\mathbb{R}^n \times [0, \infty)$, respectively, as

$$\Phi_0 = \{f \in C(\mathbb{R}^n, \mathbb{R}) \mid \inf_{\mathbb{R}^n}(f - \phi_0) > -\infty\},$$

$$\Psi_0 = \{g \in C(\mathbb{R}^n \times [0, \infty), \mathbb{R}) \mid \inf_{\mathbb{R}^n\times[0,T]}(g - \phi_0) > -\infty, \quad \text{for all } T > 0\}.$$

Theorem 3.1. (a) *The additive eigenvalue problem* (4) *has a solution* $(c, v) \in \mathbb{R} \times \Phi_0$, *and moreover the additive eigenvalue c is uniquely determined. That is, if* $(d, w) \in \mathbb{R} \times \Phi_0$ *is another solution of* (4)*, then* $d = c$.

(b) *Let* $u_0 \in \Phi_0$. *Then there exists a unique solution* $u \in \Psi_0$ *of the Cauchy problem* (2) *and* (3).

(c) *Let* $\mathcal{A}_c$ *be the Aubry set for the Hamiltonian* $H - c$ *and* $d_{H,c} := d_{H-c}$. *Let* $u \in \Psi_0$ *be the solution of* (2) *and* (3). *Assume that* (15) *holds. Then there exists a solution* $v_0 \in \Phi_0$ *of* (4) *with c being the additive eigenvalue for H such that*

$$\lim_{t\to\infty} \sup_{x\in B(0,R)} |u(x,t) + ct - v_0(x)| = 0 \quad \textit{for any } R > 0.$$

Moreover,

$$v_0(x) = \inf\{d_{H,c}(x, y) + d_{H,c}(y, z) + u_0(z) \mid z \in \mathbb{R}^n,\ y \in \mathcal{A}_c\} \quad \textit{for all } x \in \mathbb{R}^n.$$

The above result is contained in [23]. This result is a variant of those obtained by Fathi, Namah, Roquejoffre, Barles, Souganidis, Davini, Siconolfi, and others for compact manifolds Ω. We refer to [16], [26], [28], [6], [12] for previous results and developments. See also [21] for results in $\mathbb{R}^n$ and [7], [20] for similar results for viscous Hamilton–Jacobi equations. In [21], Hamilton–Jacobi equations of the form $u_t + \alpha x \cdot Du + H_0(Du) = f(x)$ are treated, where α is a positive constant and $H_0, f \in C(\mathbb{R}^n)$. It is assumed that H_0 is convex and coercive and that there is a convex function $l \in C(\mathbb{R}^n)$ such that

$$\lim_{|x|\to\infty} (l(-\alpha x) + f(x)) = \infty \quad \text{and} \quad \lim_{|\xi|\to\infty} (L_0 - l)(\xi) = \infty,$$

where L_0 is the convex conjugate of H_0. If we assume that H_0 is strictly convex, then the hypotheses of Theorem 3.1 are satisfied with the choice of $\phi_0(x) := -(1/\sigma)l(-\alpha x)$, $\phi_1(x) := -(1/\alpha)L(-\alpha x)$, $\sigma_0(x) := l(-\alpha x) + f(x) - C$, and $\sigma_1(x) := L(-\alpha x) + f(x)$, where C is a sufficiently large constant.

Another example of H which satisfies the hypotheses of Theorem 3.1 is given by $H(x, p) = H_0(x, p) - f(x)$, where $H_0 \in C(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies (15), (7), and

$$\sup_{\mathbb{R}^n \times B(0,\delta)} |H_0| < \infty \quad \text{for some } \delta > 0,$$

and $f \in C(\mathbb{R}^n)$ satisfies $\lim_{|x|\to\infty} f(x) = \infty$. A possible choice of ϕ_i, $i = 0, 1$, is as follows: $\phi_0(x) := -(\delta/2)|x|$ and $\phi_1(x) = -\delta|x|$.

4. Outline of proof of Theorem 3.1

4.1. Additive eigenvalue problem. Additive eigenvalue problem (4) appears in ergodic optimal control or the homogenization of Hamilton–Jacobi equations. In ergodic optimal control the additive eigenvalue c corresponds to averaged long-run optimal costs while c determines the effective Hamiltonian in the homogenization of

Hamilton–Jacobi equations. See [25], [13] for periodic homogenization of Hamilton–Jacobi equations.

To avoid technicalities, we assume in this subsection that $\phi_0 = 0$. One method of solving problem (4) is to approximate it by a regular problem

$$\lambda v_\lambda + H(x, Dv_\lambda) = 0 \quad \text{in } \mathbb{R}^n, \tag{16}$$

where λ is a positive constant, and then send $\lambda \to 0$ along an appropriate sequence $\lambda_j \to 0$, to obtain a solution $(c, v) \in \mathbb{R} \times \Phi_0$ of (4) as the limits

$$c := \lim_{j\to\infty} (-\lambda_j v_{\lambda_j}(0)) \quad \text{and} \quad v(x) := \lim_{j\to\infty} (v_{\lambda_j}(x) - v_{\lambda_j}(0)). \tag{17}$$

Indeed, thanks to the coercivity of H, we may build a solution $\psi_0 \in C^{0+1}(\mathbb{R}^n)$ of $H[\psi_0] \ge -C_0$ in $\mathbb{R}^n$ for some constant $C_0 > 0$ which satisfies $\phi_0 \le \psi_0$ in $\mathbb{R}^n$. If $C > 0$ is large enough, then the functions

$$f_\lambda(x) := \phi_0(x) - \lambda^{-1}C \quad \text{and} \quad g_\lambda(x) := \psi_0(x) + \lambda^{-1}C$$

are a subsolution and a supersolution of (16), respectively. The Perron method now yields a solution $v_\lambda \in C^{0+1}(\mathbb{R}^n)$ of (16) which satisfies $f_\lambda \le v_\lambda \le g_\lambda$ in $\mathbb{R}^n$. Again the coercivity of H guarantees that the family $\{v_\lambda\}_{\lambda\in(0,1)}$ is locally equi-Lipschitz in $\mathbb{R}^n$, while the inequality $f_\lambda \le v_\lambda \le g_\lambda$ in $\mathbb{R}^n$ assures that $\{\lambda v_\lambda(0)\}_{\lambda\in(0,1)}$ is bounded. These observations allow us to pass to the limit in (17).

Another approach to solving (4) is to define the additive eigenvalue $c \in \mathbb{R}$ by

$$c = \inf\{a \in \mathbb{R} \mid \text{there exists a solution } \phi \in C(\mathbb{R}^n) \text{ of } H[\phi] \le a\}$$

and then to prove that $\mathcal{A}_c \ne \emptyset$. Any pair of c and $v := d_{H,c}(\cdot, y)$, with $y \in \mathcal{A}_c$, is a solution of (4).

In what follows we *assume* by replacing H by $H - c$, where c is the additive eigenvalue for H, that $c = 0$.

4.2. Critical curves. An important tool in the weak KAM approach is the collection of critical curves for the Lagrangian L. It allows us to analyze the asymptotic behavior for large time of solutions of (2) in the viewpoint of the Lagrangian dynamical system behind (2), which may not be well-defined under our regularity assumptions on H.

According to [12], a curve $\gamma : \mathbb{R} \to \mathbb{R}^n$ is said to be *critical* for the Lagrangian L if for any $a, b \in \mathbb{R}$, with $a < b$, and any subsolution $\phi \in C(\mathbb{R}^n)$ of $H[\phi] = 0$ in $\mathbb{R}^n$,

$$\phi(\gamma(b)) - \phi(\gamma(a)) = \int_a^b L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s.$$

We denote by Γ the set of all critical curves γ. Note that, in general, if γ is a curve on $[a, b]$ and $\phi \in C(\mathbb{R}^n)$ is a subsolution of $H[\phi] = 0$ in $\mathbb{R}^n$, then

$$\phi(\gamma(b)) - \phi(\gamma(a)) \le \int_a^b L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s. \tag{18}$$

Indeed, we compute by assuming $\phi \in C^1(\mathbb{R}^n)$ that for any curve γ on $[a, b]$,

$$\phi(\gamma(b)) - \phi(\gamma(a)) = \int_a^b D\phi(\gamma(s)) \cdot \dot{\gamma}(s)\mathrm{d}s$$
$$\leq \int_a^b [L(\gamma, \dot{\gamma}) + H(\gamma, D\phi(\gamma(s))]\mathrm{d}s \leq \int_a^b L(\gamma, \dot{\gamma})\mathrm{d}s.$$

Here we have used the Fenchel inequality: $p \cdot \xi \leq H(x, p) + L(x, \xi)$ for all x, p, ξ in $\mathbb{R}^n$. The above computation can be justified by the standard mollification technique for general ϕ which is locally Lipschitz because of the coercivity of H.

Theorem 4.1. *For any $y \in \mathcal{A}$ there exists a critical curve γ such that $\gamma(0) = y$.*

Existence of critical curves is one of crucial observations in weak KAM theory. See [15], [17], [18], [12] for results on the existence of critical curves.

A main point in the proof of the above theorem is the following general observation concerning the Aubry set, which gives another definition of the Aubry set in terms of the Lagrangian L. We remark that this latter definition of the Aubry set has been employed in [15], [18], [19].

Theorem 4.2. *Let $y \in \mathbb{R}^n$. Then $y \in \mathcal{A}$ if and only if for any $\varepsilon > 0$,*

$$\inf \left\{ \int_0^t L(\gamma(s), \dot{\gamma}(s))\mathrm{d}s \;\Big|\; t \geq \varepsilon,\ \gamma \in \mathcal{C}(y, t),\ \gamma(0) = y \right\} = 0.$$

Once we have the above theorem in hand, the proof of Theorem 4.1 goes like this. For any $y \in \mathcal{A}$ and $k \in \mathbb{N}$ we may choose a curve γ_k on $[0, T_k]$, where $T_k \geq k$ such that $\gamma_k(0) = \gamma_k(T_k) = y$ and

$$\int_0^{T_k} L(\gamma_k(s), \dot{\gamma}_k(s))\mathrm{d}s < \frac{1}{k}.$$

We define the curve η_k on $[-T_k,\ T_k]$ by setting

$$\eta_k(s) = \begin{cases} \gamma_k(s + T_k) & \text{for } s \in [-T_k, 0], \\ \gamma_k(s) & \text{for } s \in [0, T_k]. \end{cases}$$

Using the observations that

$$\lim_{r\to\infty} \inf \left\{ \frac{L(x, \xi)}{|\xi|} \;\Big|\; x \in B(0, R),\ \xi \in \mathbb{R}^n \setminus B(0, r) \right\} = \infty \quad \text{for any } R > 0, \qquad (19)$$

since $H \in C(\mathbb{R}^n \times \mathbb{R}^n)$ and that $\mathcal{A}$ is compact, we may send $k \to \infty$ along a subsequence so that η_k has a limit γ in $C(\mathbb{R}, \mathbb{R}^n)$, which is a critical curve.

For any $\gamma \in \Gamma$ we have

$$\gamma(t) \in \mathcal{A} \quad \text{for all } t \in \mathbb{R}. \qquad (20)$$

This can be seen easily by recalling that for any $y \in \mathbb{R}^n \setminus \mathcal{A}$ there are functions $\phi \in C(\mathbb{R}^n)$ and $\sigma \in C(\mathbb{R}^n)$ such that $\sigma \geq 0$ in $\mathbb{R}^n$, $\sigma(y) > 0$, and $H[\phi] \leq -\sigma$ in $\mathbb{R}^n$ and using (18), with $H(x, p)$ replaced by $H(x, p) + \sigma(x)$.

4.3. Cauchy problem. To prove existence of a solution of (2) and (3), we may use the well-known formula

$$u(x,t) = \inf\left\{ \int_0^t L(\gamma(s),\dot\gamma(s))\mathrm{d}s + u_0(\gamma(0)) \mid \gamma \in \mathcal{C}(x,t) \right\} \quad \text{for } (x,t) \in \mathbb{R}^n \times (0,\infty). \tag{21}$$

We have to check if this formula really gives a solution of (2) and (3). For this the first thing to do is to check that the function u defined by (21) is a locally bounded function in $\mathbb{R}^n \times (0,\infty)$. Fix a subsolution $\phi \in C(\mathbb{R}^n)$ of $H[\phi] = 0$ in $\mathbb{R}^n$ and set $\phi_2(x) = \min\{\phi(x) - A, \phi_1(x)\}$, where $A > 0$ is a constant. If A is large enough, then ϕ_2 has the following properties: (a) $H[\phi_2] \le 0$ in $\mathbb{R}^n$ and (b) $\phi_2 \le u_0$ in $\mathbb{R}^n$. Then, for any $(x,t) \in \mathbb{R}^n \times (0,\infty)$ and $\gamma \in \mathcal{C}(x,t)$, we have

$$\phi_2(\gamma(t)) \le u_0(\gamma(0)) + \int_0^t L(\gamma,\dot\gamma)\mathrm{d}s,$$

from which we get $\phi_2(x) \le u(x,t)$.

In order to get a local upper bound of the function u defined by (21), we fix any $(x,t) \in \mathcal{A} \times (0,\infty)$ and choose a curve $\gamma \in \Gamma$ so that $\gamma(t) = x$. Existence of such a critical curve is guaranteed by Theorem 4.1. As before let $\phi \in C(\mathbb{R}^n)$ be a solution of $H[\phi] \le 0$ in $\mathbb{R}^n$. We have

$$\begin{aligned} u(x,t) &\le u_0(\gamma(0)) + \int_0^t L(\gamma,\dot\gamma)\mathrm{d}s \le u_0(\gamma(0)) + \phi(\gamma(t)) - \phi(\gamma(0)) \\ &\le \max_{\mathcal{A}} u_0 + 2\max_{\mathcal{A}} |\phi|. \end{aligned}$$

Since $H \in C(\mathbb{R}^n \times \mathbb{R}^n)$, for each $R > 0$ there are constants $\delta_R > 0$ and $C_R > 0$ such that $L(x,\xi) \le C_R$ for all $(x,\xi) \in B(0,R) \times B(0,\delta_R)$. Fix any $R > 0$ such that $\mathcal{A} \subset B(0,R)$ and any $x \in B(0,R)$. There is a $T_R > 0$, independent of x, and a curve (e.g., the line segment connecting a point in $\mathcal{A}$ and x) $\gamma_x \in C^1([0,T_R],\mathbb{R}^n)$ such that $\gamma_x(0) \in \mathcal{A}$, $\gamma_x(T_R) = x$, and $|\gamma_x(s)| \le R$, $|\dot\gamma_x(s)| \le \delta_R$ for all $s \in [0,T_R]$. Using the dynamic programming principle which states that for any $x \in \mathbb{R}^n$ and $t, s \in (0,\infty)$,

$$u(x,t+s) = \inf\left\{ \int_0^s L(\gamma(\tau),\dot\gamma(\tau))\mathrm{d}\tau + u(\gamma(0),t) \;\middle|\; \gamma \in \mathcal{C}(x,s) \right\}, \tag{22}$$

we find that

$$\begin{aligned} u(x,t+T_R) &\le \int_0^{T_R} L(\gamma_x(s),\dot\gamma_x(s))\mathrm{d}s + u(\gamma_x(0),t) \\ &\le C_R T_R + \max_{\mathcal{A}} u_0 + 2\max_{\mathcal{A}} |\phi| \quad \text{for all } t \ge 0. \end{aligned}$$

Noting that $u(x,t) \le u_0(x) + tL(x,0)$, we obtain for all $t > 0$,

$$u(x,t) \le \max\{\max_{B(0,R)} (|u_0| + T_R|L(\cdot,0)|), C_R T_R + \max_{\mathcal{A}} u_0 + 2\max_{\mathcal{A}} |\phi|\}.$$

Setting $u(x, 0) = u_0(x)$ for $x \in \mathbb{R}^n$ and making further standard estimates on u, we conclude the following theorem.

Theorem 4.3. *The function u belongs to Ψ_0 and moreover u is bounded and uniformly continuous on $B(0, R) \times [0, \infty)$ for any $R > 0$.*

Furthermore, using the dynamic programming principle (22), we have:

Theorem 4.4. *The function u is a solution of* (2) *and* (3).

Uniqueness of solutions of (2) and (3) follows from the following comparison theorem.

Theorem 4.5. *Let $u \in C(\mathbb{R}^n \times [0, \infty))$ and $v \in \Psi_0$ be a subsolution and a supersolution of* (2). *Assume that $u(\cdot, 0) \le v(\cdot, 0)$ in $\mathbb{R}^n$. Then $u \le v$ in $\mathbb{R}^n \times [0, \infty)$.*

An outline of the proof of this theorem goes like this. Let A and B be large positive constants. We set $\psi(x, t) = \phi_1(x) - At$ for $(x, t) \in \mathbb{R}^n \times [0, \infty)$. We may fix A so that ψ is a subsolution of (2). We set $u_B(x, t) = \min\{u(x, t), \psi(x, t) + B\}$ for $(x, t) \in \mathbb{R}^n \times (0, \infty)$ and observe that u_B is a subsolution of (2), that $u_B(\cdot, 0) \le v(\cdot, 0)$ in $\mathbb{R}^n$, and that for each $T > 0$,

$$\lim_{|x| \to \infty} \sup\{(u_B - v)(x, t) \mid t \in [0, T]\} = -\infty. \tag{23}$$

Applying the standard comparison result to u_B and v on the set $B(0, R) \times [0, R)$, with $R > 0$ sufficiently large, we find that $u_B \le v$ in $\mathbb{R}^n \times [0, R)$. Because of the arbitrariness of R, B, we conclude that $u \le v$ in $\mathbb{R}^n \times [0, \infty)$.

4.4. Asymptotic analysis.

4.4.1. Equilibrium points. A point y in the Aubry set $\mathcal{A}$ is called an *equilibrium point* if $\min_{p \in \mathbb{R}^n} H(y, p) = 0$ or equivalently $L(y, 0) = 0$. Under the assumption that $\mathcal{A}$ consists only of equilibrium points, the convergence assertion of Theorem 3.1 can be proved in an easy way compared to the general case. To see this, let $u \in \Psi_0$ be the solution of (2) and (3), and set $v_0(x) = \liminf_{t \to \infty} u(x, t)$ for $x \in \mathbb{R}^n$. We then observe in view of the convexity of $H(x, p)$ in p that v_0 is a solution of $H[v_0] = 0$ in $\mathbb{R}^n$. Also, we observe at least formally that $u_t \le 0$ in $\mathcal{A} \times (0, \infty)$, which can be stated correctly that the function $t \mapsto u(x, t)$ is nonincreasing in $(0, \infty)$ for any $x \in \mathcal{A}$. Now, by Theorem 4.3 and Dini's lemma, we see that the functions $u(\cdot, t)$ converge to v_0 uniformly on $\mathcal{A}$ as $t \to \infty$.

We take a small digression here and state a comparison theorem for (1) with $\Omega = \mathbb{R}^n$, a version of Theorem 2.3 for $\Omega = \mathbb{R}^n$.

Theorem 4.6. *Let $u, v \in C(\mathbb{R}^n)$ be a subsolution and a supersolution of $H = 0$ in $\mathbb{R}^n$, respectively. Assume that $u \le v$ on $\mathcal{A}$ and that $v \in \Phi_0$. Then $u \le v$ in $\mathbb{R}^n$.*

For the proof of this theorem, we may assume by a simple modification of ϕ_1 that ϕ_1 is a solution of $H[\phi_1] \le 0$ in $\mathbb{R}^n$. We then replace u by $\varepsilon\phi_1 + (1-\varepsilon)u$, with a small $\varepsilon \in (0, 1)$, so that we are in the situation that $\lim_{|x|\to\infty}(u - v)(x) = -\infty$, which allows us to work on a ball $B(0, R)$, with sufficiently large $R > 0$. Now, as in the proof of Theorem 2.3, we get $u \le v$ in $\mathbb{R}^n$. This is an outline of the proof of Theorem 4.6.

Back to the main theme, we use the same argument as the proof of Theorem 4.6 just outlined, to control the functions $u(\cdot, t)$ through their restrictions on $\mathcal{A}$ and to conclude the desired convergence of $u(\cdot, t)$ to v_0 in $\mathbb{R}^n$ as $t \to \infty$.

We remark that if $\mathcal{A}$ is a finite set, then all the points of $\mathcal{A}$ are equilibrium points. Also, if there is a function $f \in C(\mathbb{R}^n)$ such that $H(x, Df) \le \min_{p\in\mathbb{R}^n} H(x, p)$ for $x \in \mathbb{R}^n$ in the viscosity sense, then all the points of $\mathcal{A}$ are equilibrium points.

4.4.2. General case. We turn to the general case. Let $u \in C(\mathbb{R}^n \times [0, \infty))$ be a solution of (2). A formal calculation reveals that for any $\gamma \in \Gamma$, any $t, T \in [0, \infty)$ satisfying $t < T$, and any solution of $\phi \in C(\mathbb{R}^n)$ of $H[\phi] \le 0$ in $\mathbb{R}^n$,

$$\begin{aligned} u(\gamma(T), T) - u(\gamma(t), t) &= \int_t^T \big[Du(\gamma(s), s)\cdot\dot\gamma(s) + u_t(\gamma(s), s)\big]\,\mathrm{d}s \\ &= \int_t^T \big[Du(\gamma(s), s)\cdot\dot\gamma(s) - H(\gamma(s), Du(\gamma(s), s))\big]\,\mathrm{d}s \\ &\le \int_t^T L(\gamma(s), \dot\gamma(s))\mathrm{d}s = \phi(\gamma(T)) - \phi(\gamma(t)). \end{aligned}$$

Indeed, we have

Lemma 4.7. *Under the above assumptions, the function $t \mapsto u(\gamma(t), t) - \phi(\gamma(t))$ is nonincreasing on* $[0, \infty)$.

In what follows we denote by S_t, with $t \ge 0$, the semigroup generated by (2), i.e., the map $S_t\colon \Phi_0 \to \Phi_0$ defined by $S_t u_0 = u(\cdot, t)$, where u is the solution in Ψ_0 of (2) and (3).

The continuous dependence of the solution of (2) and (3) on the initial data can be stated as follows.

Theorem 4.8. *Let $f, g \in \Phi_0$ be a subsolution and a supersolution of $H = 0$ in $\mathbb{R}^n$, respectively. Assume that $f \le g$ in $\mathbb{R}^n$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $u_0, v_0 \in [f, g] \cap C(\mathbb{R}^n)$, if*

$$\max_{B(0,\delta^{-1})} (u_0 - v_0) \le \delta,$$

then

$$\sup_{(x,t)\in B(0,\varepsilon^{-1})\times[0,\infty)} (S_t u_0(x) - S_t v_0(x)) \le \varepsilon.$$

Here $[f, g]$ *denotes the space of those functions* $w : \mathbb{R}^n \to \mathbb{R}$ *which satisfy* $f \le w \le g$ *in* $\mathbb{R}^n$.

This theorem can be proved by an argument similar to the proof of Theorem 4.5, and we omit presenting it here.

For any $u_0 \in \Phi_0$, the *ω-limit set* $\omega(u_0)$ for the initial point u_0 is defined as the set consisting of those $w \in \Phi_0$ for which there exists a sequence $\{t_j\} \subset (0, \infty)$ diverging to infinity such that $S_{t_j} u_0 \to w$ in $C(\mathbb{R}^n)$ as $j \to \infty$. It is obvious from Theorem 4.3 that $\omega(u_0) \ne \emptyset$ for all $u_0 \in \Phi_0$. Another basic property of ω-limit sets, which follows from Theorems 4.8 and 4.3, is the following.

Lemma 4.9. *Let* $u_0 \in \Phi_0$ *and let* $\{t_j\}, \{r_j\} \subset (0, \infty)$ *be sequences diverging to infinity such that* $S_{t_j} u_0 \to v$ *and* $S_{t_j + r_j} u_0 \to w$ *in* $C(\mathbb{R}^n)$ *as* $j \to \infty$ *for some* $v, w \in \omega(u_0)$. *Then* $S_{r_j} v \to w$ *in* $C(\mathbb{R}^n)$ *as* $j \to \infty$.

So far, we have needed only the convexity of the Hamiltonian $H(x, p)$ in p, but not its strict convexity (15) although this point may not be clear because of the rough presentation. In the next lemma we need the strict convexity assumption (15), which guarantees that $L(x, \xi)$ and $D_\xi L(x, \xi)$ are continuous on the set $\{(x, \xi) \in \mathbb{R}^{2n} \mid L(x, \xi) < \infty\}$. The following lemma is an equivalent of [12, Lemma 5.2] and a key observation for the convergence proof.

Lemma 4.10. *Assume that* (15) *holds. Then there exist a* $\delta > 0$ *and a function* $\rho \in C([0, \infty))$, *with* $\rho(0) = 0$, *such that for any* $u_0 \in \Phi_0$, $\gamma \in \Gamma$, $\varepsilon \in (-\delta, \delta)$, *and* $t > 0$,

$$S_t u_0(\gamma(t)) \le u_0(\gamma(\varepsilon t)) + \int_{\varepsilon t}^{t} L(\gamma(s), \dot{\gamma}(s)) \mathrm{d}s + |\varepsilon t| \, \rho(|\varepsilon|).$$

Proof of Theorem 3.1 (c). We denote by $\omega(\Gamma)$ the set of all those curves γ to which there correspond a curve $\eta \in \Gamma$ and a sequence $\{t_j\}$ diverging to infinity such that $\eta(\cdot + t_j) \to \gamma$ in $C(\mathbb{R})$ as $j \to \infty$. We set $\mathcal{M} = \{\gamma(0) \mid \gamma \in \omega(\Gamma)\}$. We remark that any $\gamma \in \omega(\Gamma)$ is a critical curve for L. Consequently, we have $\mathcal{M} \subset \mathcal{A}$. Moreover, it is easily seen that for any two solutions $\phi, \psi \in C(\mathbb{R}^n)$ of $H = 0$ in $\mathbb{R}^n$, if $\phi \le \psi$ on $\mathcal{M}$, then $\phi \le \psi$ on $\mathcal{A}$.

As before we define the function $v_0 \in \Phi_0$ by

$$v_0(x) = \liminf_{t \to \infty} S_t u_0(x),$$

which is a solution of $H[v_0] = 0$ in $\mathbb{R}^n$.

We prove that $u(\cdot, t) \to v_0$ in $C(\mathbb{R}^n)$ as $t \to \infty$. To this end, it is enough to show that $w = v_0$ for all $w \in \omega(u_0)$. By the definition of v_0, we have $v_0 \le w$ in $\mathbb{R}^n$ for all $w \in \omega(u_0)$. Hence, recalling the proof in the case when $\mathcal{A}$ consists only of equilibrium points and using the remark made above, we find that it is enough to show that $w \le v_0$ in $\mathcal{M}$ for all $w \in \omega(u_0)$. (We omit here proving the formula for v_0 in the theorem.)

Fix any $w \in \omega(u_0)$ and $y \in \mathcal{M}$. Choose a curve $\gamma \in \Gamma$ and sequences $\{a_j\}$, $\{b_j\} \subset (0, \infty)$ diverging to infinity so that, as $j \to \infty$, $\gamma(a_j) \to y$ and $S_{b_j} w \to w$ in $C(\mathbb{R}^n)$. Existence of such a sequence $\{b_j\}$ is assured by Lemma 4.9. We may assume that $c_j := a_j - b_j \to \infty$ as $j \to \infty$. We fix any $s \geq 0$ and apply Lemma 4.10, with w and $\gamma(\,\cdot + c_j)$ in place of u_0 and γ, respectively, and with $t = b_j$ and $\varepsilon = s/b_j$, to obtain for sufficiently large j,

$$S_{b_j} w(\gamma(b_j + c_j)) - w(\gamma(s + c_j)) \leq v_0(\gamma(b_j + c_j)) - v_0(\gamma(s + c_j)) + s\rho(s/b_j),$$

where $\rho \in C([0, \infty))$ is the function from Lemma 4.10. Sending $j \to \infty$ yields

$$w(y) - v_0(y) \leq w(\eta(s)) - v_0(\eta(s)) \quad \text{for all } s \geq 0 \tag{24}$$

and for some $\eta \in \omega(\Gamma)$.

The final step is to show that

$$\liminf_{t \to \infty} [w(\eta(t)) - v_0(\eta(t))] \leq 0, \tag{25}$$

which yields, together with (24), $w(y) \leq v_0(y)$. To do this, we choose sequences $\{t_j\}$, $\{\tau_j\} \subset (0, \infty)$ diverging to infinity so that, as $j \to \infty$, $S_{t_j} u_0(\eta(0)) \to v_0(\eta(0))$ and $S_{t_j + \tau_j} u_0 \to w$ in $C(\mathbb{R}^n)$. We calculate by using Lemma 4.7 that

$$\begin{aligned} w(\eta(\tau_j)) - v_0(\eta(\tau_j)) &\leq |w(\eta(\tau_j)) - S_{t_j + \tau_j} u_0(\eta(\tau_j))| + S_{t_j + \tau_j} u_0(\eta(\tau_j)) - v_0(\eta(\tau_j)) \\ &\leq \max_{\mathcal{A}} |w - S_{t_j + \tau_j} u_0| + S_{t_j} u_0(\eta(0)) - v_0(\eta(0)). \end{aligned}$$

Sending $j \to \infty$, we get

$$\limsup_{j \to \infty} \big[w(\eta(\tau_j)) - v_0(\eta(\tau_j)) \big] \leq 0,$$

which shows that (25) is valid. □

References

[1] Alvarez, O., Bounded-from-below viscosity solutions of Hamilton-Jacobi equations. *Differential Integral Equations* **10** (3) (1997), 419–436.

[2] Barles, G., *Solutions de viscosité des équations de Hamilton-Jacobi*. Math. Appl. (Berlin) 17, Springer-Verlag, Paris 1994.

[3] Bardi, M., and Capuzzo-Dolcetta, I., *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*. With appendices by Maurizio Falcone and Pierpaolo Soravia, Systems Control Found. Appl. Birkhäuser Boston, Inc., Boston, MA, 1997.

[4] Barron, E. N., and Jensen, R., Semicontinuous viscosity solutions for Hamilton-Jacobi equations with convex Hamiltonians. *Comm. Partial Differential Equations* **15** (12) (1990), 1713–1742.

[5] Barron, E. N., and Jensen, R., Optimal control and semicontinuous viscosity solutions. *Proc. Amer. Math. Soc.* **113** (2) (1991), 397–402.

[6] Barles, G., and Souganidis, P. E., On the large time behavior of solutions of Hamilton-Jacobi equations. *SIAM J. Math. Anal.* **31** (4) (2000), 925–939.

[7] Barles, G., and Souganidis, P. E., Space-time periodic solutions and long-time behavior of solutions to quasi-linear parabolic equations, *SIAM J. Math. Anal.* **32** (6) (2001), 1311–1323.

[8] Crandall, M. G., Ishii, H., and Lions, P.-L., Uniqueness of viscosity solutions of Hamilton-Jacobi equations revisited. *J. Math. Soc. Japan* **39** (4) (1987), 581–596.

[9] Crandall, M. G., Ishii, H., and Lions, P.-L., User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.* **27** (1992), 1–67.

[10] Crandall, M. G., and Lions, P.-L., Condition d'unicité pour les solutions generalisées des équations de Hamilton-Jacobi du premier ordere. *C. R. Acad. Sci. Paris Sér. I Math.* **292** (1981), 183–186.

[11] Crandall, M. G., and Lions, P.-L., Viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.* **277** (1983), 1–42.

[12] Davini, A., and Siconolfi, A., A generalized dynamical approach to the large time behavior of solutions of Hamilton-Jacobi equations. Preprint, 2005.

[13] Evans, L. C., Periodic homogenisation of certain fully nonlinear partial differential equations. *Proc. Roy. Soc. Edinburgh Sect. A* **120** (3–4) (1992), 245–265.

[14] Evans, L. C., A survey of partial differential equations methods in weak KAM theory. *Comm. Pure Appl. Math.* **57** (4) (2004), 445–480.

[15] Fathi, A., Théorème KAM faible et théorie de Mather pour les systèmes lagrangiens. *C. R. Acad. Sci. Paris Sér. I Math.* **324** (9) (1997), 1043–1046.

[16] Fathi, A., Sur la convergence du semi-groupe de Lax-Oleinik. *C. R. Acad. Sci. Paris Sér. I Math.* **327** (3) (1998), 267–270.

[17] Fathi, A., *Weak KAM theorem in Lagrangian dynamics*. To appear.

[18] Fathi, A., and Siconolfi, A., Existence of C^1 critical subsolutions of the Hamilton-Jacobi equation. *Invent. Math.* **155** (2) (2004), 363–388.

[19] Fathi, A., and Siconolfi, A., PDE aspects of Aubry-Mather theory for quasiconvex Hamiltonians. *Calc. Var. Partial Differential Equations* **22** (2) (2005), 185–228.

[20] Fujita, Y., Ishii, H., and Loreti, P., Asymptotic solutions of viscous Hamilton-Jacobi equations with Ornstein-Uhlenbeck operator. *Comm. Partial Differential Equations* **31** (6) (2006), 827–848 .

[21] Fujita, Y., Ishii, H., and Loreti, P., Asymptotic solutions of Hamilton-Jacobi equations in Euclidean *n* space. *Indiana Univ. Math. J.*, to appear.

[22] Ishii, H., A generalization of a theorem of Barron and Jensen and a comparison theorem for lower semicontinuous viscosity solutions. *Proc. Roy. Soc. Edinburgh Sect. A* **131** (1) (2001), 137–154.

[23] Ishii, H., Asymptotic solutions for large time of Hamilton-Jacobi equations in Euclidean *n* space. Preprint.

[24] Lions, P.-L., *Generalized solutions of Hamilton-Jacobi equations*. Res. Notes in Math. 69, Pitman, Boston, MA, London 1982.

[25] Lions, P.-L., Papanicolaou, G., and Varadhan, S., Homogenization of Hamilton-Jacobi equations. Unpublished preprint.

[26] Namah, G., and Roquejoffre, J.-M., Remarks on the long time behaviour of the solutions of Hamilton-Jacobi equations. *Comm. Partial Differential Equations* **24** (5–6) (1999), 883–893.

[27] Rockafellar, T., *Convex Analysis*. Princeton Math. Ser. 28, Princeton University Press, Princeton, 1970.

[28] Roquejoffre, J.-M., Convergence to steady states or periodic solutions in a class of Hamilton-Jacobi equations. *J. Math. Pures Appl.* (9) **80** (1) (2001), 85–104.

Department of Mathematics, Faculty of Education and Integrated Arts and Sciences,
Waseda University, Nishi-waseda, Shinjuku-ku, Tokyo, 169-8050 Japan
E-mail: ishii@edu.waseda.ac.jp