Vortices in the Ginzburg–Landau model of superconductivity

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Abstract. We review some mathematical results on the Ginzburg–Landau model with and without magnetic field. The Ginzburg–Landau energy is the standard model for superconductivity, able to predict the existence of vortices (which are quantized, topological defects) in certain regimes of the applied magnetic field. We focus particularly on deriving limiting (or reduced) energies for the Ginzburg–Landau energy functional, depending on the various parameter regimes, in the spirit of Γ-convergence. These passages to the limit allow to perform a sort of dimension-reduction and to deduce a rather complete characterization of the behavior of vortices for energy-minimizers, in agreement with the physics results. We also describe the behavior of energy critical points, the stability of the solutions, the motion of vortices for solutions of the gradient-flow of the Ginzburg–Landau energy, and show how they are also governed by those of the limiting energies.

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1. Introduction

1.1. Presentation of the Ginzburg–Landau model. We are interested in describing mathematical results on the two-dimensional Ginzburg–Landau model. This is a model of great importance and recognition in physics (with several Nobel prizes awarded for it: Landau, Ginzburg, Abrikosov). It was introduced by Ginzburg and Landau (see [15]) in the 1950s as a phenomenological model to describe superconductivity. Superconductivity was itself discovered in 1911 by Kammerling Ohnes. It consists in the complete loss of resistivity of certain metals and alloys at very low temperatures. The two most striking consequences of it are the possibility of permanent *superconducting currents* and the particular behavior that, when the material is submitted to an external magnetic field, that field gets expelled from it. Aside from explaining these phenomena, and through the very influential work of Abrikosov [1], the Ginzburg–Landau model allowed to predict the possibility of a *mixed state* in type-II superconductors where triangular vortex lattices appear. These vortices – a vortex can be described in a few words as a quantized amount of vorticity of the superconducting current localized near a point – have since been the objects of many observations and experiments.

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The Ginzburg–Landau theory has also been justified as a limit of the Bardeen–Cooper–Schrieffer (BCS) quantum theory, which explains superconductivity by the existence of "Cooper pairs" of superconducting electrons.

In addition to its importance in the modelling of superconductivity, the Ginzburg–Landau model turns out to be the simplest case of a gauge theory, and vortices to be the simplest case of topological solitons (for these aspects see [34], [21] and the references therein); moreover, it is mathematically extremely close to the Gross–Pitaevskii model for superfluidity and models for rotating Bose–Einstein condensates in which quantized vortices are also essential objects, to which the Ginzburg–Landau techniques have been successfully exported.

The 2D Ginzburg–Landau model leads (after various suitable rescalings) to describing the state of the superconducting sample submitted to the external field h_{ex}, below the critical temperature, through the energy functional

$$G_\varepsilon(u, A) = \frac{1}{2}\int_\Omega |\nabla_A u|^2 + |\mathrm{curl}\, A - h_{ex}|^2 + \frac{1}{2\varepsilon^2}\left(1 - |u|^2\right)^2. \tag{1}$$

In this expression, Ω is a two-dimensional open subset of $\mathbb{R}^2$, which in our study is always assumed for simplicity to be smooth, bounded and simply connected. One can imagine it represents the section of an infinitely long cylinder, or a thin film.

The first unknown u is a *complex*-valued function, called "order parameter" in physics, where it is generally denoted ψ. It is a condensed wave function, indicating the local state of the material or the phase in the Landau theory approach of phase transitions: $|u|^2$ is the density of "Cooper pairs" of superconducting electrons (responsible for superconductivity in the BCS approach). With our normalization $|u| \le 1$ and where $|u| \simeq 1$ the material is in the superconducting phase, while where $|u| = 0$, it is in the normal phase (i.e. behaves like a normal conductor), the two phases being able to coexist in the sample.

The second unknown is A, the electromagnetic vector-potential of the magnetic field, a function from Ω to $\mathbb{R}^2$. The magnetic field in the sample is deduced by $h = \mathrm{curl}\, A = \partial_1 A_2 - \partial_2 A_1$, it is thus a real-valued function in Ω. The notation ∇_A denotes the covariant gradient $\nabla - iA$; $\nabla_A u$ is thus a vector with complex components.

The *superconducting current* is a real vector given by $\langle iu, \nabla_A u\rangle$ where $\langle\, .\, , .\rangle$ denotes the scalar-product in $\mathbb{C}$ identified with $\mathbb{R}^2$, it may also be written as $\frac{i}{2}\left(u\overline{\nabla_A u} - \bar{u}\nabla_A u\right)$, where the bar denotes complex conjugation.

The parameter $h_{\mathrm{ex}} > 0$ represents the intensity of the applied field (assumed to be perpendicular to the plane of Ω). Finally, the parameter ε is the inverse of the "Ginzburg–Landau parameter" usually denoted κ, a non-dimensional parameter depending on the material only. We will be interested in the regime of small ε or $\kappa \to +\infty$, corresponding to high-κ superconductors (also called the London limit). In this limit, the characteristic size of the vortices, which is ε, tends to 0 and vortices become point-like.

The stationary states of the system are the critical points of G_ε, or the solutions of the Ginzburg–Landau equations (Euler–Lagrange equations associated to G_ε):

$$\text{(GL)}\begin{cases} -(\nabla_A)^2 u = \dfrac{1}{\varepsilon^2} u(1-|u|^2) & \text{in } \Omega \\ -\nabla^\perp h = \langle iu, \nabla_A u\rangle & \text{in } \Omega \\ h = h_{\text{ex}} & \text{on } \partial\Omega \\ \nu \cdot \nabla_A u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\nabla^\perp$ denotes the operator $(-\partial_2, \partial_1)$, and ν the outer unit normal to $\partial\Omega$.

The Ginzburg–Landau equations and functional are invariant under $\mathbb{U}(1)$-gauge transformations (it is an Abelian gauge-theory):

$$\begin{cases} u \mapsto ue^{i\Phi}, \\ A \mapsto A + \nabla\Phi. \end{cases} \tag{2}$$

The physically relevant quantities are those that are gauge-invariant, such as the energy G_ε, $|u|$, h, etc.

For more on the model and on the physics, we refer to the physics literature, in particular [53], [13]. For more reference on the results we present here, we refer to our monograph with E. Sandier [47].

We will also mention results on the simplified Ginzburg–Landau model, without magnetic field. It consists in taking $A = 0$ and $h_{\text{ex}} = 0$, then the energy reduces to

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \tag{3}$$

with still $u \colon \Omega \to \mathbb{C}$. Critical points of this energy are solutions of

$$-\Delta u = \frac{u}{\varepsilon^2}(1-|u|^2). \tag{4}$$

It is a complex-valued version of the Allen–Cahn model for phase-transitions. The first main study of this functional was done by Bethuel–Brezis–Hélein in the book [6], where they replace the effect of the applied field h_{ex} by a fixed Dirichlet boundary condition. Since then, this model has been extensively studied.

1.2. Vortices. A vortex is an object centered at an isolated zero of u, around which the phase of u has a nonzero winding number, called the *degree of the vortex*. When ε is small, it is clear from (1) that $|u|$ prefers to be close to 1, and a scaling argument hints that $|u|$ is different from 1 in regions of characteristic size ε. A typical vortex centered at a point x_0 "looks like" $u = \rho e^{i\varphi}$ with $\rho(x_0) = 0$ and $\rho = f(\frac{|x-x_0|}{\varepsilon})$ where $f(0) = 0$ and f tends to 1 as $r \to +\infty$, i.e. its characteristic core size is ε, and

$$\frac{1}{2\pi}\int_{\partial B(x_0, R\varepsilon)} \frac{\partial\varphi}{\partial\tau} = d \in \mathbb{Z}$$

is an integer, called the *degree of the vortex*. For example $\varphi = d\theta$ where θ is the polar angle centered at x_0 yields a vortex of degree d. We have the important relation

$$\operatorname{curl} \nabla\varphi = 2\pi \sum_i d_i \delta_{a_i} \tag{5}$$

where the a_i's are the centers of the vortices, the d_i's their degrees, and δ the Dirac mass.

In the limit $\varepsilon \to 0$ vortices become *point-like* or more generally, in any dimension, *codimension* 2 singularities (see [30], [7]) – to be compared with the case of real-valued phase-transition models (Allen–Cahn), where the order parameter u is real-valued, leading to codimension 1 singular sets in the limit.

1.3. Critical fields. When an external magnetic field is applied to a superconductor, several responses can be observed depending on the intensity of the field h_{ex}.

There are three main critical values of h_{ex} or *critical fields* H_{c_1}, H_{c_2}, and H_{c_3}, for which phase-transitions occur. Below the first critical field, which is of order $O(|\log\varepsilon|)$ (as first established by Abrikosov), the superconductor is everywhere in its superconducting phase $|u| \sim 1$ and the magnetic field does not penetrate (this is called the Meissner effect or Meissner state). At H_{c_1}, the first vortice(s) appear. Between H_{c_1} and H_{c_2}, the superconducting and normal phases (in the form of vortices) coexist in the sample, and the magnetic field penetrates through the vortices. This is called the *mixed state*. The higher $h_{\text{ex}} > H_{c_1}$, the more vortices there are. Since they repel each other, they tend to arrange in these triangular Abrikosov lattices in order to minimize their repulsion. Reaching $H_{c_2} \sim \frac{1}{\varepsilon^2}$, the vortices are so densely packed that they overlap each other, and at H_{c_2} a second phase transition occurs, after which $|u| \sim 0$ inside the sample, i.e. all superconductivity in the bulk of the sample is lost.

In the interval $[H_{c_2}, H_{c_3}]$ however, superconductivity persists near the boundary, this is called *surface superconductivity*. Above $H_{c_3} = O\left(\frac{1}{\varepsilon^2}\right)$ (defined in decreasing fields), the sample is completely in the normal phase $u \equiv 0$, the magnetic field completely penetrates, and decreasing the field below H_{c_3}, surface superconductivity is observed.

1.4. Questions, results and methods. The main question is to understand mathematically the behavior above, and in particular:

– To understand the vortices and their repartition, interaction.

– To understand the influence of the boundary conditions and/or of the applied field.

– To find the asymptotic values of the critical fields (as $\varepsilon \to 0$).

– To prove compactness results and derive *limiting energies/reduced problems*, thus following the strategy of Γ-convergence. This enables to understand the behavior of global minimizers (or energy minimizers) and their vortices. In order to achieve this, one needs to find *lower bounds* for the energy, together with matching upper bounds.

– To understand and find local minimizers. This is done through a special "local minimization in energy sectors" method.

– To understand the behavior of critical points of the energy (i.e. solutions which are not necessarily stable), that is to pass to the limit $\varepsilon \to 0$ in the Ginzburg–Landau equations (GL). The method used here is to pass to the limit in the "stress-energy tensor".

– To derive the limiting motion law of vortices, and to understand its link with the reduced energies mentioned above.

2. Mathematical tools

Various methods were introduced to describe vortices, since [6]. A crucial difference between the analysis for (1) and the one for (3) is that for the case with magnetic field (1) we really need to be able to handle numbers of vortices which are *unbounded* as $\varepsilon \to 0$. We designed tools able to capture vortices for arbitrary maps u (not necessarily solutions), and to treat possibly unbounded numbers.

Let us describe the two main technical tools which we use throughout: the *"vortex ball construction"*, yielding the lower bounds on the energy, and the *vorticity measures*, which serve to describe vortex-densities instead of individual vortices.

2.1. The vortex-ball construction. This serves to obtain lower bounds for bounded or unbounded numbers of vortices. The idea is that, whatever the map u, for topological reasons, a vortex of degree d confined in a ball of radius r should cost at least an energy $\pi d^2 \log \frac{r}{\varepsilon}$. Then, since there may be a large number of these vortices, one must find a way to add up those estimates. It is done following the ball-growth method of Sandier [38] and Jerrard [22] (which consists in growing annuli of same conformal type and merging them appropriately). The best result to date is the following:

Theorem 2.1 (see [47]). *Let (u, A) be a configuration such that $E_\varepsilon(|u|) \le C\varepsilon^{\alpha-1}$ with $\alpha > 0$, then for any $r \in (\varepsilon^{\frac{\alpha}{2}}, 1)$, and ε small enough, there exists a finite collection of disjoint closed balls $\{B_i\}_i$ of centers a_i, of sum of the radii r, covering $\{|u| \le 1 - \varepsilon^{\frac{\alpha}{4}}\} \cap \{x \in \Omega, \operatorname{dist}(x, \partial\Omega) \ge \varepsilon\}$, and such that*

$$\frac{1}{2}\int_{\bigcup_i B_i} |\nabla_A u|^2 + |\operatorname{curl} A|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \ge \pi D\left(\log \frac{r}{D\varepsilon} - C\right) \tag{6}$$

where $D = \sum_i |d_i|$ and $d_i = \deg(u, \partial B_i)$.

This lower bound is very general, it does not require any hypothesis on (u, A) other than a reasonable (but quite large) upper bound on its energy, and it is in fact sharp (examples where it is can be constructed).

2.2. The vorticity measures. Recall that a complex-valued map u can be written in polar coordinates $u = \rho e^{i\varphi}$ with a phase φ which can be multi-valued. Given a configuration (u, A), we define its *vorticity* by

$$\mu(u, A) = \operatorname{curl} \langle iu, \nabla_A u\rangle + \operatorname{curl} A. \tag{7}$$

Formally

$$\langle iu, \nabla u\rangle = \rho^2 \nabla\varphi \simeq \nabla\varphi,$$

considering that $\rho = |u| \simeq 1$. Taking the curl of this expression and using (5), one would get the approximate (formal) relation

$$\mu(u, A) \simeq 2\pi \sum_i d_i \delta_{a_i} \tag{8}$$

where a_i's are the vortices of u and d_i's their degrees. Thus we see why the quantity μ corresponds to a vorticity-measure of the map u (just like the vorticity for fluids). The following theorem gives a rigorous content to (8).

Theorem 2.2 (see [24] and [47]). *The $(a_i, d_i)'s$ being given by the previous theorem, we have*

$$\Big\| \mu(u, A) - 2\pi \sum_i d_i \delta_{a_i} \Big\|_{(C_0^{0,\gamma}(\Omega))^*} \le C r^\gamma G_\varepsilon^0(u, A) \quad \text{for all } 0 < \gamma < 1,$$

where G_ε^0 is the energy when $h_{\mathrm{ex}} = 0$.

The previous theorem allowed to give a control on the mass of $2\pi \sum_i d_i \delta_{a_i}$ as measures. This one ensures that if r is taken small enough, $\mu(u, A)$ and $2\pi \sum_i d_i \delta_{a_i}$ are close in a weak norm. Combining the two yields a compactness result on the vorticity $\mu(u, A)$, if rescaled by dividing by the number of vortices. It also ensures the limiting vorticity is a bounded Radon measure. This is the limiting "vortex-density" we are looking to characterize in various situations.

Remark 2.3. When the Ginzburg–Landau equations (GL) are satisfied, taking the curl of the second relation, we find that the vorticity and the induced field are linked by the relation

$$\begin{cases} -\Delta h + h = \mu(u, A) & \text{in } \Omega, \\ h = h_{\mathrm{ex}} & \text{on } \partial\Omega. \end{cases} \tag{9}$$

Thus the knowledge of the vorticity is equivalent to that of the induced field h.

3. Global minimization (Γ-convergence type) results

3.1. Results for E_ε

3.1.1. In two dimensions. For the two-dimensional simplified model (3), the main result of [6] can be written in the following form.

Theorem 3.1 (Bethuel–Brezis–Hélein [6]). *Let Ω be a strictly starshaped simply connected domain of $\mathbb{R}^2$ and $g\colon \partial\Omega \to \mathbb{S}^1$ a smooth map of degree $d > 0$. If u_ε minimizes E_ε among maps with values g on $\partial\Omega$, then, as $\varepsilon \to 0$, up to extraction of a subsequence, there exist d distinct points $a_1, \dots, a_d \in \Omega$ such that $u_\varepsilon \to u_*$ in $C^k_{\mathrm{loc}}\big(\Omega\setminus \bigcup_i \{a_i\}\big)$ where*

1. *u_* is a harmonic map from $\Omega\setminus\{a_1, \dots, a_d\}$ to $\mathbb{S}^1$ with $u_* = g$ on $\partial\Omega$ and with degree $d_i = 1$ around each a_i,*
2. *$(a_1, \dots, a_d)$ is a minimizer of the renormalized energy W with $d_i = 1$,*
3. *$E_\varepsilon(u_\varepsilon) \geq \pi d|\log\varepsilon| + W(a_1, \dots, a_d) + o(1)$.*

Here W denotes a function of the points $a_i \in \Omega$ (depending also on the degrees), called *"renormalized energy"* and which has a form

$$W(x_1, \dots, x_n) = -\pi \sum_{i\neq j} d_i d_j \log|x_i - x_j| + \text{terms of interaction with the boundary.}$$

W corresponds to the finite part of the energy left when subtracting the "infinite" self-interaction cost of the vortices $\pi d|\log\varepsilon|$, i.e. to the interaction between the vortices (vortices of same sign repel, of opposite sign attract).

This result can be phrased as a Γ-convergence result (in the sense of DeGiorgi):

Proposition 3.2 (Γ-convergence of E_ε).

1. *For any family $\{u_\varepsilon\}_\varepsilon$ such that $E_\varepsilon(u_\varepsilon) \leq C|\log\varepsilon|$ and $u_\varepsilon = g$ on $\partial\Omega$; up to extraction, there exists a finite family (a_i, d_i) of n points + degrees such that $\sum_{i=1}^n d_i = d$ and*

$$\operatorname{curl}\langle iu_\varepsilon, \nabla u_\varepsilon\rangle \rightharpoonup 2\pi \sum_{i=1}^n d_i \delta_{a_i},$$

$$E_\varepsilon(u_\varepsilon) \geq \pi \sum_{i=1}^n |d_i||\log\varepsilon| + W(a_1, \dots, a_n) + o(1) \quad \text{as } \varepsilon \to 0.$$

2. *For all distinct a_i's and $d_i = \pm 1$, there exists u_ε such that*

$$E_\varepsilon(u_\varepsilon) \leq \pi n|\log\varepsilon| + W(a_1, \dots, a_n) + o(1).$$

Fixing the degrees $d_i = \pm 1$ and the number of vortices n, this result states exactly that $E_\varepsilon - \pi n|\log\varepsilon|$ Γ-converges to W. This reduces the dimension of the problem, by reducing the minimization of E_ε to the simple one of the limiting energy W, which is a function on a finite dimensional set.

3.1.2. In higher dimensions. Three-dimensional as well as higher-dimensional versions of Theorem 3.1 have been given, in particular by Lin–Rivière [30], Sandier [39], Bethuel–Brezis–Orlandi [7]. Jerrard and Soner gave a Γ-convergence formulation (i.e. analogous to Proposition 3.2), later improved by Alberti–Baldo–Orlandi [2]. Here E_ε refers to the n-dimensional version of the energy (3). When $n = 3$ the vortex-set (or zero-set of u) is a set of lines, in higher dimensions it is a set of codimension 2, and the vorticity is best described in the language of currents.

Theorem 3.3 (Jerrard–Soner [24]). *Let $\{u_\varepsilon\}_\varepsilon$ be a family such that $E_\varepsilon(u_\varepsilon) \le C|\log\varepsilon|$; up to extraction, there exists an integer-multiplicity rectifiable $(n-2)$-dimensional current J such that*

$$\mu_\varepsilon(u_\varepsilon) := *d\langle iu_\varepsilon, du_\varepsilon\rangle \rightharpoonup 2\pi J \quad \text{in } (C_0^{0,\gamma}(\Omega))^*$$

for all $\gamma < 1$ (in the language of differential forms) and

$$\varliminf_{\varepsilon\to 0} \frac{E_\varepsilon(u_\varepsilon)}{|\log\varepsilon|} \ge \pi \|J\|(\Omega),$$

where $\|J\|(\Omega)$ is the total mass of the current.

The total mass of the current corresponds in dimension 3 to the total length of the vortex-lines. The result of [30], [39] essentially states that minimizers of E_ε have a vorticity μ_ε which converges to minimizers of the length $\|J\|(\Omega)$, thus have vortices which converge to straight lines or minimal connections (or codimension 2 minimal currents in higher dimension). The result of [7] generalizes it to critical points and proves that critical points of E_ε have vortices which converges to stationary varifolds.

Observe that the situation is quite different from the dimension 2, because the main order $|\log\varepsilon|$ of the energy already gives a nontrivial limiting problem: the mass of the limiting object J; in contrast with the 2D problem which only leads to minimizing the number of points (one needs to go to the next order in the energy to get an interesting problem: the minimization of W).

3.2. Global minimization results for G_ε

3.2.1. Close to H_{c_1}. Let us introduce h_0 the solution of

$$\begin{cases} -\Delta h_0 + h_0 = 0 & \text{in } \Omega, \\ h_0 = 1 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

and

$$C(\Omega) = (2 \max |h_0 - 1|)^{-1}. \tag{11}$$

We also introduce the set $\Lambda = \{x \in \Omega / h_0(x) = \min h_0\}$ and we will assume here for simplicity that it is reduced to only one point called p, and denote $Q(x) = \langle D^2 h_0(p)x, x\rangle$, assumed to be definite positive. With these notations, a first essential result is the asymptotic formula for H_{c_1} (confirming physical predictions that $H_{c_1} = O(|\log \varepsilon|)$):

$$H_{c_1} = C(\Omega)|\log \varepsilon| + O(1). \tag{12}$$

Theorem 3.4 ([48], [47]). *Assume $h_{\text{ex}} \le H_{c_1} + O(\log\ |\log \varepsilon|)$, then for $h_{\text{ex}} \in (H_n, H_{n+1})$ where H_n has the expansion*

$$H_n = C(\Omega)\left(|\log \varepsilon| + (n-1)\log \frac{|\log \varepsilon|}{n}\right) + \textit{lower order terms,}$$

global minimizers of G_ε have exactly n vortices of degree 1, $a_i^\varepsilon \to p$ as $\varepsilon \to 0$ and the $\tilde{a_i^\varepsilon} = \sqrt{\frac{h_{\text{ex}}}{n}}(a_i^\varepsilon - p)$ converge as $\varepsilon \to 0$ to a minimizer of

$$w_n(x_1, \dots, x_n) = -\pi \sum_{i \ne j} \log |x_i - x_j| + \pi n \sum_{i=1}^{n} Q(x_i). \tag{13}$$

Through this theorem we see that the behavior is as expected: below $H_{c_1} = H_1$ there are no vortices in energy minimizers (in addition it was proved in [49] that the minimizer is unique), then at H_{c_1} the first vortex becomes favorable, close to the point p. Then, there is a sequence of additional critical fields $H_2, H_3, \dots$ separated by increments of $\log\ |\log \varepsilon|$, for which a second, third, etc, vortex becomes favorable. Each time the optimal vortices are located close to p as $\varepsilon \to 0$, and after blowing-up at the scale $\sqrt{\frac{h_{\text{ex}}}{n}}$ around p, they converge to configurations which minimize w_n in $\mathbb{R}^2$. Now, w_n, which appears as a limiting energy (after that rescaling) contains a repulsion term like W, and a confinement term due to the applied field. It is a standard two-dimensional interaction, however rigorous results on its minimization are hard to obtain as soon as $n \ge 3$. When Q has rotational symmetry, numerical minimization (see Gueron–Shafrir [17]) yields very regular shapes (regular polygons for $n \le 6$, regular stars) which look very much like the birth of a triangular lattice as n becomes large. All these results are in very good agreement with experimental observations.

3.2.2. Global minimizers in the intermediate regime. In the next higher regime of applied field, the result is the following:

Theorem 3.5 ([47]). *Assume h_{ex} satisfies, as $\varepsilon \to 0$,*

$$\log\ |\log \varepsilon| \ll h_{\text{ex}} - H_{c_1} \ll |\log \varepsilon|.$$

Then there exists $1 \ll n_\varepsilon \ll h_{\text{ex}}$ *such that*

$$h_{\text{ex}} \sim C(\Omega)\left(|\log\varepsilon| + n_\varepsilon \log\frac{|\log\varepsilon|}{n_\varepsilon}\right)$$

and if $(u_\varepsilon, A_\varepsilon)$ *minimizes* G_ε, *then*

$$\frac{\tilde{\mu}(u_\varepsilon, A_\varepsilon)}{2\pi n_\varepsilon} \rightharpoonup \mu_0$$

where $\tilde{\mu}(u_\varepsilon, A_\varepsilon)$ *is the push-forward of the measure* $\mu(u_\varepsilon, A_\varepsilon)$ *under the blow-up* $x \mapsto \sqrt{\frac{h_{\text{ex}}}{n_\varepsilon}}(x-p)$, *and* μ_0 *is the unique minimizer over probability measures of*

$$I(\mu) = -\pi\int_{\mathbb{R}^2\times\mathbb{R}^2} \log|x-y|\,d\mu(x)\,d\mu(y) + \pi\int_{\mathbb{R}^2} Q(x)\,d\mu(x). \tag{14}$$

Here, n_ε corresponds to the expected optimal number of vortices. In [47] our result is really phrased as a Γ-convergence of G_ε in the regime $1 \ll n \ll h_{\text{ex}}$, reducing the minimization of G_ε to that of the limiting energy I. The problem of minimizing I is a classical one in potential theory. Its minimizer μ_0 is a probability measure of constant density over a subdomain of $\mathbb{R}^2$ (typically a disc or an ellipse). This result is in continuous connection with Theorem 3.4, except $n_\varepsilon \gg 1$. Again, vortices in the minimizers converge to p as $\varepsilon \to 0$, and when one blows up at the right scale $\sqrt{\frac{h_{\text{ex}}}{n_\varepsilon}}$ around p, one obtains a uniform density of vortices μ_0 in a subdomain of $\mathbb{R}^2$.

3.2.3. Global minimizers in the regime n_ε proportional to h_{ex}. This happens in the next regime: $h_{\text{ex}} \sim \lambda|\log\varepsilon|$ with $\lambda > C(\Omega)$.

Theorem 3.6 ([42], [47]). *Assume* $h_{\text{ex}} = \lambda|\log\varepsilon|$ *where* $\lambda > 0$ *is a constant independent of* ε. *As* $\varepsilon \to 0$, $\frac{G_\varepsilon}{h_{\text{ex}}^2}$ Γ*-converges to*

$$E_\lambda(\mu) = \frac{\|\mu\|}{2\lambda} + \frac{1}{2}\int_\Omega |\nabla h_\mu|^2 + |h_\mu - 1|^2,$$

defined over bounded Radon measures which are in $H^{-1}(\Omega)$, *where* $\|\mu\|$ *is the total mass of* μ *and* h_μ *is the solution to*

$$\begin{cases} -\Delta h_\mu + h_\mu = \mu & \text{in } \Omega, \\ h_\mu = 1 & \text{on } \partial\Omega. \end{cases} \tag{15}$$

Consequently, if $(u_\varepsilon, A_\varepsilon)$ *minimizes* G_ε, *then*

$$\frac{\mu(u_\varepsilon, A_\varepsilon)}{h_{\text{ex}}} \rightharpoonup \mu_*,$$

μ_* *being the unique minimizer over* $H^{-1}(\Omega) \cap (C_0^0(\Omega))^*$ *of* E_λ.

Observe also that

$$E_\lambda(\mu) = \frac{1}{2\lambda}\int_\Omega |\mu| + \frac{1}{2}\int_{\Omega\times\Omega} G(x,y)\, d(\mu-1)(x)\, d(\mu-1)(y) \tag{16}$$

where G is the solution to $-\Delta G + G = \delta_y$ with $G = 0$ on $\partial\Omega$. That way, the similarity with I is more apparent.

Again, by Γ-convergence, we reduce to minimizing the limiting energy E_λ on the space of bounded Radon measures on Ω. It turns out that this problem is dual, in the sense of convex duality, to an obstacle problem:

Proposition 3.7. *μ minimizes E_λ if and only if h_μ is the minimizer for*

$$\min_{\substack{h\geq 1-\frac{1}{2\lambda}\\ h=1 \text{ on } \partial\Omega}} \int_\Omega |\nabla h|^2 + h^2. \tag{17}$$

The solution of the obstacle problem (17) is well-known, and given by a variational inequality (see [28]). Obstacle problems are a particular type of *free-boundary problems*, the free-boundary here being the boundary of the coincidence set

$$\omega_\lambda = \left\{ x \in \Omega / h(x) = 1 - \frac{1}{2\lambda} \right\}.$$

Then h verifies $-\Delta h + h = 0$ outside of ω_λ, so ω_λ is really the support of μ_*, on which μ_* is equal to the constant density $\left(1 - \frac{1}{2\lambda}\right) dx$. An easy analysis of this obstacle problem yields the following:

1. $\omega_\lambda = \varnothing$ (hence $\mu_* = 0$) if and only if $\lambda < C(\Omega)$, where $C(\Omega)$ was given by (11). (This corresponds to the case $h_{\mathrm{ex}} < H_{c_1}$.)

2. For $\lambda = C(\Omega)$, $\omega_\lambda = \{p\}$. This is the case when $h_{\mathrm{ex}} \sim H_{c_1}$ at leading order. In the scaling chosen here $\mu_* = 0$ but the true behavior of the vorticity is ambiguous unless going to the next order term as done in Theorems 3.4 and 3.5.

3. For $\lambda > C(\Omega)$, the measure of ω_λ is nonzero, so the limiting vortex density $\mu_* \neq 0$. Moreover, as λ increases (i.e. as h_{ex} does), ω_λ increases. When $\lambda = +\infty$, ω_λ becomes Ω and $\mu_* = 1$, this corresponds to the case $h_{\mathrm{ex}} \gg |\log\varepsilon|$ of the next subsection.

3.2.4. Global minimizers in the regime $|\log\varepsilon| \ll h_{\mathrm{ex}} \ll \varepsilon^{-2}$. For applied fields much larger than $|\log\varepsilon|$ but below H_{c_2}, even though the number of vortices becomes very large, the minimization problem becomes local and can be solved by blowing-up and using Theorem 3.6. The energy-density and the vortex repartition are thus found to be uniform, as seen in:

Theorem 3.8 ([41], [47]). *Assume, as* $\varepsilon \to 0$*, that* $|\log \varepsilon| \ll h_{\rm ex} \ll 1/\varepsilon^2$*. Then, if* $(u_\varepsilon, A_\varepsilon)$ *minimizes* G_ε*, and letting* $g_\varepsilon(u, A)$ *denote the energy-density* $\frac{1}{2}\big(|\nabla_A u|^2 + |h - h_{\rm ex}|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2\big)$*, we have*

$$\frac{2g_\varepsilon(u_\varepsilon, A_\varepsilon)}{h_{\rm ex} \log \frac{1}{\varepsilon\sqrt{h_{\rm ex}}}} \rightharpoonup dx \quad \text{as } \varepsilon \to 0$$

in the weak sense of measures, where dx *denotes the two-dimensional Lebesgue measure; and thus*

$$\min_{(u,A)\in H^1\times H^1} G_\varepsilon(u, A) \sim \frac{|\Omega|}{2} h_{\rm ex} \log \frac{1}{\varepsilon\sqrt{h_{\rm ex}}} \quad \text{as } \varepsilon \to 0,$$

where $|\Omega|$ *is the area of* Ω*. Moreover*

$$\frac{h_\varepsilon}{h_{\rm ex}} \to 1 \qquad \text{in } H^1(\Omega)$$

$$\frac{\mu(u_\varepsilon, A_\varepsilon)}{h_{\rm ex}} \to dx \quad \text{in } H^{-1}(\Omega).$$

In Theorems 3.5, 3.6 and 3.8 we find an optimal limiting density which is constant on its support (ω_λ or Ω). This provides a first (but very incomplete) confirmation of the Abrikosov lattices of vortices observed and predicted in physics.

3.2.5. Global minimizers in higher applied field. Here, we will present the situation with decreasing applied field. For large enough applied field, the only solution is the (trivial) normal one $u \equiv 0$, $h \equiv h_{\rm ex}$.

Giorgi and Phillips have proved in [16] that this is the case for $h_{\rm ex} \geq C\varepsilon^{-2}$.

Theorem 3.9 (Giorgi–Phillips [16]). *There exists a constant* C *such that if* $h_{\rm ex} \geq C\varepsilon^{-2}$ *and* ε *is small enough, then the only solution to* (GL) *is the normal one* $u \equiv 0$*,* $h \equiv h_{\rm ex}$*.*

This result implies the upper bound $H_{c_3} \leq C\varepsilon^{-2}$ for that constant C.

Decreasing the applied field to H_{c_3}, a bifurcation from the normal solution of a branch of solutions with surface superconductivity occurs. The linear analysis of this bifurcation was first performed in the half-plane by De Gennes [13], then by Bauman–Phillips–Tang [4] in the case of a disc; and for general domains, formally by Chapman [11], Bernoff–Sternberg [5], Lu and Pan [33], then rigorously by Del Pino–Felmer–Sternberg [14], Helffer–Morame [19], Helffer–Pan [18]. The nucleation of surface superconductivity takes place near the point of maximal curvature of the boundary, and the asymptotics for H_{c_3} is given by the following result.

Theorem 3.10.

$$H_{c_3} \sim \frac{\varepsilon^{-2}}{\beta_0} + \frac{C_1}{\beta_0^{3/2}} \max(\mathrm{curv}(\partial\Omega))\varepsilon^{-1},$$

where β_0 is the smallest eigenvalue of a Schrödinger operator with magnetic field in the half-plane.

The behavior of energy minimizers for $H_{c_2} \leq h_{\text{ex}} \leq H_{c_3}$ has been studied by Pan [37] (see also Almog), who showed that, as known by physicists, minimizers present surface superconductivity which spreads to the whole boundary, with exponential decay of $|u|$ from the boundary of the domain.

At H_{c_2}, one goes from surface superconductivity to bulk-superconductivity. It was established by Pan [37] that

$$H_{c_2} = \varepsilon^{-2}.$$

Qualitative results on bulk-superconductivity below H_{c_2} were obtained in [44], where we established in particular how bulk-superconductivity increases (in average) as h_{ex} is lowered immediately below H_{c_2}.

4. Local minimizers: branches of solutions

The techniques developed to describe energy-minimizers also allow to find branches of locally minimizing (hence stable, and physically observable) solutions, with prescribed numbers of vortices. This is a problem of inverse type: given critical points or minimizers of the limiting energy w, can we find critical points / local minimizers of G_ε which converge to it? For answers on that question regarding E_ε, see the book of Pacard and Rivière [36].

Theorem 4.1 ([49], [47]). *For $\varepsilon < \varepsilon_0$, and for any n and h_{ex} belonging to appropriate intervals, there exists a locally minimizing critical point $(u_\varepsilon, A_\varepsilon)$ of G_ε such that u_ε has exactly n zeroes $a_1^\varepsilon, \dots, a_n^\varepsilon$ and there exists $R > 0$ such that $|u_\varepsilon| \geq \frac{1}{2}$ in $\Omega \setminus \bigcup_i B(a_i^\varepsilon, R\varepsilon)$, with $\deg(u_\varepsilon, \partial B(a_i^\varepsilon, R)) = 1$. Moreover:*

1. *If n and h_{ex} are constant, independent of ε, up to extraction of a subsequence, the configuration $(a_1^\varepsilon, \dots, a_n^\varepsilon)$ converges as $\varepsilon \to 0$ to a minimizer of the function*

$$R_{n,h_{\text{ex}}} = -\pi \sum_{i \neq j} \log|x_i - x_j| + \pi \sum_{i,j} S_\Omega(x_i, x_j) + 2\pi h_{\text{ex}} \sum_{i=1}^{n} (h_0 - 1)(x_i).$$

   where S_Ω is the regular part of a Green function associated to Ω.

2. *If $n = O(1)$ and $h_{\text{ex}} \to \infty$, up to extraction of a subsequence, the configuration of the $\tilde{a}_i^\varepsilon = \sqrt{\frac{h_{\text{ex}}}{n}}(a_i^\varepsilon - p)$ converges as $\varepsilon \to 0$ to a minimizer of w_n.*

3. *If $n \to \infty$ and $h_{\text{ex}} \to \infty$, then denoting again $\tilde{a}_i^\varepsilon = \sqrt{\frac{h_{\text{ex}}}{n}}(a_i^\varepsilon - p)$,*

$$\frac{1}{n} \sum_{i=1}^{n} \delta_{\tilde{a}_i^\varepsilon} \rightharpoonup \mu_0$$

   where μ_0 is the unique minimizer of I (defined in (14)).

The method of the proof consists in finding these solutions as local minimizers by minimizing G_ε over some open sets of the type $U_n = \{(u, A)/\pi(n-1)|\log\varepsilon| < G^0_\varepsilon(u, A) < \pi(n+1)|\log\varepsilon|\}$. Minimizing over U_n consists, roughly speaking, in minimizing over configurations with n vortices, the difficulty is in proving that the minimum over U_n is achieved at an interior point (this comes from the quantization of the energetic cost of vortices), thus yielding a local energy minimizer.

We thus show the multiplicity of locally minimizing solutions, for a given h_{ex}, in a wide range (from $h_{\mathrm{ex}} = O(1)$ to $h_{\mathrm{ex}} \gg |\log\varepsilon|$): essentially, solutions with 0, 1, 2, 3, ... vortices coexist and are all stable, even if not energy-minimizing.

We also have derived multiple "renormalized energies" $R_{n,h_{\mathrm{ex}}}$, w_n, $I(\mu)$ corresponding to the three regimes above. Observe that w_n corresponds somewhat to the limit of $R_{n,h_{\mathrm{ex}}}$ as $h_{\mathrm{ex}} \to \infty$, while I is a continuum limit as $n \to \infty$ (but still $n \ll h_{\mathrm{ex}}$) of w_n. E_λ can also be seen as the limit as both n and h_{ex} tend to ∞ but n/h_{ex} not tending to 0. Thus these limiting or renormalized energies are not only valid for global minimization, but also for local minimization.

5. Critical points approach

The issue here is to derive conditions on limiting vortices or vortex-densities just assuming that we start from a family of solutions to (GL) or critical points of G_ε, not necessarily stable. This strategy was already implemented for the functional E_ε in [6], leading to

Theorem 5.1 (Bethuel–Brezis–Hélein [6]). *If u_ε is a sequence of solutions of* (4) *in* Ω *with $u_\varepsilon = g$ on $\partial\Omega$,* $\deg(g) = d > 0$*, and $E_\varepsilon(u_\varepsilon) \le C|\log\varepsilon|$ then, as $\varepsilon \to 0$ and up to extraction of a subsequence, there exist distinct points $a_1, \dots, a_n \in \Omega$, and degrees $d_1, \dots, d_n$ with $\sum_{i=1}^n d_i = d$, such that $u_\varepsilon \to u_*$ in $C^k_{\mathrm{loc}}\big(\Omega\setminus\bigcup_i\{a_i\}\big)$ where u_* is a harmonic map from $\Omega\setminus\{a_1, \dots, a_n\}$ to $\mathbb{S}^1$ with $u_* = g$ on $\partial\Omega$ and with degree d_i around each a_i. Moreover $(a_1, \dots, a_n)$ is a critical point of W (the d_i's being fixed).*

Thus, the vortices of critical points of E_ε converge to critical points of the limiting energy W.

It is a corresponding result that we obtain for the vortex-densities for G_ε. The strategy consists similarly in passing to the limit $\varepsilon \to 0$, not in (GL), but in the stationarity relation

$$\frac{d}{dt}_{|t=0} G_\varepsilon(u \circ \chi_t, A \circ \chi_t) = 0$$

satisfied for the critical points (with χ_t a one-parameter family of diffeomorphisms such that $\chi_0 = Id$). That relation is equivalent by Noether's theorem to a relation of the form

$$\operatorname{div} T_\varepsilon = 0$$

where T_ε is called the "stress-energy" or "energy-momentum" tensor. For the present energy-functional

$$T_\varepsilon = \frac{1}{2}\begin{pmatrix} |\partial_1^A u|^2 - |\partial_2^A u|^2 & 2\langle \partial_1^A u, \partial_2^A u\rangle \\ 2\langle \partial_1^A u, \partial_2^A u\rangle & |\partial_2^A u|^2 - |\partial_1^A u|^2 \end{pmatrix} + \left(\frac{h^2}{2} - \frac{(1-|u|^2)^2}{2\varepsilon^2}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $\partial_j^A = \partial_j - iA_j$.

In what follows we assume that $(u_\varepsilon, A_\varepsilon)$ are sequences of critical points of G_ε such that $G_\varepsilon(u_\varepsilon, A_\varepsilon) \le C\varepsilon^{-\alpha}$, $\alpha < \frac{1}{3}$, and n_ε is defined as $\sum_i |d_i|$ where the d_i's are the degrees of the balls of total radius $r = \varepsilon^{2/3}$ given by Theorem 2.1.

Theorem 5.2 ([43], [47]). *Let $(u_\varepsilon, A_\varepsilon)$ and n_ε be as above. If n_ε vanishes in a neighborhood of* 0 *then $\mu(u_\varepsilon, A_\varepsilon)$ tends to* 0 *in $W^{-1,p}(\Omega)$ for some $p \in (1, 2)$. If not, then going to a subsequence*

$$\frac{\mu(u_\varepsilon, A_\varepsilon)}{n_\varepsilon} \to \mu \tag{18}$$

in $W^{-1,p}(\Omega)$ for some $p \in (1, 2)$ where μ is a measure. Moreover, one of the two following possibilities occur (*after extraction of a subsequence if necessary*).

1. *If $n_\varepsilon = o(h_{\mathrm{ex}})$ then*
$$\mu \nabla h_0 = 0. \tag{19}$$

2. *If $h_{\mathrm{ex}}/n_\varepsilon \to \lambda \ge 0$, then, letting h_μ be the solution of* (15)*, the symmetric* 2*-tensor T_μ with coefficients*
$$T_{ij} = \frac{1}{2}\begin{pmatrix} |\partial_1 h_\mu|^2 - |\partial_2 h_\mu|^2 & 2\partial_1 h_\mu \partial_2 h_\mu \\ 2\partial_1 h_\mu \partial_2 h_\mu & |\partial_2 h_\mu|^2 - |\partial_1 h_\mu|^2 \end{pmatrix} - \frac{h_\mu^2}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
is divergence-free in finite part.

In the latter case, if μ is such that $h_\mu \in H^1(\Omega)$ then T_μ is in L^1 and divergence-free in the sense of distributions. Moreover $|\nabla h_\mu|^2$ is in $W^{1,q}_{\mathrm{loc}}(\Omega)$ for any $q \in [1, +\infty)$.

If we assume in addition that $\mu \in L^p(\Omega)$ for some $p > 1$, then

$$\mu \nabla h_\mu = 0.$$

Finally, if we assume $\nabla h_\mu \in C^0(\Omega) \cap W^{1,1}(\Omega)$ (*this is the case if μ is in L^p, for some $p > 1$ for instance*)*, then h_μ is in $C^{1,\alpha}(\Omega)$ for any $\alpha \in (0, 1)$ and $0 \le h_\mu \le 1$. In this case*

$$\mu = h_\mu \mathbf{1}_{\{|\nabla h_\mu| = 0\}}, \tag{20}$$

(*where* **1** *stands for the characteristic function*) *and thus μ is a nonnegative L^∞ function.*

To sum up, the limiting condition is $\mu \nabla h_0$ in the first case, it means that when there are too few of them, vortices all concentrate at the critical points of h_0 at the limit. In the second case, it is a weak form of the relation $\mu \nabla h_\mu = 0$ (which cannot be written as such when h_μ is not regular enough, counterexamples can be built).

We obtained an analogous result for critical points of E_ε with possibly large numbers of vortices.

Also, once more, the result has a higher-dimensional version for the functional E_ε: as mentioned in Section 3.1.2, it was proved in [30], [7] that the vorticities for critical points of E_ε converge to stationary varifolds, i.e. critical points for the length/area.

6. Study of the dynamics

The philosophy that has been successful in the minimization approach has been to extract limiting reduced energies (most often depending on some parameter regimes). These energies come up as Γ-limits, thus giving the behavior of energy-minimizers, but we have seen that they are not only relevant for energy-minimizers, but also for critical points ("critical points converge to critical points") and for local minimizers. It turns out that these limiting energies are also relevant for the study of dynamical problems, such as that of the heat-flow of (GL) and (4).

6.1. Energy-based method

6.1.1. Abstract argument. In [46], [50], we gave criteria to determine when a family of energies F_ε converges to its Γ-limit F in a sort of C^1 or C^2 sense which allows to pass to the limit in the associated gradient-flow (we called this "Γ-convergence of gradient flows"). The abstract situation is the following: assume that F_ε Γ-converges to F for the sense of convergence S (the sense of convergence can be a weak convergence or a convergence of some nonlinear quantity), that means in particular

$$\varliminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \geq F(u) \quad \text{when } u_\varepsilon \overset{S}{\rightharpoonup} u, \tag{21}$$

and consider the gradient of F_ε with respect to some Hilbert structure X_ε, denoted $\nabla_{X_\varepsilon} F_\varepsilon$. The question is to find conditions to get that solutions of the gradient flow $\partial_t u_\varepsilon = -\nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon)$ converge (in the sense S) to a solution of the gradient flow of F with respect to some structure Y to be determined. In the problem on Ginzburg–Landau vortices, F_ε should be taken to be $E_\varepsilon - \pi n |\log \varepsilon|$ and $F = W$ (see Proposition 3.2), the sense of convergence to be considered is $u_\varepsilon \rightharpoonup^S u = (a_1, \dots, a_n)$ if $\operatorname{curl} \langle i u_\varepsilon, \nabla u_\varepsilon \rangle \rightharpoonup 2\pi \sum_{i=1}^n d_i \delta_{a_i}$, where the $d_i = \pm 1$ and n are fixed a priori. In that case the limiting flow is a finite dimensional one, so the proof of existence of its solution is easy.

The two sufficient abstract conditions are that there should exist another Hilbert structure Y on the space where F is defined, satisfying the following:

1) For a subsequence such that $u_\varepsilon(t) \overset{S}{\rightharpoonup} u(t)$ for every $t \in [0, T)$, we have for all $s \in [0, T)$,

$$\varliminf_{\varepsilon\to 0} \int_0^s \|\partial_t u_\varepsilon\|_{X_\varepsilon}^2(t)\,dt \geq \int_0^s \|\partial_t u\|_Y^2(t)\,dt. \tag{22}$$

2) For any $u_\varepsilon \overset{S}{\rightharpoonup} u$

$$\varliminf_{\varepsilon\to 0} \|\nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon\|_{X_\varepsilon}^2 \geq \|\nabla_Y F(u)\|_Y^2. \tag{23}$$

These conditions suffice in the case where F is defined on a finite-dimensional space, to derive that if u_ε solves

$$\partial_t u_\varepsilon = -\nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon) \quad \text{on } [0, T) \tag{24}$$

with $u_\varepsilon(0) \overset{S}{\rightharpoonup} u_0$, and is well-prepared in the sense that $F_\varepsilon(u_\varepsilon(0)) = F(u_0) + o(1)$, then $u_\varepsilon(t) \overset{S}{\rightharpoonup} u(t)$, where u is the solution to

$$\begin{cases} \partial_t u = -\nabla_Y F(u), \\ u(0) = u_0. \end{cases}$$

The proof of this abstract result is rather elementary: for all $t < T$ we may write

$$\begin{aligned} F_\varepsilon(u_\varepsilon(0)) - F_\varepsilon(u_\varepsilon(t)) &= -\int_0^t \langle \nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon(s)), \partial_t\, u_\varepsilon(s)\rangle_{X_\varepsilon}\,ds \\ &= \frac{1}{2}\int_0^t \|\nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon)\|_{X_\varepsilon}^2 + \|\partial_t u_\varepsilon\|_{X_\varepsilon}^2\,ds \\ &\geq \frac{1}{2}\int_0^t \|\nabla_Y F(u)\|_Y^2 + \|\partial_t u\|_Y^2\,ds - o(1) \\ &\geq \int_0^t -\langle \nabla_Y F(u(s)), \partial_t u(s)\rangle_Y\,ds - o(1) \qquad (25) \\ &= F(u(0)) - F(u(t)) - o(1), \end{aligned}$$

hence

$$F(u(0)) - F(u(t)) \leq F_\varepsilon(u_\varepsilon(0)) - F_\varepsilon(u_\varepsilon(t)) + o(1).$$

But by well-preparedness, $F_\varepsilon(u_\varepsilon(0)) = F(u(0)) + o(1)$, thus

$$F_\varepsilon(u_\varepsilon(t)) \leq F(u(t)) + o(1).$$

But $F_\varepsilon \overset{\Gamma}{\longrightarrow} F$ implies $\varliminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon(t)) \geq F(u(t))$. Therefore we must have equality everywhere and in particular equality in the Cauchy–Schwarz type relation (25), that is

$$\frac{1}{2}\int_0^t \|\nabla_Y F(u)\|_Y^2 + \|\partial_t u\|_Y^2\,ds = \int_0^t -\langle \nabla_Y F(u(s)), \partial_t u(s)\rangle_Y$$

or

$$\int_0^t \|\nabla F(u) + \partial_t u\|_Y^2 \, ds = 0.$$

Hence, we conclude that $\partial_t u = -\nabla_Y F(u)$, for a.e. $t \in [0, T)$.

The method should and can be extended to infinite-dimensional limiting spaces and to the case where the Hilbert structures X_ε and Y (in particular Y) depend on the point, forming a sort of Hilbert manifold structure. In fact we can write down an analogous abstract result using the theory of "minimizing movements" of De Giorgi formalized by Ambrosio–Gigli–Savarè [3], a notion of gradient flows on structures which are not differentiable but simply metric structures.

6.1.2. The result for Ginzburg–Landau without magnetic field. Applying this abstract method to $F_\varepsilon = E_\varepsilon - \pi n |\log \varepsilon|$ and $F = W$ (with a prescribed number of vortices, and prescribed degrees), we retrieve the dynamical law of vortices which had been first established by Lin and Jerrard–Soner by PDE methods:

Theorem 6.1 ([29], [25], [46]). *Let u_ε be a family of solutions of*

$$\frac{\partial_t u}{|\log \varepsilon|} = \Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2) \quad \text{in } \Omega$$

with either

$$u_\varepsilon = g \ \text{ on } \partial\Omega \quad \text{or} \quad \frac{\partial u_\varepsilon}{\partial n} = 0 \ \text{ on } \partial\Omega$$

such that

$$\operatorname{curl} \langle i u_\varepsilon, \nabla u_\varepsilon \rangle(0) \rightharpoonup 2\pi \sum_{i=1}^n d_i \delta_{a_i^0} \quad \text{as } \varepsilon \to 0$$

with a_i^0 distinct points in Ω, $d_i = \pm 1$, and

$$E_\varepsilon(u_\varepsilon)(0) - \pi n |\log \varepsilon| \leq W(a_i^0) + o(1). \tag{26}$$

Then there exists $T^* > 0$ such that for all $t \in [0, T^*)$,

$$\operatorname{curl} \langle i u, \nabla u \rangle(t) \rightharpoonup 2\pi \sum_{i=1}^n d_i \delta_{a_i(t)}$$

as $\varepsilon \to 0$, with

$$\begin{cases} \dfrac{da_i}{dt} = -\dfrac{1}{\pi} \nabla_i W(a_1(t), \dots, a_n(t)), \\ a_i(0) = a_i^0, \end{cases} \tag{27}$$

where T^* is the minimum of the collision time and exit time of the vortices under this law.

Thus, as expected, vortices move along the gradient flow for their interaction W, and this reduces the PDE to a finite dimensional evolution (a system of ODE's).

The difficulty is in proving that the abstract conditions (22)–(23) hold in the Ginzburg–Landau setting. For example the first relation (22) relates the velocity of underlying vortices to $\partial_t u_\varepsilon$ and can be read

$$\varliminf_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|}\int_{[0,t]\times\Omega} |\partial_t u_\varepsilon|^2\,ds \geq \pi \sum_i \int_0^t |d_t\, a_i|^2\,ds \tag{28}$$

assuming $\operatorname{curl}\langle iu, \nabla u_\varepsilon\rangle(t) \rightharpoonup 2\pi\sum_i d_i\delta_{a_i(t)}$, as $\varepsilon \to 0$, for all t and $d_i = \pm 1$. This turns out to hold as a general relation (as proved in [45]), without requiring the configurations to solve any particular equation; it is related to the topological nature of the vortices.

6.1.3. Dynamical law for Ginzburg–Landau with magnetic field. By the same method, we obtained the dynamics of a bounded number of vortices for the full Ginzburg–Landau equations with magnetic field, i.e. the gradient-flow of (1), for large applied fields (the result for bounded applied fields had been obtained by Spirn [52]).

Assuming that $h_{\rm ex} = \lambda|\log\varepsilon|$, $0 < \lambda < \infty$, we obtained in [46] that, for energetically well-prepared solutions $(u_\varepsilon, A_\varepsilon)$ of (GL), such that $\mu(u_\varepsilon, A_\varepsilon)(0) \rightharpoonup 2\pi\sum_{i=1}^n d_i\delta_{a_i^0}$, with $d_i = \pm 1$, we have for all $t \in [0, T^*)$,

$$\mu(u_\varepsilon, A_\varepsilon)(t) \rightharpoonup 2\pi\sum_{i=1}^n d_i\delta_{a_i(t)}$$

with the dynamical law

$$\frac{da_i}{dt} = -d_i\lambda\nabla h_0(a_i(t)), \quad a_i(0) = a_i^0$$

for all i, where T^* is the minimum of the collision time and of the exit time from Ω for this law of motion.

6.1.4. Stability of critical points. In [50], we extended the "Γ-convergence of gradient flows" method above to the second order, i.e. we gave conditions on the second derivatives of the energies F_ε Γ-converging to F which ensure that critical points of F_ε converge to critical points of F (condition (23) above already ensures it) and that *moreover* stable critical points (in the sense of nonnegative Hessian) of F_ε converge to stable critical points of F (and more generally bounding from below the Morse index of the critical points of F_ε by that of those of F). The abstract condition is roughly the following: for any family u_ε of critical points of F_ε such that $u_\varepsilon \overset{S}{\rightharpoonup} u$; for any V, we can find $v_\varepsilon(t)$ defined in a neighborhood of $t = 0$, such that $\partial_t v_\varepsilon(0)$

depends on V in a linear and one-to-one manner, and

$$v_\varepsilon(0) = u_\varepsilon \tag{29}$$

$$\lim_{\varepsilon\to 0} \frac{d}{dt}_{|t=0} F_\varepsilon(v_\varepsilon(t)) = \frac{d}{dt}_{|t=0} F(u+tV) = dF(u).V \tag{30}$$

$$\lim_{\varepsilon\to 0} \frac{d^2}{dt^2}_{|t=0} F_\varepsilon(v_\varepsilon(t)) = \frac{d^2}{dt^2}_{|t=0} F(u+tV) = \langle D^2F(u)V, V\rangle. \tag{31}$$

We show that these conditions hold for (3) and deduce the result

Theorem 6.2 ([50]). *Let u_ε be a family of solutions of* (4) *such that $E_\varepsilon(u_\varepsilon) \le C|\log\varepsilon|$, with either Dirichlet or homogeneous Neumann boundary conditions. Then, there exists a family of points $a_1, \dots, a_n$ and nonzero integers $d_1, \dots, d_n$ such that, up to extraction of a subsequence,*

$$\text{curl}\,(iu_\varepsilon, \nabla u_\varepsilon) \rightharpoonup 2\pi \sum_{i=1}^n d_i \delta_{a_i},$$

where $(a_1, \dots, a_n)$ is a critical point of W. Moreover, if u_ε is a stable solution of (4) *then $(a_1, \dots, a_n)$ is a stable critical point of W; and more generally, denoting by n_ε^+ the dimension of the space spanned by eigenvectors of $D^2E_\varepsilon(u_\varepsilon)$ associated to positive eigenvalues and n^+ the dimension of the space spanned by eigenvectors of $D^2W(a_i)$ associated to positive eigenvalues* (*resp. n_ε^- and n^- for negative eigenvalues*), *we have, for ε small enough,*

$$n_\varepsilon^+ \ge n^+, \quad n_\varepsilon^- \ge n^-. \tag{32}$$

One of the interesting consequences of the theorem is the following, a consequence of the fact that the renormalized energy in Neumann boundary condition has no nontrivial stable critical point.

Theorem 6.3. *Let u_ε be a family of nonconstant solutions of* (4) *with homogeneous Neumann boundary condition $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$, such that $E_\varepsilon(u_\varepsilon) \le C|\log\varepsilon|$, then for ε small enough, u_ε is unstable.*

This shows that the model without magnetic field (4) cannot stabilize vortices, contrarily to the one with nonzero applied magnetic field.

This extended a result of Jimbo–Morita [26] (see also Jimbo–Sternberg [27] with magnetic field) valid for any ε but for convex domains.

6.2. PDE-based results. Most results on convergence of solutions of the Ginzburg–Landau flow to solutions of the flow for limiting energies were proved by PDE-based methods. We briefly review them.

6.2.1. Heat flow in higher dimensions. The convergence of the flow for E_ε is also true in higher dimensions where the limiting energy-density is length/surface, it was established (see [8], [31]) that the limit of the parabolic evolution of E_ε is a Brakke flow (a weak form of mean-curvature flow, which is the expected gradient flow of the limiting energy).

6.2.2. Other flows. The Schrödinger flow of (4), also called the Gross–Pitaevskii equation, is considered in superfluids, nonlinear optics and Bose–Einstein condensation. The limiting dynamical law of vortices is still the corresponding one (i.e. the Hamiltonian flow) for the limiting renormalized energy

$$\frac{da_i}{dt} = -\frac{1}{\pi}\nabla_i^{\perp} W(a_1, \dots, a_n).$$

The convergence was proved, still with well-prepared assumptions, by Colliander–Jerrard [12] on a torus, and by Lin–Xin [32] in the whole plane. In the case of the wave flow, the analogous limiting dynamical law was established by Jerrard in [23].

6.3. Collision issues. The result of Theorem 6.1 is valid only up to collision time under the law (27), but collisions do happen if there are vortices of opposite degrees. The question of understanding the collisions and extending the motion law passed them is delicate. Bethuel–Orlandi–Smets [9], [10] treated this question, as well as other issues of non well-prepared data, vortex-splitting and phase-vortex interaction in infinite domains.

In [51], the collision problem was approached with the idea of basing the study on the energy, like for the "Γ-convergence of gradient flow". We proved that when several vortices become very close to each other (but not too close) a dynamical law after blow-up can be derived through the same method presented above. When vortices become too close to apply this, we focused on evaluating energy dissipation rates, through the study of the perturbed Ginzburg–Landau equation

$$\Delta u + \frac{1}{\varepsilon^2} u(1 - |u|^2) = f_\varepsilon \quad \text{in } \Omega, \tag{33}$$

with Dirichlet or Neumann boundary data, where f_ε is given in $L^2(\Omega)$ (the instantaneous energy-dissipation rate in the dynamics is exactly $|\log\varepsilon| \|f_\varepsilon\|^2_{L^2(\Omega)}$). We prove that the "energy-excess" (meaning the difference between $E_\varepsilon - \pi n|\log\varepsilon|$ and the renormalized energy W of the underlying vortices) is essentially controlled by $C\|f_\varepsilon\|^2_{L^2}$. We then show that when u solves (33) and has vortices which become very close, forming what we call an "unbalanced cluster" in the sense that $\sum_i d_i^2 \neq \left(\sum_i d_i\right)^2$ in the cluster (see [51] for a precise definition), then a lower bound for $\|f_\varepsilon\|_{L^2}$ must hold:

Theorem 6.4 ([51]). *Let u_ε solve* (33) *with $E_\varepsilon(u_\varepsilon) \le C|\log\varepsilon|$, $|\nabla u_\varepsilon| \le \frac{M}{\varepsilon}$ and $|u_\varepsilon| \le 1$. There exists $l_0 > 0$ such that, if u_ε has an unbalanced cluster of vortices at the scale $l < l_0$ then*

$$\|f_\varepsilon\|^2_{L^2(\Omega)} \ge \min\left(\frac{C}{l^2|\log\varepsilon|}, \frac{C}{l^2\log^2 l}\right). \tag{34}$$

In particular, when vortices get close to each other, say two vortices of opposite degrees for example, then they form an unbalanced cluster of vortices at scale

$l =$ their distance, and the relation (34) gives a large energy-dissipation rate (scaling like $1/l^2$). This serves to show that such a situation cannot persist for a long time and we are able to prove that the vortices collide and disappear in time $Cl^2 + o(1)$, with all energy-excess dissipating in that time. Thus after this time $o(1)$, the configuration is again "well-prepared" and Theorem 6.1 can be applied again, yielding the dynamical law with the remaining vortices, until the next collision, etc.

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