The weak-coupling limit of large classical and quantum systems

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Abstract. In this contribution we illustrate the delicate transition from the microscopic description of a particle system, given in terms of fundamental equations as the Newton or the Schrödinger equation, to the reduced kinetic picture, given in terms of the Boltzmann and Landau equations which are obtained under suitable scaling limits. Special emphasis is given to the so called weak-coupling limit.

The content of the lecture is mostly devoted to the very many open problems, rather than to the few known results.

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1. The Boltzmann and Landau equations

The present section is largely discursive: its scope is the heuristic introduction of the Boltzmann and Landau equations on the basis of physical arguments.

In 1872 Ludwig Boltzmann, starting from the mathematical model of elastic balls and using mechanical and statistical considerations, established an evolution equation to describe the behavior of a rarefied gas. The starting point of the Boltzmann analysis is to renounce to study the behavior of a gas in terms of the detailed motion of the molecules which constitute it because of their huge number. It is rather better to investigate a function $f(x, v)$ which is the probability density of a given particle, where x and v denote position and velocity of such a particle. Actually $f(x, v)dxdv$ is often confused with the fraction of molecules falling in the cell of the pase space of size $dx\,dv$ around x, v. The two concept are not exactly the same but they are asymptotically equivalent (when the number of particles is diverging) if a law of large numbers holds.

The Boltzmann equation is the following:

$$(\partial_t + v \cdot \nabla_x) f = Q(f, f) \tag{1.1}$$

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where Q, the collision operator, is defined for $\lambda > 0$ by

$$Q(f, f) = \lambda^{-1} \int dv_1 \int_{S_+} dn\, (v - v_1) \cdot n\, [f(x, v') f(x, v_1') - f(x, v) f(x, v_1)] \tag{1.2}$$

and

$$v' = v - n[n \cdot (v - v_1)], \quad v_1' = v_1 + n[n \cdot (v - v_1)]. \tag{1.3}$$

Also n (the impact parameter) is a unitary vector and $S_+ = \{n \mid n \cdot (v - v_1) \geq 0\}$. Note that v', v_1' are the outgoing velocities after a collision of two elastic balls with incoming velocities v and v_1 and centers x and $x + dn$, being d the diameter of the spheres. Obviously the collision takes place if $n \cdot (v - v_1) \geq 0$. Equations (1.3) are consequence of the energy, momentum and angular momentum conservation. Note also that d does not enter in equation (1.1) as a parameter.

As fundamental features of equation (1.1) we have the conservation in time of the following five quantities

$$\iint dx dv f(x, v; t) v^\alpha \tag{1.4}$$

with $\alpha = 0, 1, 2$ expressing conservation of the probability, momentum and energy.

Moreover Boltzmann introduced the (kinetic) entropy defined by

$$H(f) = \int dx \int dv f \log f(x, v) \tag{1.5}$$

and proved the famous H-theorem asserting the decreasing of $H(f(t))$ along the solutions to equation (1.1).

Finally, in case of bounded domains, the distribution defined for $\beta > 0$:

$$M(v) = \text{const}\, e^{-\beta v^2},$$

called Maxwellian distribution, is stationary for the evolution given by equation (1.1). In addition M minimizes H among all distributions with zero mean velocity, and given energy.

In conclusion Boltzmann was able to introduce an evolutionary equation with the remarkable properties of expressing mass, momentum, energy conservations, but also the trend to the thermal equilibrium. In other words he tried to conciliate the Newton laws with the second principle of Thermodynamics.

Boltzmann's heuristic argument in deriving equation (1.1) is, roughly speaking, the following. The molecular system we are considering consists of N identical particles of diameter d in the whole space $\mathbb{R}^3$ and we denote by $x_1, v_1, \ldots, x_N, v_N$ a state of the system, where x_i and v_i indicate the position and the velocity of the particle i. The particles cannot overlap, that is the centers of two particles cannot be at distance smaller than the diameter d.

The particles are moving freely up to the first contact instant, that is the first time in which two particles arrive at distance d. Then the pair interacts performing

an elastic collision. This means that they change instantaneously their velocities, according to the conservation of the energy, linear and angular momentum. After the first collision the system goes on by iterating the procedure. Here we neglect triple collisions because unlikely. The evolution equation for a tagged particle is of the form

$$(\partial_t + v \cdot \nabla_x) f = \text{Coll} \tag{1.6}$$

where Coll denotes the variation of f due to the collisions. We have

$$\text{Coll} = G - L \tag{1.7}$$

where L and G (loss and gain term respectively) are the negative and positive contribution to the variation of f due to the collisions. More precisely $Ldxdvdt$ is the probability of our test particle to disappear from the cell $dxdv$ of the phase space because of a collision in the time interval $(t, t + dt)$ and $Gdxdvdt$ is the probability to appear in the same time interval for the same reason. Let us now consider the sphere of center x with radius d and a point $x + dn$ over the surface, where n denotes the generic unit vector. Consider also the cylinder with base area $dS = d^2 dn$ and height $|V|dt$ along the direction of $V = v_2 - v$.

Then a given particle (say particle 2) with velocity v_2, can contribute to L because it can collide with our test particle in the time dt, provided it is localized in the cylinder and if $V \cdot n \leq 0$. Therefore the contribution to L due to the particle 2 is the probability of finding such a particle in the cylinder (conditioned to the presence of the first particle in x). This quantity is $f_2(x, v, x + nd, v_2)|(v_2 - v) \cdot n|d^2 dn dv_2 dt$, where f_2 is the joint distribution of two particles. Integrating in dn and dv_2 we obtain that the total contribution to L due to any predetermined particle is:

$$d^2 \int dv_2 \int_{S_-} dn f_2(x, v, x + nd, v_2)|(v_2 - v) \cdot n| \tag{1.8}$$

where S_- is the unit hemisphere $(v_2 - v) \cdot n < 0$. Finally we obtain the total contribution multiplying by the total number of particles:

$$L = (N - 1)d^2 \int dv_2 \int_{S_-} dn f_2(x, v, x + nd, v_2)|(v_2 - v) \cdot n|. \tag{1.9}$$

The gain term can be derived analogously by considering that we are looking at particles which have velocities v and v_2 after the collisions so that we have to integrate over the hemisphere $S_+ = (v_2 - v) \cdot n > 0$:

$$G = (N - 1)d^2 \int dv_2 \int_{S_+} dn f_2(x, v, x + nd, v_2)|(v_2 - v) \cdot n|. \tag{1.10}$$

Summing G and $-L$ we get

$$\text{Coll} = (N - 1)d^2 \int dv_2 \int dn f_2(x, v, x + nd, v_2)(v_2 - v) \cdot n. \tag{1.11}$$

which, however, is not a very useful expression because the time derivative of f is expressed in term of another object namely f_2. An evolution equation for f_2 will imply f_3, the joint distribution of three particles and so on up to arrive to the total particle number N. Here the basic Boltzmann's main assumption enters, namely that two given particles are uncorrelated if the gas is rarefied, namely:

$$f(x, v, x_2, v_2) = f(x, v) f(x_2, v_2). \tag{1.12}$$

Condition (1.12), called *propagation of chaos*, seems contradictory at a first sight: if two particles collide, correlations are created. Even though we could assume equation (1.12) at some time, if the test particle collides with the particle 2, such an equation cannot be satisfied anymore after the collision.

Before discussing the propagation of chaos hypothesis, we first analyze the size of the collision operator. We remark that, in practical situations for a rarefied gas, the combination $Nd^3 \approx 10^{-4}\text{cm}^3$ (that is the volume occupied by the particles) is very small, while $Nd^2 = O(1)$. This implies that $G = O(1)$. Therefore, since we are dealing with a very large number of particles we are tempted to perform the limit $N \to \infty$ and $d \to 0$ in such a way that $d^2 = O(N^{-1})$. As a consequence the probability that two tagged particles collide (which is of the order of the surface of a ball, that is $O(d^2)$), is negligible. However the probability that a given particle performs a collision with any one of the remaining $N-1$ particles (which is $O(Nd^2) = O(1)$) is not negligible. Condition (1.12) is referring to two preselected particles (say particle 1 and particle 2) so that it is not unreasonable to conceive that it holds in the limiting situation in which we are working.

However we cannot insert (1.12) in (1.11) because this latter equation refer both to instants before and after the collision and, if we know that a collision took place, we certainly cannot invoke (1.12). Hence we assume (1.12) in the loss term and work over the gain term to keep advantage of the factorization property which will be assumed *only* before the collision.

Coming back to equation (1.10) for the outgoing pair velocities v, v_2 (satisfying the condition $(v_2 - v) \cdot n > 0$) we make use of the continuity property

$$f_2(x, v, x + nd, v_2) = f_2(x, v', x + nd, v_2') \tag{1.13}$$

where the pair v', v_2' is pre-collisional. On f_2 expressed before the collision we can reasonably apply condition (1.12) obtaining:

$$\begin{aligned} G - L = (N-1)r^2 \int dv_2 \int_{S_-} dn (v - v_2) \cdot n \\ \cdot [f(x, v') f(x - nd, v_2') - f(x, v) f(x + nd, v_2)] \end{aligned} \tag{1.14}$$

after a change $n \to -n$ in the Gain term. This transforms the pair v', v_2' from a pre-collisional to a post-collisional pair.

Finally, in the limit $N \to \infty$, $r \to 0$, $Nd^2 = \lambda^{-1}$ we find equation (1.1) where Q, the collision operator, has the form (1.2). The parameter λ, called *mean*

free path, represents, roughly speaking, the typical length a particle can cover without undergoing any collision.

Equation (1.1) is has a statistical nature and it is not equivalent to the Hamiltonian dynamics from which it has been derived. In particular, due to the H-Theorem, it is not time reversal.

The heuristic arguments we have developed so far can be extended to different potentials than that of the hard-sphere systems. If the particles interact via a two-body interaction $\phi = \phi(x)$ the resulting Boltzmann equation is equation (1.1) with

$$Q(f, f) = \int dv_1 \int_{S_+} dn B(v - v_1; n) [f' f_1' - f f_1], \tag{1.15}$$

where we are using the usual short hand notation:

$$f' = f(x, v'), \quad f_1' = f(x, v_1'), \quad f = f(x, v), \quad f_1 = f(x, v_1) \tag{1.16}$$

and $B = B(v - v_1; n)$ is a suitable function of the relative velocity and the impact parameter, proportional to the cross-section relative to the potential ϕ. Another equivalent, some times convenient way to express Q is

$$Q(f, f) = \int dv_1 \int dv' \int dv_1' W(v, v_1 | v', v_1') [f' f_1' - f f_1] \tag{1.17}$$

with

$$W(v, v_1 | v', v_1') = w(v, v_1 | v', v_1') \cdot \delta(v + v_1 - v' + v_1') \delta\left(\frac{1}{2}(v^2 + v_1^2 - (v')^2 + (v_1')^2)\right). \tag{1.18}$$

and w a suitable kernel. All the qualitative properties as the conservation laws and the H-theorem are obviously still valid.

The arguments we have used in deriving the Boltzmann equation are delicate and require a more rigorous and deeper analysis. If we want that the Boltzmann equation is not a phenomenological model, derived by assumptions *ad hoc* and justified by its practical relevance, but rather a consequence of a mechanical model, we must derive it rigorously from a logical and mathematical viewpoint. In particular the propagation of chaos should be not an hypothesis but the statement of a theorem.

Many scientists, among them Loschmidt, Zermelo and Poincaré, outlined inconsistencies between the irreversibility of the equation and the reversible character of the Hamiltonian dynamics. Boltzmann argued the statistical nature of his equation and his answer to the irreversibility paradox was that *most* of the configurations behave as expected by the thermodynamical laws. However he did not have the probabilistic tools for formulating in a precise way the statements of which he had a precise intuition. In 1949 H. Grad [25] stated clearly the limit $N \to \infty$, $d \to 0$, $Nd^2 \to$ const, where N is the number of particles and d is the diameter of the molecules, in which the Boltzmann equation is expected to hold. This limit is usually called the Boltzmann-Grad (or low-density limit).

The problem of a rigorous derivation of the Boltzmann equation was an open and challenging problem for a long time. O. E. Lanford [31] showed that, although for a very short time, the Boltzmann equation can be derived starting from the mechanical model of the hard-sphere system. The proof has a deep content but is relatively simple from a technical view point. Later on [28] it has been proved that this technique can be adapted to prove a result holding globally in time, but for the special situation of a rare claud of gas expanding in the vacuum.

We address the reader to references [31], [14] and [19] for a deeper discussion on the validation problem of the Boltzmann equation. We also warmly suggest the monograph [13] for a critical and historical discussion on the Boltzmann equation and the scientist who conceived it.

A preliminary problem to the validation of the Boltzmann equation for an arbitrary time interval, still open in general, is the construction of a global solution, hopefully unique. See [14] and [42] for the state of art of existence problems at present times. We just mention that the most general result we have up to now is due to Di Perna and Lions [16] who showed the existence of suitable weak solutions to equation (1.1). However we still do not know whether such solutions, which preserve mass, momentum and satisfy the H-theorem, are unique and preserve also the energy.

The Boltzmann equation works for rarefied gas, however one can ask whether a useful kinetic picture can be invoked for dense gas. Here we want to describe a situation in which the gas particles are weakly interacting, but $N = O(r^{-3})$ being $r \ll 1$ the interaction length of the particles. To express the weakness of the interaction, we assume that the two-particle potential is $O(\sqrt{r})$. In this case we want to compute the total momentum variation for a unit time. Note that the force is $O\left(\frac{1}{\sqrt{r}}\right)$ but acts on the time interval $O(r)$. The momentum variation due to the single scattering is therefore $O(\sqrt{r})$. The number of particles met by a test particles is $O\left(\frac{1}{r}\right)$. Hence the total momentum variation for unit time is $O\left(\frac{1}{\sqrt{r}}\right)$. However this variation, in case of homogeneous gas and symmetric force, should be zero in the average. If we compute the variance, it should be $\frac{1}{r}O(\sqrt{r})^2 = O(1)$. As a consequence of this central limit type of argument we expect that the kinetic equation which holds in the limit (if any), should be a diffusion equation in velocity variable.

A more convincing argument will be presented in the next section. For the moment let us now argue at level of kinetic equation. Consider the collision operator in the form (1.17). Suppose that $\varepsilon > 0$ is a small parameter. To express the fact that the transferred momentum is small, we rescale w as $\frac{1}{\varepsilon^3} w\left(\frac{p}{\varepsilon}\right)$. In addition we also rescale the mean-free path inverse by a factor $\frac{1}{\varepsilon}$ to take into account the high density situation. The collision operator becomes:

$$Q_\varepsilon(f, f) = \frac{1}{\varepsilon^4} \int dv_1 \int dp w\left(\frac{p}{\varepsilon}\right) \delta(p^2 + (v - v_1) \cdot p)[f' f_1' - f f_1] \qquad (1.19)$$

$$
\begin{aligned}
&= \frac{1}{2\pi\varepsilon^2}\int dv_1 \int dp\ w(p) \int_{-\infty}^{+\infty} ds e^{is(p^2\varepsilon+(v-v_1)\cdot p)} \\
&\quad \cdot [f(v+\varepsilon p) f(v_1-\varepsilon p) - f(v) f(v_1)] \\
&= \frac{1}{2\pi\varepsilon}\int dv_1 \int dp\ w(p) \int_0^1 d\lambda \int_{-\infty}^{+\infty} ds e^{is(p^2\varepsilon+(v-v_1)\cdot p)} \\
&\quad \cdot p(\nabla_v - \nabla_{v_1}) f(v+\varepsilon\lambda p) f(v_1-\varepsilon\lambda p).
\end{aligned}
$$

Here the smooth function w, which modulates the collision, is assumed depending only on p trough its modulus. The δ appearing in equation (1.19) expresses the energy conservation.

To outline the behavior of $Q_\varepsilon(f, f)$ in the limit $\varepsilon \to 0$, we introduce a test function φ for which, after a change of variables (here $(\cdot,\cdot)$ denotes the scalar product in $L_2(v)$):

$$
\begin{aligned}
(\varphi, Q_\varepsilon(f,f)) &= \frac{1}{2\pi\varepsilon}\int dv \int dv_1 \int dp\ w(p) \int_0^1 d\lambda \int_{-\infty}^{+\infty} ds e^{is(p^2(\varepsilon-2\varepsilon\lambda)+(v-v_1)\cdot p)} \\
&\quad \cdot \varphi(v-\varepsilon\lambda p)\ p\cdot(\nabla_v - \nabla_{v_1}) f f_1 \\
&= \frac{1}{2\pi\varepsilon}\int dv \int dv_1 \int dp w(p) \int_0^1 d\lambda \int_{-\infty}^{+\infty} ds e^{is(v-v_1)\cdot p} \\
&\quad \cdot [\varphi(v) + \varepsilon p\cdot\nabla_v\varphi(v)] p\cdot(\nabla_v - \nabla_{v_1}) f f_1 \\
&\quad + \frac{1}{2\pi}\int dv \int dv_1 \int dp\ w(p) \int_{-\infty}^{+\infty} ds e^{is(v-v_1)\cdot p)}\varphi(v) \\
&\quad \cdot isp^2 \int_0^1 d\lambda(1-2\lambda)\ p\cdot(\nabla_v - \nabla_{v_1}) f f_1 + O(\varepsilon).
\end{aligned}
\tag{1.20}
$$

Note now that the term $O(\varepsilon^{-1})$ vanishes because of the symmetry $p \to -p$ (w is even). Also the imaginary part of the $O(1)$ term is vanishing, being null the integral in $d\lambda$. As a result:

$$
\begin{aligned}
(\varphi, Q_\varepsilon(f,f)) = \frac{1}{2\pi}\int dv \int dv_1 \int dp\ w(p) \int_{-\infty}^{+\infty} ds\ e^{is(v-v_1)\cdot p)} \\
\cdot p\cdot\nabla_v\varphi\ p\cdot(\nabla_v - \nabla_{v_1}) f f_1 + O(\varepsilon).
\end{aligned}
\tag{1.21}
$$

Therefore we have recovered the kinetic equation (1.1) with a new collision operator

$$
Q_L(f,f) = \int dv_1 \nabla_v\, a(\nabla_v - \nabla_{v_1}) f f_1, \tag{1.22}
$$

where $a = a(v - v_1)$ denotes the matrix

$$
a_{i,j}(V) = \int dp\ w(p)\ \delta(V\cdot p)\ p_i p_j. \tag{1.23}
$$

This matrix can be handled in a better way by introducing polar coordinates:

$$a_{i,j}(V) = \frac{1}{|V|}\int dp\ |p|w(p)\ \delta(\hat{V}\cdot\hat{p})\ \hat{p}_i\hat{p}_j \cdot \frac{B}{|V|}\int d\hat{p}\ \delta(\hat{V}\cdot\hat{p})\ \hat{p}_i\hat{p}_j, \tag{1.24}$$

where $\hat{V}$ and $\hat{P}$ are the versor of V and p respectively and

$$B = \int_0^{+\infty} drr^3 w(r). \tag{1.25}$$

Note that B is the only parameter describing the interaction appearing in the equation. Finally a straightforward computation yields:

$$a_{i,j}(V) = \frac{B}{|V|}(\delta_{i,j} - \hat{V}_i\hat{V}_j). \tag{1.26}$$

The collision operator Q_L has been introduced by Landau ([32]) for the study of a weakly interacting dense plasma. Note that the qualitative properties of the solutions to the Landau equation are the same as for the Boltzmann equation as regards the basic conservation laws and the H-theorem.

A rigorous derivation of the Landau equation starting from the Boltzmann equation in the grazing collision limit (that is what we presented here at a formal level), has been obtained in [1], [24] and [43] for spatially homogeneous solutions. The (diverging) asymptotics for the Coulomb forces is discussed in [15] and in [43]. However, in the present lecture, we are interested in deriving the Landau equation in terms of particle systems. In the next section we present a formal derivation outlining the difficulties in trying a rigorous proof.

2. Weak-coupling limit for classical systems

We consider a classical system of N identical particles of unitary mass. Positions and velocities are denoted by $q_1, \ldots, q_N$ and $v_1, \ldots, v_N$. The Newton equations reads as:

$$\frac{d}{d\tau}q_i = v_i, \quad \frac{d}{d\tau}v_i = \sum_{\substack{j=1,\ldots,N:\\ j\neq i}} F(q_i - q_j). \tag{2.1}$$

Here $F = -\nabla\phi$ denotes the interparticle (conservative) force, ϕ the two-body interaction potential and τ the time.

We are interested in a situation where the number of particles N is very large and the interaction quite moderate. In addition we look for a reduced or macroscopic description of the system. Namely if q and τ refer to the system seen in a microscopic

scale, we introduce $\varepsilon > 0$ a small parameter expressing the ratio between the macro and microscales. Indeed it is often convenient to rescale equation (2.1) in terms of the macroscopic variables

$$x = \varepsilon q, \quad t = \varepsilon \tau$$

whenever the physical variables of interest are varying on such scales and are almost constant on the microscopic scales. Therefore, rescaling the potential according to

$$\phi \to \sqrt{\varepsilon}\phi, \tag{2.2}$$

system (2.1), in terms of the (x, t) variables, becomes

$$\frac{d}{dt}q_i = v_i, \quad \frac{d}{dt}v_i = -\frac{1}{\sqrt{\varepsilon}} \sum_{\substack{j=1,\dots,N: \\ j\neq i}} \nabla\phi\Big(\frac{x_i - x_j}{\varepsilon}\Big). \tag{2.3}$$

Note that the velocities are automatically unscaled. Moreover we also assume that $N = O(\varepsilon^{-3})$, namely the density is $O(1)$.

Let $W^N = W^N(X_N, V_N)$ be a probability distribution on the phase space of the system. Here (X_N, V_N) denote the set of positions and velocities:

$$X_N = x_1, \dots, x_N, \quad V_N = v_1, \dots, v_N.$$

Then from equations (2.3) we obtain the following Liouville equation:

$$(\partial_t + V_N \cdot \nabla_N) W^N(X_N, V_N) = \frac{1}{\sqrt{\varepsilon}} \big(T_N^\varepsilon W^N\big)(X_N, V_N), \tag{2.4}$$

where $V_N \cdot \nabla_N = \sum_{i=1}^N v_i \cdot \nabla_{x_i}$ and $(\partial_t + V_N \cdot \nabla_N)$ is the usual free stream operator. Also, we have introduced the operator

$$(T_N^\varepsilon W^N)(X_N, V_N) = \sum_{0<k<\ell\leq N} (T_{k,\ell}^\varepsilon W^N)(X_N, V_N), \tag{2.5}$$

with

$$T_{k,\ell}^\varepsilon W^N = \nabla\phi\Big(\frac{x_k - x_\ell}{\varepsilon}\Big) \cdot (\nabla_{v_k} - \nabla_{v_\ell}) W^N. \tag{2.6}$$

To investigate the limit $\varepsilon \to 0$ it is convenient to introduce the BBKGY hierarchy for the j- particle distributions defined as

$$\begin{aligned} f_j^N(X_j, V_j) = \int dx_{j+1} \dots \int dx_N \int dv_{j+1} \dots \int dv_N \\ \cdot W^N(X_j, x_{j+1}, \dots, x_N; V_j, v_{j+1}, \dots, v_N) \end{aligned} \tag{2.7}$$

for $j = 1, \dots, N-1$. Obviously, we set $f_N^N = W^N$. Note that BBKGY stands for Bogoliubov, Born, Green, Kirkwood and Yvon, the names of physicists who introduced independently this system of equations (see e.g. [2] and [19]).

From now on we shall suppose that, due to the fact that the particles are identical, the objects which we have introduced (W^N, f_j^N) are all symmetric in the exchange of particles.

A partial integration of the Liouville equation (2.4) and standard manipulations give us the following hierarchy of equations (for $1 \le j \le N$):

$$\Big(\partial_t + \sum_{k=1}^{j} v_k \cdot \nabla_k\Big) f_j^N = \frac{1}{\sqrt{\varepsilon}} T_j^\varepsilon f_j^N + \frac{N-j}{\sqrt{\varepsilon}} C_{j+1}^\varepsilon f_{j+1}^N. \tag{2.8}$$

The operator C_{j+1}^ε is defined as

$$C_{j+1}^\varepsilon = \sum_{k=1}^{j} C_{k,j+1}^\varepsilon, \tag{2.9}$$

and

$$\begin{aligned} &C_{k,j+1}^\varepsilon f_{j+1}(x_1, \dots, x_j; v_1, \dots, v_j) \\ &\quad = -\int dx_{j+1} \int dv_{j+1} F\Big(\frac{x_k - x_\ell}{\varepsilon}\Big) \nabla_{v_k} f_{j+1}(x_1, x_2, \dots, x_{j+1}; v_1, \dots, v_{j+1}). \end{aligned} \tag{2.10}$$

$C_{k,j+1}^\varepsilon$ describes the "collision" of particle k, belonging to the j-particle subsystem, with a particle outside the subsystem, conventionally denoted by the number $j+1$ (this numbering uses the fact that all particles are identical). The total operator C_{j+1}^ε takes into account all such collisions. The dynamics of the j-particle subsystem is governed by three effects: the free-stream operator, the collisions "inside" the subsystem (the T term), and the collisions with particles "outside" the subsystem (the C term).

We finally fix the initial value $\{f_j^0\}_{j=1}^N$ of the solution $\{f_j^N(t)\}_{j=1}^N$ assuming that $\{f_j^0\}_{j=1}^N$ is factorized, that is, for all $j = 1, \dots, N$

$$f_j^0 = f_0^{\otimes j}, \tag{2.11}$$

where f_0 is a given one-particle distribution function. This means that the states of any pair of particles are statistically uncorrelated at time zero. Of course such a statistical independence is destroyed at time $t > 0$. Dynamics creates correlations and equation (2.8) shows that the time evolution of f_1^N is determined by the knowledge of f_2^N which turns out to be dependent on f_3^N and so on. However, since the interaction between two given particle is going to vanish in the limit $\varepsilon \to 0$, we can hope that such statistical independence is recovered in the same limit. Note that the physical meaning of the propagation of chaos here is quite different from that arising in the contest of the Boltzmann equation. Here two particles can interact but the effect of the collision is small, while in a low-density regime the effect of a collision between two given particles is large but quite unlikely.

Therefore we expect that in the limit $\varepsilon \to 0$ the one-particle distribution function f_1^N converges to the solution of a suitable nonlinear kinetic equation f which we are going to investigate.

If we expand $f_j^N(t)$ as a perturbation of the free flow $S(t)$ defined as

$$(S(t)f_j)(X_j, V_j) = f_j(X_j - V_j t, V_j), \tag{2.12}$$

we find

$$\begin{aligned} f_j^N(t) = S(t)f_j^0 + \frac{N-j}{\sqrt{\varepsilon}} \int_0^t S(t-t_1) C_{j+1}^{\varepsilon} f_{j+1}^N(t_1) dt_1 \\ + \frac{1}{\sqrt{\varepsilon}} \int_0^t S(t-t_1) T_j^{\varepsilon} f_j^N(t_1) dt_1. \end{aligned} \tag{2.13}$$

We now try to keep information on the limit behavior of $f_j^N(t)$. Assuming for the moment that the time evolved j-particle distributions $f_j^N(t)$ are smooth (in the sense that the derivatives are uniformly bounded in ε), then

$$\begin{aligned} &C_{j+1}^{\varepsilon} f_{j+1}^N(X_j; V_j; t_1) \\ &\quad = -\varepsilon^3 \sum_{k=1}^{j} \int dr \int dv_{j+1} F(r) \cdot \nabla_{v_k} f_{j+1}(X_j, x_k - \varepsilon r; V_j, v_{j+1}, t_1). \end{aligned} \tag{2.14}$$

Assuming now, quite reasonably, that

$$\int dr F(r) = 0, \tag{2.15}$$

we find that

$$C_{j+1}^{\varepsilon} f_{j+1}^N(X_j; V_j; t_1) = O(\varepsilon^4)$$

provided that $D_v^2 f_{j+1}^N$ is uniformly bounded. Since

$$\frac{N-j}{\sqrt{\varepsilon}} = O\big(\varepsilon^{7/2}\big)$$

we see that the second term in the right-hand side of (2.13) does not give any contribution in the limit.

Moreover

$$\begin{aligned} &\int_0^t S(t-t_1) T_j^{\varepsilon} f_j^N(t_1) dt_1 \\ &\quad = \sum_{i \neq k} \int_0^t dt_1 F\Big(\frac{(x_i - x_k) - (v_i - v_k)(t-t_1)}{\varepsilon} \Big) g(X_j, V_j; t_1) \end{aligned} \tag{2.16}$$

where g is a smooth function.

Obviously the above time integral is $O(\varepsilon)$ so that also the last term in the right-hand side of (2.13) does not give any contribution in the limit. Then we are facing the alternative: either the limit is trivial or the time evolved distributions are not smooth. This is indeed a bed new because, if we believe that the limit is not trivial (actually we expect to get the Landau equation, according to the previous discussion) a rigorous proof of this fact seems problematic.

The difficulty in obtaining a-priori estimates induce us to exploit the full series expansion of the solution, namely

$$f_1^N(t) = \sum_{n\geq 0}\sum_{G_n} K(G_n)\int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \cdot \left[S(t-t_1)O_1 S(t_1-t_2)\ldots O_n S(t_n)\right] f_m^0. \tag{2.17}$$

Here O_j is either an operator C or T expressing a creation of a new particle or a recollision between two particles respectively. G_n is a graph namely a sequence of indices

$$(r_1, l_1),\ (r_2, l_2), \ldots, (r_n, l_n)$$

where (r_j, l_j), $r_j < l_j$ is the pair of indices of the particles involved in the interaction at time t_j. $m-1$ is the number of particles created in the process. It is convenient to represent the generic graph in the following way.

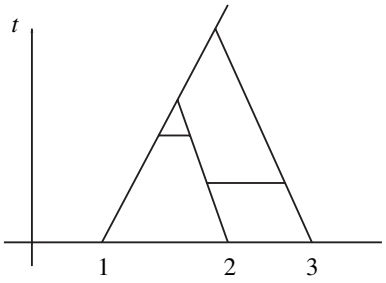

Figure 1

Here the legs of the graph denotes the particles and the nodes the creation of new particles (operators C). Recollisions (operators T) are represented by horizontal links. For instance the graph in the figure is

$$(1,2),\quad (1,3),\quad (1,3),\quad (2,3)$$

($m=3$), and the integrand in equation (2.17) in this case is

$$\left[S(t-t_1)C_{1,2}S(t_1-t_2)C_{1,3}S(t_2-t_3)T_{1,3}S(t_3-t_4)T_{2,3}S(t_4)\right] f_3^0. \tag{2.18}$$

Note that the knowledge of the graph determines completely the sequence of operators in the right-hand side of (2.17). Finally the factor $K(G_n)$ takes into account

the divergences:

$$K(G_n) = O\left(\left(\frac{1}{\sqrt{\varepsilon}}\right)^n \varepsilon^{-3(m-1)}\right). \tag{2.19}$$

We are not able to analyze the asymptotic behaviour of each term of the expansion (2.17) however we can compute the limit for $\varepsilon \to 0$ of the few terms up to the second order (in time). We have:

$$\begin{aligned} &g^N(x_1, v_1; t) \\ &\quad = f^0(x_1 - v_1 t, v_1) + \frac{N-1}{\sqrt{\varepsilon}} \int_0^t S(t-t_1) C^{\varepsilon}_{1,2} S(t_1) f_2^0 dt_1 \\ &\qquad + \frac{(N-1)}{\varepsilon} \frac{(N-2)}{\varepsilon} \sum_{j=1,2} \int_0^t \int_0^{t_1} dt_2 S(t-t_1) C^{\varepsilon}_{1,2} S(t_1 - t_2) C^{\varepsilon}_{j,3} S(t_2) f_3^0 \\ &\qquad + \frac{N-1}{\varepsilon} \int_0^t dt_1 \int_0^{t_1} dt_2 S(t-t_1) C^{\varepsilon}_{1,2} S(t_1 - t_2) T^{\varepsilon}_{1,2} S(t_2) f_2^0. \end{aligned} \tag{2.20}$$

Here the right-hand side of (2.20) defines g^N.

The second and third term in (2.20) corresponding to the graphs in Figure 2 are indeed vanishing as follows by the use of the previous arguments. The most interesting term is the last one (collision–recollision) in Figure 3.

Figure 2

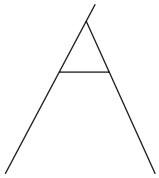

Figure 3

To handle this term we denote by $w = v_1 - v_2$ the relative velocity and note that, for

a given function u,

$$\begin{aligned} &S(t_1-t_2)T^{\varepsilon}_{1,2}u(x_1,x_2;v_1,v_2)\\ &\quad=-F\Big(\frac{(x_1-x_2)-w(t_1-t_2)}{\varepsilon}\Big)\\ &\qquad\cdot[(\nabla_{v_1}-\nabla_{v_2})u](x_1-v_1(t_1-t_2),x_2-v_2(t_1-t_2);v_1,v_2) \qquad (2.21)\\ &\quad=-F\Big(\frac{(x_1-x_2)-w(t-t_1)}{\varepsilon}\Big)\\ &\qquad\cdot(\nabla_{v_1}-\nabla_{v_2}+(t_1-t_2)(\nabla_{x_1}-\nabla_{x_2}))S(t_1-t_2)u(x_1,x_2;v_1,v_2) \end{aligned}$$

Therefore the last term in the right-hand side of (2.21) is

$$\begin{aligned} &\frac{N-1}{\varepsilon}\int_0^t dt_1 S(t-t_1)\int_0^{t_1}dt_2\int dx_2\int dv_2\\ &\quad\cdot F\Big(\frac{x_1-x_2}{\varepsilon}\Big)\cdot\nabla_{v_1}F\Big(\frac{(x_1-x_2)-w(t_1-t_2)}{\varepsilon}\Big) \qquad (2.22)\\ &\qquad\cdot(\nabla_{v_1}-\nabla_{v_2}+(t_1-t_2)(\nabla_{x_1}-\nabla_{x_2}))S(t_1)f_2^0(x_1,x_2;v_1,v_2). \end{aligned}$$

Setting now $r=\frac{x_1-x_2}{\varepsilon}$ and $s=\frac{t_1-t_2}{\varepsilon}$ then

$$\begin{aligned} &g^N(x_1,v_1)\\ &\quad=(N-1)\varepsilon^3\int_0^t dt_1\int_0^{\frac{t_1}{\varepsilon}}ds\int dr\int dv_2\ F(r)\cdot\nabla_{v_1}\cdot F(r-ws) \qquad (2.23)\\ &\qquad\cdot(\nabla_{v_1}-\nabla_{v_2}+\varepsilon s(\nabla_{x_1}-\nabla_{x_2}))S(t_1-\varepsilon s)f_2^0(x_1,x_2;v_1,v_2)+O(\sqrt{\varepsilon}). \end{aligned}$$

The formal limit is of (2.20) is

$$g(t)=S(t)f_0+\int_0^t dt_1 S(t-t_1)\nabla_{v_1}a(v_1-v_2)\ (\nabla_{v_1}-\nabla_{v_2})\ S(t_1)f_2^0, \qquad (2.24)$$

where (using $F(r)=-F(-r)$) the matrix a is given by

$$\begin{aligned} a(w)&=\int dr\int_0^{+\infty}ds\ F(r)\otimes F(r-ws)\\ &=\frac{1}{2}\int dr\int_{-\infty}^{+\infty}ds\ F(r)\otimes F(r-ws)\\ &=\frac{1}{2}\left(\frac{1}{2\pi}\right)^3\int_{-\infty}^{+\infty}ds\int dk k\otimes k\ \hat{\phi}(k)^2e^{i(w\cdot k)s}\\ &=\left(\frac{1}{8\pi}\right)^2\int dk k\otimes k\ \hat{\phi}(k)^2\delta(w\cdot k). \end{aligned} \qquad (2.25)$$

Here the interaction potential ϕ has been assumed spherically symmetric. Therefore the matrix a has the same form (1.26) with B given by

$$B=\left(\frac{1}{8\pi}\right)^2\int_0^{+\infty}drr^3\hat{\phi}(r)^2. \qquad (2.26)$$

Consider now the Landau equation

$$(\partial_t + v \cdot \nabla_x) f = Q_L(f, f) \tag{2.27}$$

with the collision operator Q_L given by (1.22) and the matrix a given by

$$a_{i,j}(V) = \frac{B}{|V|}(\delta_{i,j} - \hat{V}_i \hat{V}_j), \tag{2.28}$$

B being defined by (2.26). We obtain the following (infinite) hierarchy of equations

$$(\partial_t + V_j \cdot \nabla_{X_j}) f_j = C_{j+1} f_{j+1} \tag{2.29}$$

for the quantities

$$f_j(t) = f(t)^{\otimes j}, \tag{2.30}$$

where $f(t)$ solves equation (2.27). Accordingly $C_{j+1} = \sum_k C_{k,j+1}$, where

$$C_{k,j+1} f_{j+1}(x_1, \ldots, x_j; v_1, \ldots, v_j) = \prod_{r \neq k} f(x_r.v_r) Q_L(f, f)(x_k, v_k). \tag{2.31}$$

Therefore f has the following series expansion representation

$$\begin{aligned} f(t) = \sum_{n \geq 0} \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \\ \cdot \left[S(t - t_1) C_2 S(t_1 - t_2) C_3 \ldots C_n S(t_n) \right] f_{n+1}^0. \end{aligned} \tag{2.32}$$

As matter of fact we showed the formal convergence of g^N to the first two terms of the expansion (2.32), namely we have an agreement between the particle system (2.17) and the solution to the Landau equation (2.32) at least up to the first order in time (or second order in the potential). Although the above arguments can be made rigorous under suitable assumption on the initial condition f_0 and the potential ϕ, it seems difficult to show the convergence of the whole series. On the other hand it is clear that the graphs which should contribute in the limit are those formed by a collision–recollision sequence like:

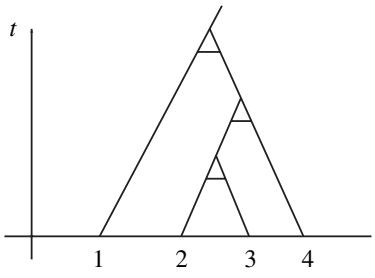

Figure 4

For those terms it is probably possible to show the convergence. For instance the case in the figure has the asymptotics

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \big[S(t-t_1)C_{1,2}S(t_1-t_2)C_{2,3}S(t_2-t_3)C_{3,4}S(t_3)\big] f_4^0. \quad (2.33)$$

However the proof that all other graphs are vanishing in the limit is not easy. Even more difficult is a uniform control of the series expansion (2.17), even for short times. As we shall see in the next section, something more can be obtained for quantum systems under the same scaling limit. Note finally that the diffusion coefficient found here given by equation (2.26), is different from that obtained in the grazing collision limit (see (1.25)). Indeed the transition kernel w appearing in equation (1.18) is in general different from $\hat{\phi}^2$. Actually to recover the Landau equation by a low-density limit (to get the Boltzmann equation) and then selecting the grazing collision part, is not equivalent to the direct, and more physical, weak-coupling limit.

3. Weak-coupling limit for quantum systems

We consider now the quantum analog of the system considered in Section 2, namely N identical quantum particles with unitary mass in $\mathbb{R}^3$.

The interaction between particles is still a two-body potential ϕ so that the total potential energy is taken as

$$U(x_1, \dots, x_N) = \sum_{i<j} \phi(x_i - x_j). \quad (3.1)$$

The associated Schrödinger equation reads

$$i\partial_t \Psi(X_N, t) = -\frac{1}{2}\Delta_N \Psi(X_N, t) + U(X_N)\Psi(X_N, t), \quad (3.2)$$

where $\Delta_N = \sum_{i=1}^N \Delta_i$, Δ_i is the Laplacian with respect to the x_i variables, $X_N = x_1, \dots, x_N$ and $\hbar$ is normalized to unity.

As for the classical system considered in Section 2 we rescale the equation and the potential by

$$x \to \varepsilon x, \quad t \to \varepsilon t, \quad \phi \to \sqrt{\varepsilon}\phi. \quad (3.3)$$

The resulting equation is,

$$i\varepsilon\partial_t \Psi^\varepsilon(X_N, t) = -\frac{\varepsilon^2}{2}\Delta_N \Psi^\varepsilon(X_N, t) + U_\varepsilon(X_N)\, \Psi^\varepsilon(X_N, t), \quad (3.4)$$

where

$$U_\varepsilon(x_1, \dots, x_N) = \sqrt{\varepsilon} \sum_{i<j} \phi\Big(\frac{x_i - x_j}{\varepsilon}\Big). \quad (3.5)$$

We want to analyze the limit $\varepsilon \to 0$ in the above equations, when $N = \varepsilon^{-3}$.

Note that this limit looks, at a first sight, similar to a semiclassical (or high frequence) limit. It is not so: indeed the potential varies on the same scale of the typical oscillations of the wave functions so that the scattering process is a genuine quantum process. Obviously, due to the oscillations, we do not expect that the wave function does converge to something in the limit. The right quantity to look at was introduced by Wigner in 1922 [44] to deal with kinetic problems. It is called the Wigner transform (of Ψ^ε) and is defined as

$$W^N(X_N, V_N) = \left(\frac{1}{2\pi}\right)^{3N} \int dY_N \; e^{iY_N \cdot V_N} \overline{\Psi}^\varepsilon \left(X_N + \frac{\varepsilon}{2} Y_N\right) \Psi^\varepsilon \left(X_N - \frac{\varepsilon}{2} Y_N\right). \quad (3.6)$$

As it is standard, W^N satisfies a transport-like equation, completely equivalent to the Schrödinger equation:

$$(\partial_t + V_N \cdot \nabla_N) W^N(X_N, V_N) = \frac{1}{\sqrt{\varepsilon}} \big(T_N^\varepsilon W^N\big)(X_N, V_N). \quad (3.7)$$

The operator T_N^ε on the right-hand-side of (3.7) plays the same role of the classical operator denoted with the same symbol in Section 2. It is

$$(T_N^\varepsilon W^N)(X_N, V_N) = \sum_{0<k<\ell\leq N} (T_{k,\ell}^\varepsilon W^N)(X_N, V_N), \quad (3.8)$$

where each $T_{k,\ell}^\varepsilon$ describes the interaction of particle k with particle ℓ:

$$\begin{aligned}(T_{k,\ell}^\varepsilon W^N)&(X_N, V_N) \\ &= \frac{1}{i} \left(\frac{1}{2\pi}\right)^{3N} \int dY_N \, dV'_N \; \mathrm{e}^{iY_N \cdot (V_N - V'_N)} \\ &\quad \cdot \left[\phi\left(\frac{x_k - x_\ell}{\varepsilon} - \frac{y_k - y_\ell}{2})\right) - \phi\left(\frac{x_k - x_\ell}{\varepsilon} + \frac{y_k - y_\ell}{2}\right)\right] W^N(X_N, V'_N).\end{aligned} \quad (3.9)$$

Equivalently, we may write

$$\begin{aligned}(T_{k,\ell}^\varepsilon W^N)&(X_N, V_N) \\ &= -i \sum_{\sigma=\pm 1} \sigma \int \frac{dh}{(2\pi)^3} \; \hat{\phi}(h) \; \mathrm{e}^{i\frac{h}{\varepsilon}(x_k - x_\ell)} \\ &\quad \cdot W^N\big(x_1, \ldots, x_N; v_1, \ldots, v_k - \sigma\tfrac{h}{2}, \ldots, v_\ell + \sigma\tfrac{h}{2}, \ldots, v_N\big).\end{aligned} \quad (3.10)$$

Note that $T_{k,\ell}^\varepsilon$ is a pseudodifferential operator which formally converge, at fixed ε, for $\hbar \to 0$ (here $\hbar = 1$) to its classical analog. Note also that in (3.10), "collisions" may take place between *distant* particles ($x_k \neq x_\ell$). However, such distant collisions are penalized by the highly oscillatory factor $\exp(ih(x_k - x_\ell)/\varepsilon)$. These oscillations

turn out to play a crucial role throughout the analysis, and they explain why collisions tend to happen when $x_k = x_\ell$ in the limit $\varepsilon \to 0$.

The formalism we have introduced is similar to the one of the classical case so that we proceed as before by transforming equation (3.7) into a hierarchy of equations. We introduce the partial traces of the Wigner transform W^N, denoted by f_j^N. They are defined through the following formula, valid for $j = 1, \dots, N-1$:

$$f_j^N(X_j, V_j) = \int dx_{j+1} \dots \int dx_N \int dv_{j+1} \dots \int dv_N \cdot W^N(X_j, x_{j+1}, \dots, x_N; V_j, v_{j+1}, \dots, v_N). \tag{3.11}$$

Obviously, we set $f_N^N = W^N$. The function f_j^N is the kinetic object that describes the state of the j particles subsystem at time t.

Due to the fact that the particles are identical, the wave function Ψ, as well as W^N and f_j^N, are assumed to be symmetric in the exchange of particle, a property that is preserved in time.

Proceeding then as in the derivation of the BBKGY hierarchy for classical systems, we readily transform equation (3.7) into the following hierarchy:

$$\Big(\partial_t + \sum_{k=1}^{j} v_k \cdot \nabla_k\Big) f_j^N(X_j, V_j) = \frac{1}{\sqrt{\varepsilon}} T_j^\varepsilon f_j^N + \frac{N-j}{\sqrt{\varepsilon}} C_{j+1}^\varepsilon f_{j+1}^N, \tag{3.12}$$

where

$$C_{j+1}^\varepsilon = \sum_{k=1}^{j} C_{k,j+1}^\varepsilon, \tag{3.13}$$

and $C_{k,j+1}^\varepsilon$ is defined by

$$\begin{aligned} &C_{k,j+1}^\varepsilon f_{j+1}^N(X_j; V_j) \\ &\quad = -i \sum_{\sigma=\pm 1} \sigma \int \frac{dh}{(2\pi)^3} \int dx_{j+1} \int dv_{j+1}\, \hat{\phi}(h)\, e^{i\frac{h}{\varepsilon}(x_k - x_{j+1})} \\ &\qquad \cdot f_{j+1}^N\big(x_1, x_2, \dots, x_{j+1}; v_1, \dots, v_k - \sigma\tfrac{h}{2}, \dots, v_{j+1} + \sigma\tfrac{h}{2}\big). \end{aligned} \tag{3.14}$$

As before the initial value $\{f_j^0\}_{j=1}^N$ is assumed completely factorized: for all $j = 1, \dots, N$, we suppose

$$f_j^0 = f_0^{\otimes j}, \tag{3.15}$$

where f_0 is a one-particle Wigner function, and f^0 is assumed to be a probability distribution.

In the limit $\varepsilon \to 0$, we expect that the j-particle distribution function $f_j^N(t)$, that solves the hierarchy (3.12) with initial data (3.15), tends to be factorized for all times: $f_j^N(t) \sim f(t)^{\otimes j}$ (propagation of chaos).

As for the classical case, if f_{j+1} is smooth, then

$$\begin{aligned} &C^{\varepsilon}_{k,j+1} f^N_{j+1}(X_j; V_j) \\ &\quad = -i\varepsilon^3 \sum_{\sigma=\pm 1} \sigma \int \frac{dh}{(2\pi)^3} \hat{\phi}(h) \int dr \int dv_{j+1}\, e^{ih\cdot r} \\ &\qquad \cdot f^N_{j+1}\big(X_j, x_k - \varepsilon r;\ v_1, \dots, v_k - \sigma\tfrac{h}{2}, \dots, v_{j+1} + \sigma\tfrac{h}{2}\big) = O(\varepsilon^4). \end{aligned} \tag{3.16}$$

Indeed, setting $\varepsilon = 0$ in the integrand, the integration over r produces $\delta(h)$. As a consequence the integrand is independent of σ and the sum vanishes. Therefore the integral is $O(\varepsilon)$. Also

$$\begin{aligned} &\frac{1}{\sqrt{\varepsilon}} \int_0^t dt_1 S(t-t_1) T_{r.k} f^N_j(t_1) \\ &\quad = -i \sum_{\sigma=\pm 1} \sigma \int_0^t dt_1 \frac{dh}{(2\pi)^3} \hat{\phi}(h) \\ &\qquad \cdot e^{i\frac{h}{\varepsilon}\cdot(x_r - x_k) - (v_r - v_k)(t-t_1)} f^N_j(X_j - V_j(t-t_1); V_j; t_1) \end{aligned} \tag{3.17}$$

is weakly vanishing, by a stationary phase argument (see [4]). Therefore we are in the same situation as for the classical case for which we are led to analyze the asymptotics of g^N (see (2.20)) which means to study the limit of the collision–recollision term:

$$\frac{N-1}{\varepsilon} \int_0^t dt_1 \int_0^{t_1} d\tau_1\ S(t-t_1) C_{1,2} S(t_1 - \tau_1) T_{1.2} S(\tau_1) f^0_2. \tag{3.18}$$

Explicitly it looks as follows:

$$\begin{aligned} &-\frac{N-1}{\varepsilon} \sum_{\sigma,\sigma'=\pm 1} \sigma\sigma' \int_0^t dt_1 \int_0^{t_1} d\tau_1 \int dx_2 \int dv_2 \int \frac{dh}{(2\pi)^3} \int \frac{dk}{(2\pi)^3} \\ &\quad \cdot \hat{\phi}(h)\ \hat{\phi}(k)\ \mathrm{e}^{i\frac{h}{\varepsilon}\cdot(x_1 - x_2 - v_1(t-t_1))} \mathrm{e}^{i\frac{k}{\varepsilon}\cdot(x_1 - x_2 - v_1(t-t_1) - (v_1 - v_2 - \sigma h)(t_1 - \tau_1))} \\ &\qquad \cdot f^0_2\left(x_1 - v_1 t + \sigma\tfrac{h}{2} t_1 + \sigma'\tfrac{k}{2}\tau_1, x_2 - v_2 t_1 - \sigma\tfrac{h}{2} t_1 - \sigma'\tfrac{k}{2}\tau_1;\right. \\ &\qquad\quad \left. \cdot\ v_1 - \sigma\tfrac{h}{2} - \sigma'\tfrac{k}{2}, v_2 + \sigma\tfrac{h}{2} + \sigma'\tfrac{k}{2}\right). \end{aligned} \tag{3.19}$$

By the change of variables

$$t_1 - \tau_1 = \varepsilon s_1 \ (\text{i.e. } \tau_1 = t_1 - \varepsilon s_1), \quad \xi = (h+k)/\varepsilon, \tag{3.20}$$

we have

$$\begin{aligned} (3.19) = -(N-1)\,\varepsilon^3 &\sum_{\sigma,\sigma'=\pm 1} \sigma\sigma' \int_0^t dt_1 \int_0^{t_1/\varepsilon} ds_1 \int dx_2 \int dv_2 \int \frac{d\xi}{(2\pi)^3} \int \frac{dk}{(2\pi)^3} \\ &\cdot \hat{\phi}(-k + \varepsilon\xi_1)\ \hat{\phi}(k)\ \mathrm{e}^{i\xi\cdot\left(x_1 - x_2 - v_1(t-t_1)\right)} \mathrm{e}^{-is_1 k\cdot(v_1 - v_2 - \sigma(-k+\varepsilon\xi))} f^0_2(\dots), \end{aligned}$$

In the limit $\varepsilon \to 0$, the above formula gives the asymptotics

$$(3.19)\quad \underset{\varepsilon\to 0}{\sim} -\sum_{\sigma,\sigma'=\pm 1} \sigma\sigma' \int_0^t dt_1 \int dv_2 \int \frac{dk}{(2\pi)^3} \tag{3.21}$$
$$|\hat{\phi}(k)|^2 \left(\int_0^{+\infty} e^{-is_1 k\cdot(v_1-v_2+\sigma k)}\, ds_1\right) f_2^0\left(x_1 - v_1 t - (\sigma-\sigma')\tfrac{k}{2}t_1,\right.$$
$$\left. x_1 - v_1(t-t_1) - v_2 t_1 + (\sigma-\sigma')\tfrac{k}{2}t_1;\, v_1 + (\sigma-\sigma')\tfrac{k}{2},\, v_2 - (\sigma-\sigma')\tfrac{k}{2}\right).$$

In [4], we completely justify formula (3.21) and its forthcoming consequences.

Now, we turn to identifying the limiting value obtained in (3.21). To do so, we observe that symmetry arguments allow us to replace the integral in s by its real part:

$$\mathrm{Re}\int_0^{\infty} e^{-is_1 k\cdot(v_1-v_2+\sigma k)}\, ds_1 = \pi\delta(k\cdot(v_1-v_2+\sigma k)). \tag{3.22}$$

Using formula (3.22) we realize that the contribution $\sigma = -\sigma'$ in (3.21) gives rise to the gain term:

$$\int_0^t dt_1 \int dv_2 \int d\omega\, B(\omega, v_1-v_2) \cdot f_2^0(x_1 - v_1(t-t_1) - v_1' t_1, x_2 - v_2(t-t_1) - v_2' t_1;\, v_1', v_2'), \tag{3.23}$$

where

$$B(\omega, v) = \frac{1}{8\pi^2}|\omega\cdot v|\,|\hat{\phi}(\omega\,(\omega\cdot v))|^2. \tag{3.24}$$

Similarly, the term $\sigma = \sigma'$ in (2.2) yields the loss term:

$$\int_0^t dt_1 \int dv_2 \int d\omega\, B(\omega, v_1-v_2) f_2^0(x_1 - v_1 t, x_2 - v_2(t-t);\, v_1, v_2). \tag{3.25}$$

By the same arguments used in the previous section we can conclude that the full series expansion (2.17) (of course for the present quantum case) agrees, up to the second order in the potential, with

$$S(t)f_0 + \int_0^t dt_1 S(t-t_1) Q(S(t_1)f_0, S(t_1)f_0) \tag{3.26}$$

where

$$Q(f,f) = \int dv_1 \int d\omega\, B(\omega, v-v_1)[f'f_1' - ff_1] \tag{3.27}$$
$$= \int dv_1 \int dh |\hat{\phi}(h)|^2 \delta((h\cdot(v-v_1+h))[f(v+h)f(v_1-h) - f(v)f(v_1)].$$

In other words the kinetic equation which comes out is the Boltzmann equation with cross-section B.

We note once more that the δ function in equation (3.27) expresses the energy conservation, while the momentum conservation is automatically satisfied.

Note that the cross-section B is the only quantum factor in the purely classical expression (3.27). It retains the quantum features of the elementary "collisions".

An important comment is in order. Why is the kinetic equation for quantum systems of Boltzmann type in contrast with the classical case where we got a diffusion? The answer is related to the asymptotics of a single scattering (see [35], [36] and [8]). For quantum systems the probability of a zero angle scattering is finite (that is a sort of tunnel effect), while for a classical particle we have surely a small deviation from the free motion. Therefore a quantum particle, in this asymptotic regime, is going to perform a jump process (in velocity) rather than a diffusion.

From a mathematical view point we observe that [4] proves more than agreement up to second order. We indeed consider the subseries (of the full series expansion expressing $f_j^N(t)$) formed by *all* the collision–recollision terms. In other words, we consider the subseries of $f_j^N(t)$ given by

$$\sum_{n\geq 1} \sum_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n} \varepsilon^{-4n} \int_0^t dt_1 \int_0^{t_1} d\tau_1 \; S(t-t_1) C^{\varepsilon}_{\alpha_1,\beta_1} S(t_1-\tau_1) T^{\varepsilon}_{\alpha_1,\beta_1} \cdots \int_0^{\tau_{n-1}} dt_n \int_0^{t_n} d\tau_n \; S(\tau_{n-1}-t_n) C^{\varepsilon}_{\alpha_n,\beta_n} S(t_n-\tau_n) T^{\varepsilon}_{\alpha_n,\beta_n} S(\tau_n) f^0_{j+n+1}. \tag{3.28}$$

Here the sum runs over all possible choices of the particles number α's and β's, namely we sum over the subset of graphs of the form in Figure 4. We establish in [4] that the subseries (3.28) is indeed absolutely convergent, for short times, uniformly in ε. Moreover, we prove that it approaches the corresponding complete series expansion obtained by solving iteratively the Boltzmann equation with collision operator given by equation (3.27) extending and making rigorous the above argument.

However, this does not completely finishes the proof yet: the true series expansion of $f_j^N(t)$ contains many more terms than those we consider in (3.28) and we are not able to show uniform bound on the full series. Thus a mathematical justification of the quantum Boltzmann equation is a still an open and difficult problem. More recently we proved in [7], although under severe assumptions on the potential, that all other terms than those considered in the subseries (3.28) are vanishing in the limit, but this is, unfortunately, not yet conclusive.

4. The weak coupling limit in the Bose–Einstein or the Fermi–Dirac statistics

From a physical viewpoint it is certainly more realistic to consider particles obeying the Fermi–Dirac or Bose–Einstein statistics, than considering the Maxwell–Boltzmann situation. In this case, the starting point still is the rescaled Schrödinger equation (3.4),

or the equivalent hierarchy (3.12). The only new point is that we cannot take a totally uncorrelated initial datum as in (3.15). Indeed, the Fermi–Dirac or Bose–Einstein statistics yield correlations even at time zero. In this perspective, the most uncorrelated states one can introduce, and that do not violate the Fermi–Dirac or Bose–Einstein statistics, are the so-called quasi-free states. They have, in terms of the Wigner formalism, the following form:

$$f_j(x_1, v_1, \dots, x_j, v_j) = \sum_{\pi \in \mathcal{P}_j} \theta^{s(\pi)} \, f_j^{\pi}(x_1, v_1, \dots, x_j, v_j), \tag{4.1}$$

where each f_j^{π} has the value

$$f_j^{\pi}(x_1, v_1, \dots, x_j, v_j) = \int dy_1 \dots dy_j \int dw_1 \dots dw_j \; e^{i(y_1 \cdot v_1 + \dots + y_j \cdot v_j)} \tag{4.2}$$

$$\prod_{k=1}^{j} e^{-\frac{i}{\varepsilon} w_k \cdot (x_k - x_{\pi(k)})} \; e^{-\frac{i}{2} w_k \cdot (y_k + y_{\pi(k)})} \; f\Big(\frac{x_k + x_{\pi(k)}}{2} + \varepsilon \frac{y_k - y_{\pi(k)}}{4}, w_k \Big)$$

and f is a given one-particle Wigner function. Here $\mathcal{P}_j$ denotes the group of all the permutations of j objects and π its generic element.

Note that the Maxwell–Boltzmann case treated so far is recovered by the contribution due the permutation $\pi =$ identity.

Note also that quasi-free states converge weakly to the completely factorized states as $\varepsilon \to 0$, that is a physically obvious fact because the quantum statistics become irrelevant in the semiclassical limit. However the dynamics take place on the scale ε so that the effects of the statistics are present in the limit. Indeed it is expected that the one-particle distribution function $f_1^N(t)$ converges to the solution of the following cubic Boltzmann equation:

$$(\partial_t + v \cdot \nabla_x) \, f(t, x, v) = Q_{\mathrm{w},\theta}(f, f, f)(t, x, v), \tag{4.3}$$

$$\begin{aligned} Q_\theta(f, f, f)(t, x, v) = \int dv_1 \, d\omega \; B_\theta(\omega, v - v_1) \\ \cdot \big[f(x, v') f(x, v_1')(1 + 8\pi^3 \theta f(x, v) \, f(x, v_1)) \\ - f(x, v) f(x, v_1)(1 + 8\pi^3 \theta f(x, v') f(x, v_1')) \big]. \end{aligned} \tag{4.4}$$

Here $\theta = +1$ or $\theta = -1$, for the Bose–Einstein or the Fermi–Dirac statistics respectively. Finally, B_θ is the symmetrized or antisymmetrized cross-section derived from B (see (3.24)) in a natural way.

As we see, the modification of the statistics transforms the quadratic Boltzmann equation of the Maxwell–Boltzmann case, into a cubic one (fourth order terms cancel). Also, the statistics affects the form of the cross-section and B has to be (anti)symmetrized into B_θ. The collision operator (4.4) has been introduced by Uehling and Uhlembeck in 1933 on the basis of purely phenomenological considerations [41].

Plugging in the hierarchy (3.12) an initial datum satisfying (4.1), we can follow the same procedure as for the Maxwell–Boltzmann statistics: we write the full perturbative series expansion expressing $f_j^N(t)$ in terms of the initial datum and try to analyse its asymptotic behaviour.

As we did before, we first restrict our attention to those terms of degree less than two in the potential.

The analysis up to second order is performed in [5]. We actually recover here equation (4.3), (4.4) with the suitable B_θ. Now the number of terms to control is much larger due to the sum over all permutations that enters the definition (4.1) of the initial state. Also, the asymptotics is much more delicate. In particular, we stress the fact that the initial datum brings its own highly oscillatory factors in the process, contrary to the Maxwell–Boltzmann case where the initial datum is uniformly smooth, and where the oscillatory factors simply come from the collision operators T and C. In [5] we consider the second order graphs

Figure 5

which, because of the permutation of initial state, yields various terms: two of them are bilinear in the initial condition f_0, and twelve are trilinear in f_0. Some of these terms vanish in the limit due to a non-stationary phase argument. Others give rise to truly diverging contributions (negative powers of ε). However, when grouping the terms in the appropriate way, those terms are seen to cancel each other. Last, some terms give the collision operator (4.4). The computation is heavy and hence we address the reader to [5] for the details.

This ends up the analysis of terms up to second order in the potential.

Obviously, as for the Maxwell–Boltzmann case, we could try to re-sum the dominant terms. This would lead to analyzing a true subseries of the complete series expansion expressing $f_j^N(t)$. We do not see any conceptual difficulty, however, this resummation procedure has not been explicitly done in [5].

We mention that a similar analysis, using commutator expansions in the framework of the second quantization formalism, has been performed in [27] (following [26]) in the case of the van Hove limit for lattice systems (that is the same as the weak-coupling limit, yet without rescaling the distances). For more recent formal results in this direction, but in the context of the weak-coupling limit, we also quote [22].

We finally observe that the initial value problem for equation (4.3) is somehow

trivial for Fermions. Indeed we have the a-priori bounds $f \leq \frac{1}{(8\pi)^3}$ making everything easy. For Bosons the situation is much more involved even for the spatially homogeneous case. The statistics favours large values of f and it is not clear whether the equation can explain dynamical condensation. See, for the mathematical side, references [33], [34].

Summarizing, the main scope of this lecture is to show why in the weak-coupling limit, the one-particle distribution function is expected to converge to a solution of Landau equation or Boltzmann equation, for classical and quantum system respectively. From a rigorous view point very little is known.

5. Concluding remarks

Other scaling limits yielding different kinetic equations are of course possible. We address the reader to the excellent reference [38] where the various scales and the corresponding kinetic equations are discussed. Here we analyzed in some detail the weak-coupling, however, as mentioned in Section 1, the low-density limit (or the Boltzmann-Grad limit) yields the usual Boltzmann equation for classical systems and this result has been proved for short times. It is natural to investigate what happens, in the same scaling limit, to a quantum system. Here the scaling is

$$t \to \varepsilon t, \quad x \to \varepsilon x, \quad \phi \to \phi, \quad N = \varepsilon^{-2}. \tag{5.1}$$

In other words, the density of obstacles is ε, which is a rarefaction regime, but the potential is unscaled and keeps an $O(1)$ amplitude. Now due to the fact that the density is vanishing, the particles are too rare to make the statistical correlations effective. As a consequence, we expect that the Maxwell–Boltzmann, Bose–Einstein, and Fermi–Dirac situations, all give rise to the same Boltzmann equation along the low-density limit.

As a matter of fact, the expected Boltzmann equation still is a quadratic Boltzmann equation in that case, namely

$$(\partial_t + v \cdot \nabla_x) f(t, x, v) = Q_\ell(f, f)(t, x, v), \tag{5.2}$$

$$Q_\ell(f, f)(t, x, v) \tag{5.3}$$
$$= \int dv_1 \, d\omega \, B_\ell(\omega, v - v_1) \left[f(t, x, v') f(t, x, v_1') - f(t, x, v) f(t, x, v_1) \right].$$

Here, the index "ℓ" refers to "low-density".

The factor $B_\ell(\omega, v - v_1)$ is the cross-section. In the low-density limit, collisions take place at a large energy (contrary to the weak-coupling situation), and at a distance of order ε. For this reason, the cross section B_ℓ is computed at large energy, and via the quantum rules. In other words, it agrees with the *full* Born series expansion of

quantum scattering, namely

$$B_\ell(\omega, v) = \frac{1}{8\pi^2}|\omega \cdot v|\,|\hat{\phi}(\omega\,(\omega \cdot v))|^2 + \sum_{n\geq 3} B_\ell^{(n)}(\omega, v), \tag{5.4}$$

where each $B_\ell^{(n)}(\omega, v)$ is an explicitly known function, which is n-linear in ϕ (see [37]). Note that the convergence of the Born series expansion (4.8) is well-known for potentials satisfying a smallness assumption. As it is seen on these formulae, the only difference between the low-density and the weak-coupling regimes (at least for Maxwell–Boltzmann particles) lies in the form of the cross-section. Note also that

$$B_\ell(\omega, v) = B(\omega, v) + O([\phi]^3), \tag{5.5}$$

which reflects the fact that the weak-coupling regime involves only low-energy phenomena.

The analysis of the partial series of the dominant terms (uniform bounds and convergence as for the weak-coupling limit) has been performed in [6] (on the basis of [11] and [12]).

Related problem connected with the ones discussed here are the homogeneization of the distribution function of a single particle in a random distribution of obstacles $\boldsymbol{c} = \{c_1, \ldots, c_N\}$. The basic equations are

$$\dot{x}(t) = v(t), \quad \dot{v}(t) = -\sum_j \nabla\phi(x(t) - c_j) \tag{5.6}$$

for a classical particle and, for a quantum particle

$$i\,\partial_t\psi = -\frac{1}{2}\Delta\psi + \sum_j \phi(x - c_j)\psi. \tag{5.7}$$

We are interested in the behavior of

$$f_\varepsilon(x, v; t) = \mathbb{E}[f_{\boldsymbol{c}}(x, v; t)] \tag{5.8}$$

where $f_{\boldsymbol{c}}(t)$ is the time evolved classical distribution function or the Wigner transform of ψ according to equations (5.6) or (5.7) respectively, under the action of the obstalcle configuration $\boldsymbol{c}$. Finally $\mathbb{E}$ denotes the expectation with respect to the obstacle distribution. For the low-density scaling under a Poisson distribution of obstacles (this is the so called Lorentz model) we obtain, for classical systems, a linear Boltzmann equation (see [23], [39], [3], [17], [10]). It is also known that the system does not homogenize to a jump process given by a linear Boltzmann equation in case of a periodic distribution of obstacles [9]. For the weak-coupling limit we obtain, by a central-limit type of argument, a linear Landau equation as it is shown in [29] and [18].

As regards the corresponding weak-coupling quantum problem, the easiest case is when ϕ is a Gaussian process. The kinetic equation is still a linear Boltzmann

equation. The first result, holding for short times, has been obtained in [39] (see also [30]). More recently this result has been extended to arbitrary times [21]. The technique of [21] can been applied to deal with a Poisson distribution of obstacles. Obviously the cross section appearing in the Boltzmann equation is that computed in the Born approximation. Finally in [20] the low-density case has been successfully approached. The result is a linear Boltzmann equation with the full cross-section.

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