Symmetry of entire solutions for a class of semilinear elliptic equations

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Abstract. We discuss a conjecture of De Giorgi concerning the one dimensional symmetry of bounded, monotone in one direction, solutions of semilinear elliptic equations of the form $\Delta u = W'(u)$ in all $\mathbb{R}^n$.

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1. Introduction

In 1978 De Giorgi [13] made the following conjecture about bounded solutions of a certain semilinear equation:

Conjecture (De Giorgi). Let $u \in C^2(\mathbb{R}^n)$ be a solution of

$$\Delta u = u^3 - u, \tag{1}$$

such that

$$|u| \leq 1, \quad u_{x_n} > 0$$

in the whole $\mathbb{R}^n$. Is it true that all the level sets of u are hyperplanes, at least if $n \leq 8$?

The problem originates in the theory of phase transitions and it is closely related to the theory of minimal surfaces. As we explain later, the conjecture is sometimes referred to as "the ε version of the Bernstein problem for minimal graphs". This relation with the Bernstein problem is the reason why $n \leq 8$ appears in the conjecture.

De Giorgi's conjecture is often considered with the additional natural hypothesis

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1. \tag{2}$$

Under the much stronger assumption that the limits in (2) are uniform in x', the conjecture is known as the "Gibbons conjecture". This conjecture was first proved for $n \leq 3$ by Ghoussoub and Gui in [18] and then for all dimensions n independently by Barlow, Bass and Gui [4], Berestycki, Hamel and Monneau [6] and Farina [16].

The first positive partial result on the De Giorgi conjecture was established in 1980 by Modica and Mortola [30]. They proved the conjecture in dimension $n = 2$

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under the additional hypothesis that the level sets $\{u = s\}$ are equi-Lipschitz in the x_2 direction. Their proof used a Liouville-type theorem for elliptic equations in divergence form, due to Serrin, for the bounded ratio

$$\sigma := \frac{u_{x_1}}{u_{x_2}}.$$

In 1997 Ghoussoub and Gui [18] proved De Giorgi's conjecture for $n = 2$. They used a different Liouville-type theorem for σ developed by Berestycki, Caffarelli and Nirenberg in [5] for the study of symmetry properties of positive solutions of semilinear elliptic equations in half spaces. This theorem does not require for σ to be bounded, but rather a compatibility condition between the growth of σ and the degeneracy of the coefficients of the equation.

Using similar techniques, Ambrosio and Cabre [2] extended these results up to dimension $n = 3$. Also, Ghoussoub and Gui showed in [19] that the conjecture is true for $n = 4$ or $n = 5$ for a special class of solutions that satisfy an anti-symmetry condition.

In 2003 I proved in [33] that the conjecture is true in dimension $n \leq 8$ under the additional hypothesis (2). The proof is nonvariational and uses the sliding method for a special family of radially symmetric functions.

If the level sets of u are assumed to be Lipschitz in the x_n direction, then it was shown by Barlow, Bass and Gui [4] and later in [33] that the solutions are planar in all dimensions.

It is not known whether or not the conjecture is true for all dimensions. Jerison and Monneau [22] showed that the existence of a symmetric minimizer for the energy associated with (1) in $\mathbb{R}^{n-1}$ implies the existence of a counter-example to the conjecture of the De Giorgi in $\mathbb{R}^n$. However, existence of such global minimizer has not been proved.

2. Phase transitions

Equations of type (1) arise in variational problems associated with the energy

$$J(u, \Omega) = \int_\Omega \frac{1}{2}|\nabla u|^2 + W(u)\, dx, \quad |u| \leq 1, \tag{3}$$

where $W \in C^2$ is a double well potential with minima at ± 1,

$$W(\pm 1) = W'(\pm 1) = 0, \quad W > 0 \quad \text{on } (-1, 1),$$

$$W''(-1) > 0, \quad W''(1) > 0.$$

We say that u is a local minimizer for J in Ω if

$$J(u, \Omega) \leq J(u + \varphi, \Omega)$$

for any $\varphi \in C_0^\infty(\Omega)$. Local minimizers of (3) satisfy the Euler–Lagrange equation

$$\Delta u = W'(u). \tag{4}$$

Equation (1) is obtained for the particular choice of the potential

$$W(t) = \frac{1}{4}(1 - t^2)^2.$$

The behavior of minimizers at ∞ is given by the properties of blow down solutions

$$u_\varepsilon(x) = u\left(\frac{x}{\varepsilon}\right).$$

These rescalings are local minimizers for the ε energy functional

$$J_\varepsilon(u_\varepsilon) = \int \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon} dx.$$

This is a typical energy modeling the phase separation phenomena within the van der Walls–Cahn–Hilliard theory [8]. In this context, u_ε represents the density of a multi-phase fluid, where the zero points of W correspond to stable fluid phases and the free energy J_ε depends both on the density potential and the density gradient.

One expects that u_ε has a transition region of $\mathcal{O}(\varepsilon)$ thickness which approaches a minimal surface as $\varepsilon \to 0$. The intuition behind this comes from the following calculation. By the coarea formula

$$J_\varepsilon(u_\varepsilon, \Omega) \geq \int_\Omega |\nabla u_\varepsilon| \sqrt{2W(u_\varepsilon)} dx = \int_{-1}^{1} \sqrt{2W(s)} \mathcal{H}^{n-1}(\{u_\varepsilon = s\} \cap \Omega) ds.$$

Heuristically, $J_\varepsilon(u_\varepsilon, \Omega)$ is minimized if, in the interior of Ω, the level sets $\{u_\varepsilon = s\}$ are (almost) minimal and

$$|\nabla u_\varepsilon| = \frac{1}{\varepsilon}\sqrt{2W(u_\varepsilon)}. \tag{5}$$

This equation suggests that "the profile" of u_ε behaves like

$$u_\varepsilon(x) \simeq g\left(\frac{d_{\{u=0\}}(x)}{\varepsilon}\right)$$

where g is the solution of the ordinary differential equation

$$g' = \sqrt{2W(g)}, \quad g(0) = 0,$$

and $d_{\{u=0\}}$ represents the signed distance to the 0 level surface of u.

The asymptotic behavior of u_ε was first studied by Modica and Mortola in [31] and Modica in [25] within the framework of Γ-convergence. Later, Modica [27], Sternberg [36] and many authors [3], [17], [24], [29], [32], [37] generalized these results for minimizers with volume constraint.

Modica proved in [25] that as $\varepsilon \to 0$, u_ε has a subsequence

$$u_{\varepsilon_k} \to \chi_E - \chi_{E^c} \quad \text{in } L^1_{\text{loc}} \tag{6}$$

where E is a set with minimal perimeter. Actually, the convergence in (6) is better, as it was shown by Caffarelli and Cordoba in [10]. They proved a uniform density estimate for the level sets of local minimizers u_ε of J_ε i.e, if $u_\varepsilon(0) = 0$, then

$$\frac{|\{u_\varepsilon > 0\} \cap B_\delta|}{|B_\delta|} \geq C \tag{7}$$

for $\varepsilon \leq \delta$, $C > 0$ universal. In particular, this implies that in (6) the level sets $\{u_{\varepsilon_k} = \lambda\}$ converge uniformly on compact sets to ∂E.

Next we recall some known facts about sets with minimal perimeter.

3. Minimal surfaces

The Plateau problem consist in finding a surface of least area (the minimal surface) among those bounded by a given curve. De Giorgi studied this problem by looking at hypersurfaces in $\mathbb{R}^n$ as boundaries of sets. Thus, for a measurable set E, he defined the perimeter of E in a domain $\Omega \subset \mathbb{R}^n$ (or the area of ∂E in Ω) as the total variation of $\nabla \chi_E$ in Ω, i.e.

$$P_\Omega(E) = \int_\Omega |\nabla \chi_E| := \sup \left| \int_E \operatorname{div} g \, dx \right|,$$

where the supremum is taken over all vector fields $g \in C^1_0(\Omega)$ with $\|g\|_{L^\infty} \leq 1$.

It is not difficult to show existence to the Plateau problem in this context of minimal boundaries. It is much more difficult to prove that the sets so obtained are actually regular except possibly for a closed singular set.

The main idea to prove "almost everywhere" regularity uses an improvement of flatness theorem due to De Giorgi [14], [21]. Caffarelli and Cordoba gave a different proof in [11] using nonvariational techniques.

Theorem 3.1 (De Giorgi). *Suppose that E is a set having minimal perimeter in $\{|x'| < 1,\ |x_n| < 1\}$, $0 \in \partial E$ and assume that ∂E is "flat", i.e.*

$$\partial E \subset \{|x_n| < \varepsilon\},$$

$\varepsilon \leq \varepsilon_0$, ε_0 small universal.

Then, possibly in a different system of coordinates, ∂E can be trapped in a flatter cylinder

$$\{|y'| \leq \eta_2\} \cap \partial E \subset \{|y_n| \leq \varepsilon \eta_1\},$$

with $0 < \eta_1 < \eta_2$ universal constants.

This theorem implies that flat minimal surfaces are $C^{1,\alpha}$, and therefore analytic by elliptic regularity theory.

The question of whether or not all points of a minimal surface are regular is closely related to the Bernstein problem. Singular points can exist if and only if there exist nonplanar entire minimal surfaces (or minimal cones). Simons [35] proved that in dimension $n \leq 7$ entire minimal surfaces are planar. Bombieri, De Giorgi and Giusti showed in [7] that the Simons cone

$$\{x \in \mathbb{R}^8 : \ x_1^2 + x_2^2 + x_3^2 + x_4^2 < x_5^2 + x_6^2 + x_7^2 + x_8^2\}$$

is minimal in $\mathbb{R}^8$. Moreover, if the minimal surface is assumed to be a "graph" in some direction, then there are nonplanar minimal graphs only in dimension $n \geq 9$.

Finally we mention that an entire minimal surface that is a "graph" and has at most linear growth at ∞ is planar.

4. Symmetry of minimizers

It is natural to ask if some properties of minimal surfaces hold also for local minimizers of (3) or solutions of (4). Actually, the conjecture of De Giorgi corresponds to such a question. Results in this direction were obtained by several authors.

Caffarelli and Cordoba proved the uniform density estimate (7) for the level sets of local minimizers u_ε of J_ε. Modica proved in [28] that solutions of (4) satisfy a monotonicity formula for the energy functional,

$$\frac{J(u, B_R)}{R^{n-1}} \quad \text{increases with } R.$$

I proved in [33] an improvement of flatness theorem for local minimizers of (3) which corresponds to the flatness theorem of De Giorgi for minimal surfaces. It asserts that, if a level set is trapped in a flat cylinder whose height is greater than some given θ_0, then it is trapped in a flatter cylinder in the interior (the flatness depends on θ_0).

Theorem 4.1 (Savin [33]). *Suppose that u is a local minimizer of* (3) *in the cylinder* $\{|x'| < l, \ |x_n| < l\}$, *and assume that the* 0 *level set is "flat",*

$$\{u = 0\} \subset \{|x'| < l, \ |x_n| < \theta\},$$

and contains the point 0. *Then there exist small constants* $0 < \eta_1 < \eta_2 < 1$ *depending only on n such that:*

Given $\theta_0 > 0$ *there exists* $\varepsilon_1(\theta_0) > 0$ *depending on n, W and* θ_0 *such that if*

$$\frac{\theta}{l} \leq \varepsilon_1(\theta_0), \quad \theta_0 \leq \theta$$

then

$$\{u = 0\} \cap \{|\pi_\xi x| < \eta_2 l,\ |x \cdot \xi| < \eta_2 l\}$$

is included in a flatter cylinder

$$\{|\pi_\xi x| < \eta_2 l,\ |x \cdot \xi| < \eta_1 \theta\}$$

for some unit vector ξ *(*π_ξ *denotes the projection along* ξ*).*

The proof uses the fact that at large scales the level sets behave like minimal surfaces and at small scales the equation behaves like Laplace's equation. The ideas are based on a viscosity solution proof of the flatness theorem of De Giorgi (see [34]). As a corollary of the above theorem we obtain that, if the level sets are asymptotically flat at ∞ then they are, in fact, hyperplanes.

Corollary. *Let* u *be a local minimizer of* J *in* $\mathbb{R}^n$ *with* $u(0) = 0$. *Suppose that there exist sequences of positive numbers* θ_k, l_k *and unit vectors* ξ_k *with*

$$\frac{\theta_k}{l_k} \to 0, \quad l_k \to \infty$$

such that

$$\{u = 0\} \cap \{|\pi_{\xi_k} x| < l_k,\ |x \cdot \xi_k| < l_k\} \subset \{|x \cdot \xi_k| < \theta_k\}.$$

Then the 0 *level set is a hyperplane.*

Proof. Fix $\theta_0 > 0$, and choose k large such that

$$\frac{\theta_k}{l_k} \leq \varepsilon \leq \varepsilon_1(\theta_0).$$

If $\theta_k \geq \theta_0$ then, by the theorem above, $\{u = 0\}$ is trapped in a flatter cylinder. We apply the theorem repeatedly till the height of the cylinder becomes less than θ_0.

In some system of coordinates we obtain

$$\{u = 0\} \cap \{|y'| < l_k',\ |y_n| < l_k'\} \subset \{|y_n| \leq \theta_k'\}$$

with

$$\theta_0 \geq \theta_k' \geq \eta_1 \theta_0, \quad \frac{\theta_k'}{l_k'} \leq \frac{\theta_k}{l_k} \leq \varepsilon,$$

hence

$$l_k' \geq \frac{\eta_1 \theta_0}{\varepsilon}.$$

We let $\varepsilon \to 0$ and obtain $\{u = 0\}$ is included in an infinite strip of width θ_0. The corollary is proved since θ_0 is arbitrary. □

As a consequence we have the following theorem.

Theorem 4.2. *Suppose that u is a local minimizer of J in $\mathbb{R}^n$ and $n \leq 7$. Then the level sets of u are hyperplanes.*

Alberti, Ambrosio and Cabre showed in [1] that monotone solutions of (4) satisfying (2) are in fact local minimizers for the energy (3) (see also the paper of Jerison and Monneau [22]), hence we obtain:

Theorem 4.3. *Let $u \in C^2(\mathbb{R}^n)$ be a solution of*

$$\Delta u = W'(u) \tag{8}$$

such that

$$|u| \leq 1, \quad \partial_{x_n} u > 0, \quad \lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1. \tag{9}$$

a) *If $n \leq 8$ then the level sets of u are hyperplanes.*

b) *If the 0 level set has at most linear growth at ∞ then the level sets of u are hyperplanes.*

The methods developed in [33] are quite general and can be applied for other types of nonlinear, possibly degenerate elliptic equations. Recently Valdinoci, Sciunzi and Savin [40] proved the theorems above for the energy

$$J_p(u, \Omega) := \int_\Omega \frac{1}{p} |\nabla u|^p + W(u) dx,$$

and the corresponding p-Laplace equation

$$\Delta_p u = W'(u), \quad \Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u).$$

Similar results can be obtained for solutions of nonlinear reaction–diffusion equations of the type

$$F(D^2 u) = f(u), \quad u \in C^2(\mathbb{R}^n),\ u_{x_n} > 0,\ \lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1, \tag{10}$$

where F is uniformly elliptic, and F, f are such that there exists a one dimensional solution g which solves the equation in all directions, i.e.,

$$F(D^2 g(x \cdot \nu) = f(g(x \cdot \nu)), \quad \text{for all } \nu \in \mathbb{R}^n, |\nu| = 1.$$

If the rescaled level sets $\varepsilon_k\{u = 0\}$ converge uniformly on compact sets as $\varepsilon_k \to 0$, then the limiting surface satisfies a uniformly elliptic equation (depending on F, f) for its second fundamental form (instead of the minimal surface equation). Following ideas from [33] one can prove a Liouville property for level sets of solutions of (10). More precisely, if $\{u = 0\}$ stays above a hyperplane $x \cdot \nu = \text{const.}$, then u depends only on one variable i.e,

$$u(x) = g(x \cdot \nu).$$

Moreover, if F is smooth, then the level sets satisfy an improvement of the flatness theorem. This implies that if the 0 level set is asymptotically flat at ∞, then u depends only on one variable. In particular, if the 0 level set is a Lipschitz graph in the x_n direction, then the same conclusion holds.

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