Recent developments in elliptic partial differential equations of Monge–Ampère type

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Abstract. In conjunction with applications to optimal transportation and conformal geometry, there has been considerable research activity in recent years devoted to fully nonlinear, elliptic second order partial differential equations of a particular form, given by functions of the Hessian plus a lower order matrix function. Regularity is determined through the behaviour of this function with respect to the gradient variables. We present a selection of second derivative estimates and indicate briefly their application to optimal transportation and conformal deformation of Riemannian manifolds.

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1. Introduction

In conjunction with applications to optimal transportation and conformal geometry, there has been considerable research activity in recent years devoted to fully nonlinear, elliptic second order partial differential equations of the form,

$$\mathcal{F}[u] := F\{D^2u + A(\cdot, u, Du)\} = B(\cdot, u, Du), \tag{1.1}$$

in domains Ω in Euclidean n-space, $\mathbb{R}^n$, as well as their extensions to Riemannian manifolds. Here the functions $F\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $A\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$, $B\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are given and the resultant operator $\mathcal{F}$ is well-defined classically for functions $u \in C^2(\Omega)$. As customary Du and D^2u denote respectively the gradient vector and Hessian matrix of second derivatives of u, while we also use x, z, p, r to denote points in Ω, $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^n \times \mathbb{R}^n$ respectively with corresponding partial derivatives denoted, when there is no ambiguity, by subscripts. For example $F_r = \left[\frac{\partial F}{\partial r_{ij}}\right]$, $F_p = \left(\frac{\partial F}{\partial p_1}, \dots, \frac{\partial F}{\partial p_n}\right)$ etc. The operator $\mathcal{F}$ is *elliptic* with respect to u whenever the matrix

$$F_r\left\{D^2u + A(\cdot, u, Du)\right\} > 0. \tag{1.2}$$

Unless indicated otherwise we will assume the matrix A is symmetric, but it is also important to address the possibility that it is not. When $A \equiv 0$, (1.1) reduces to the

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well-studied Hessian equation,

$$\mathcal{F}[u] = F\left(D^2u\right) = B, \tag{1.3}$$

while for $F(r) = \det r$, we obtain a *Monge–Ampère equation* of the form

$$\mathcal{F}[u] = \det\left\{D^2u + A(\cdot, u, Du)\right\} = B(\cdot, u, Du), \tag{1.4}$$

which is preserved under coordinate changes, unlike the standard Monge–Ampère equation, when $A \equiv 0$. The operator $\mathcal{F}$ in (1.4) is elliptic with respect to u whenever

$$D^2u + A(\cdot, u, Du) > 0, \tag{1.5}$$

which implies $B > 0$.

Monge–Ampère equations of the general form (1.4) arise in applications, notably in optimal transportation, through the prescription of the absolute value of the Jacobian determinant of a mapping $T = T_u : \Omega \to \mathbb{R}^n$ given by

$$T_u = Y(\cdot, u, Du), \tag{1.6}$$

where $Y : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a given vector field, that is

$$|\det DT_u| = \psi(\cdot, u, Du) \tag{1.7}$$

for a given nonnegative $\psi : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. To write (1.7) in the form (1.4) we assume that Y is differentiable and

$$\det Y_p \neq 0, \tag{1.8}$$

whence by calculation, we obtain

$$\mathcal{F}[u] = \det\{D^2u + Y_p^{-1}(Y_x + Y_z \otimes Du)\} = \psi/|\det Y_p|, \tag{1.9}$$

assuming $\mathcal{F}$ elliptic with respect to u. When the vector field Y is independent of z and generated by a *cost function* $c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, through the equations

$$c_x(\cdot, Y(\cdot, p)) = p \tag{1.10}$$

we obtain the *optimal transportation* equation,

$$\mathcal{F}[u] = \det\left\{D^2u - D_x^2c(\cdot, Y(\cdot, Du))\right\} = \psi/|\det Y_p|. \tag{1.11}$$

Note that by differentiation of (1.10) we have

$$\begin{aligned} c_{x,y}(\cdot, Y) &= Y_p^{-1}, \\ c_{xx}(\cdot, Y) &= -Y_p^{-1}Y_x. \end{aligned} \tag{1.12}$$

In conformal geometry, equations of the form (1.1) arise from the study of the Schouten tensor of a Riemannian manifold under conformal deformation of its metric.

When the functions F are homogeneous we obtain, in the special case of Euclidean space $\mathbb{R}^n$, equations of the form (1.1), where the matrix A is given by

$$A(p) = -\tfrac{1}{2}|p|^2 I + p \otimes p. \tag{1.13}$$

Here we observe that, as in equation (1.9), we may write

$$A(p) = Y_p^{-1} Y_x(\cdot, p), \tag{1.14}$$

where Y is the vector field

$$Y(x, p) = x - p/|p|^2 \tag{1.15}$$

generated by the cost function

$$c(x, y) = \log|x - y|. \tag{1.16}$$

The regularity of solutions of equations of the form (1.1) depends on the behaviour of the matrix A with respect to the p variables. Letting $\mathcal{U} \subset \Omega \times \mathbb{R} \times \mathbb{R}^n$, we say that A is *regular* in $\mathcal{U}$ if

$$D_{p_k p_\ell} A_{ij}\, \xi_i\, \xi_j\, \eta_k\, \eta_\ell \le 0 \tag{1.17}$$

in $\mathcal{U}$, for all $\xi, \eta \in \mathbb{R}^n$ with $\xi \cdot \eta = 0$; and *strictly regular* in $\mathcal{U}$ if there exists a constant $a_0 > 0$ such that

$$D_{p_k p_\ell} A_{ij}\, \xi_i\, \xi_j\, \eta_k\, \eta_\ell \le -a_0 |\xi|^2 |\eta|^2 \tag{1.18}$$

in $\mathcal{U}$, for all $\xi, \eta \in \mathbb{R}^n$ with $\xi \cdot \eta = 0$.

These conditions were introduced in [23], [31] and called there A3w, A3 respectively. As we will explain in this paper, they are the natural conditions for regularity. Note that the matrix A in (1.13) trivially satisfies (1.18) in $\mathbb{R}^n$, with $a_0 = 1$.

2. Second derivative estimates

The key estimates for classical solutions of equations of the form (1.1) are bounds for second derivatives as higher order estimates and regularity follows from the fully nonlinear theory [10], [21]. Here we present a selection of estimates for regular and strictly regular matrix functions A. For all these estimates we will assume A and B are C^2 smooth.

2.1. Interior estimates. Under the hypothesis of strict regularity we get quite strong interior estimates for very general functions F. Indeed we may assume that F is positive, increasing and concave on some convex open set $\Gamma \subset \mathbb{S}^n$, which is closed under addition of the positive cone. Here $\mathbb{S}^n$ denotes the subspace of $\mathbb{R}^n \times \mathbb{R}^n$ consisting of the symmetric matrices. Suppose also that

$$\operatorname{trace} F_r \to \infty \quad \text{as } \lambda_{\max}(r) \to \infty \tag{2.1}$$

on subsets of Γ where $F \geq \delta$ for any $\delta > 0$. The following estimate extends Theorem 4.1 and Remark 4.1 in [23].

Theorem 2.1. *Let $u \in C^4(\Omega)$, $D^2u + A \in \Gamma$, be a solution of* (1.1)*, with A strictly regular on the set $\mathcal{U} = \{(x, z, p) \mid x \in \Omega, z = u(x), p = Du(x)\}$. Then for any $\Omega' \subset\subset \Omega$ we have*

$$\sup_{\Omega'} |D^2u| \leq C, \tag{2.2}$$

where C is a constant depending on Ω', Ω, A, B and $|u|_{1;\Omega}$.

2.2. Dirichlet problem. We present a global second derivative estimate for solutions of the Dirichlet problem, or first boundary value problem, for the Monge–Ampère type equation (1.4). First we introduce a convexity condition for domains, which was fundamental in our applications to optimal transportation in [23], [31]. Namely if Ω is a connected domain in $\mathbb{R}^n$, with $\partial\Omega \in C^2$, and $A \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}^n; \mathbb{S}^n)$ we say that Ω is *A-convex* (*uniformly A-convex*), with respect to $\mathcal{U} \subset \Omega \times \mathbb{R} \times \mathbb{R}^n$, if

$$\left[D_i\, \gamma_j(x) + A_{ij,p_k}(x, z, p)\, \gamma_k(x)\right] \tau_i\, \tau_j \geq 0, \quad (\delta_0), \tag{2.3}$$

for all $x \in \partial\Omega$, $x, z, p \in \overline{\mathcal{U}}$, unit outer normal γ and unit tangent vector τ (for some $\delta_0 > 0$).

When $A \equiv 0$, A-convexity reduces to the usual convexity. We also say that a domain Ω is *A-bounded*, with respect to $\mathcal{U} \subset \Omega \times \mathbb{R} \times \mathbb{R}^n$, if there exists a function $\varphi \in C^2(\Omega)$ satisfying

$$D^2\varphi(x) + A_{p_k}(x, z, p)\, D_k\, \varphi \geq \delta_0\, I \tag{2.4}$$

for all $x \in \Omega$, $(x, z, p) \in \mathcal{U}$. Note that when $A \equiv 0$, any bounded domain is A-bounded. A domain Ω is then both uniformly A-convex and A-bounded if there exists a defining function $\varphi \in C^2(\overline{\Omega})$, satisfying $\varphi = 0$ on $\partial\Omega$, $D\varphi \neq 0$ on $\partial\Omega$, together with (2.4).

Theorem 2.2. *Let $u \in C^4(\overline{\Omega})$ be an elliptic solution of equation* (1.4) *in Ω, satisfying $u = g$ on $\partial\Omega$, where $\partial\Omega \in C^4$, $g \in C^4(\overline{\Omega})$. Suppose that A is regular on the set $\mathcal{U} = \{(x, z, p) \mid x \in \Omega, z = u(x), p = Du(x)\}$, with Ω uniformly A-convex and A-bounded with respect to $\mathcal{U}$. Then we have the estimate,*

$$\sup_{\Omega} |D^2u| \leq C, \tag{2.5}$$

where C is a constant depending on A, B, Ω, φ and $|u|_{1;\Omega}$.

2.3. Second boundary value problem. Now we turn our attention to the prescribed Jacobian equation, in the form (1.9). The second boundary value problem, or natural boundary condition, involves the prescription of the image of the mapping T_u in (1.6), that is

$$T(\Omega) = \Omega^* \tag{2.6}$$

for some given domain $\Omega^* \subset \mathbb{R}^n$. If the positive function ψ is given by

$$\psi(x, z, p) = f(x)/g \circ Y(x, z, p) \tag{2.7}$$

for positive $f, g \in C^0(\Omega), C^0(\Omega^*)$ respectively, and T is a diffeomorphism (for example when Ω is convex), we obtain the necessary condition for solvability,

$$\int_\Omega f = \int_{\Omega^*} g, \tag{2.8}$$

which is the *mass balance condition* in optimal transportation. Following our previous notions of domain convexity, we will say that Ω is *Y-convex* (*uniformly Y-convex*, *Y-bounded*) with respect to Ω^* if Ω is A-convex (uniformly A-convex, A-bounded) with respect to $\mathcal{U}_Y = \{(x, z, p) \mid x \in \Omega, Y(x, z, p) \in \Omega^*\}$, where the matrix function A is given by

$$A = Y_p^{-1}(Y_x + Y_z \otimes p) \tag{2.9}$$

as in equation (1.9). The target domain Ω^* is *Y^*-convex*, with respect to Ω, if for each $(x, z) \in \Omega \times \mathbb{R}$, the set

$$\mathcal{P}(x, z) = \{p \in \mathbb{R}^n \mid Y(x, z, p) \in \Omega^*\} \tag{2.10}$$

is *convex* in $\mathbb{R}^n$ and *uniformly Y^*-convex*, with respect to Ω, if $\mathcal{P}(x, z)$ is uniformly convex, with respect to $(x, z) \in \Omega \times \mathbb{R}$. Note that for $\partial\Omega^* \in C^2$, these concepts may also be expressed in the form (2.3), and that when Y is generated by a cost function, which happens when $Y_z \equiv 0$, by virtue of the assumed symmetry of A, they are dual to each other (see Section 3).

Theorem 2.3. *Let $u \in C^4(\overline{\Omega})$ be an elliptic solution of equation* (1.4) *in Ω, satisfying* (2.6)*, where $\partial\Omega, \partial\Omega^* \in C^4$ and Ω, Ω^* are uniformly Y-convex, Y^*-convex with respect to each other. Suppose also that Ω is Y-bounded and that A is regular on $\mathcal{U}_Y$. Then we have the estimate*

$$\sup_\Omega |D^2 u| \le C, \tag{2.11}$$

where C depends on Y, ψ, Ω, Ω^* and $|u|_{1;\Omega}$.

2.4. Remarks

1. Estimates in $C^3(\overline{\Omega})$ automatically follow from the assumed data regularity in Theorems 2.2 and 2.3, by virtue of the global $C^{2,\alpha}$ estimates [18] and [21]. Classical existence theorems then follow by the method of continuity under additional hypotheses to control the solutions and their gradients.

2. The boundary condition (2.6) is a nonlinear *oblique* boundary condition of the form

$$G(x, u, Du) := \varphi^* \circ Y(x, u, Du) = 0, \tag{2.12}$$

where φ^* is a defining function for Ω^*. If $|\nabla\varphi^*| = 1$ on $\partial\Omega^*$ we obtain, for $c^{i,j} = D_{p_i} Y^j$,

$$\chi := \gamma \,.G_p(x, u, Du) = c^{i,j}\, \gamma_i\, \gamma_j^* > 0, \tag{2.13}$$

by virtue of ellipticity, and the geometric conditions on Ω and Ω^* are used to estimate χ from below, [31].

3. The special cases $A \equiv 0$ of the standard Monge–Ampère equation in Theorems 2.2 and 2.3 are due to Ivochkina [17], Krylov [18], Caffarelli, Nirenberg and Spruck [5] (Theorem 2.2), and Caffarelli [2] and Urbas [32] (Theorem 2.3). Sharp versions for Hölder continuous inhomogeneous terms were proved by Trudinger and Wang [29] and Caffarelli [2].

4. Theorems 2.1, 2.2 and 2.3 extend to non-symmetric matrices A in two dimensions.

5. The condition of uniform A-convexity in Theorem 2.2 may be replaced by the more general condition that there exists a strict sub-solution taking the same boundary conditions, as for the case $A \equiv 0$ in [12].

3. Optimal transportation

Let Ω and Ω^* be bounded domains in $\mathbb{R}^n$ and f, g nonnegative functions in $L^1(\Omega)$, $L^1(\Omega^*)$ respectively satisfying the mass balance condition (2.8). Let $c \in C^0(\mathbb{R}^n \times \mathbb{R}^n)$ be a cost function. The corresponding Monge–Kantorovich problem of optimal transportation is to find a measure preserving mapping T_0 which maximizes (or minimizes) the cost functional,

$$\mathcal{C}(T) = \int_\Omega f(x)\, c(x, T(x))\, dx, \tag{3.1}$$

over the set $\mathcal{T}$ of measure preserving mappings T from Ω to Ω^*. A mapping T is called measure preserving if it is Borel measurable and for any Borel set $E \subset \Omega^*$,

$$\int_{T^{-1}(E)} f = \int_E g. \tag{3.2}$$

For the basic theory the reader is referred to the accounts in works such as [9], [24], [33], [34]. To fit the exposition in our previous sections, we consider maximization problems rather than minimization, noting that it is trivial to pass between them replacing c by $-c$.

3.1. Kantorovich potentials. The dual functional of Kantorovich is defined by

$$I(u, v) = \int_\Omega f(x)\, u(x)\, dx + \int_{\Omega^*} g(y)\, v(y)\, dy, \tag{3.3}$$

for $(u, v) \in K$ where

$$K = \{(u, v) \mid u \in C^0(\Omega), v \in C^0(\Omega^*), \\ u(x) + v(y) \geq c(x, y), \text{ for all } x \in \Omega, y \in \Omega^*\}. \tag{3.4}$$

It is readily shown that $\mathcal{C}(T) \leq I(u, v)$, for all $T \in \mathcal{T}$, $u, v \in K$. To solve the Monge–Kantorovich problem, we assume $c \in C^2(\mathbb{R}^n \times \mathbb{R}^n)$ and that for each $x \in \Omega$, $p \in \mathbb{R}^n$ there exists a unique $y = Y(x, p)$ satisfying (1.10), together with the corresponding condition for x replaced by $y \in \Omega^*$ and

$$|\det c_{x,y}| \geq c_0 \tag{3.5}$$

on $\Omega \times \Omega^*$ for some constant $c_0 > 0$. Then there exist semi-convex functions $(u, v) \in K$ and a mapping $T = T_u$, given by

$$T_u = Y(\cdot, Du) \tag{3.6}$$

almost everywhere in Ω, such that

$$\mathcal{C}(T) = I(u, v). \tag{3.7}$$

The functions u, v, which are uniquely determined up to additive constants, are called *potentials* and are related by

$$u(x) = \sup_{\Omega^*}\{c(x, \cdot) - v(\cdot)\}, \quad v(y) = \sup_{\Omega}\{c(\cdot, y) - u(\cdot)\}. \tag{3.8}$$

Furthermore, for positive densities f, g, the potential function u will be an elliptic solution of equation (1.11) almost everywhere in Ω (at points where it is twice differentiable). If $u \in C^2(\overline{\Omega})$, then u is a classical solution of the second boundary value problem (2.6).

3.2. Interior regularity. Consistent with our definitions in Section 2, Ω is c-convex with respect to Ω^* if $c_y(\cdot, y)(\Omega)$ is convex for all $y \in \Omega^*$ and Ω^* is c^*-convex if $c_x(x, \cdot)(\Omega^*)$ is convex for all $x \in \Omega$. As a consequence of Theorem 2.1, we have the main result in [23].

Theorem 3.1. *Let Ω, Ω^* be bounded domains in $\mathbb{R}^n$, $f \in C^2(\Omega) \cap L^\infty(\Omega)$, $g \in C^2(\Omega^*) \cap L^\infty(\Omega^*)$, $\inf f, \inf g > 0$. Let $c \in C^4(\mathbb{R}^n \times \mathbb{R}^n)$ and Ω^* be c^*-convex with respect to Ω. Suppose that A is strictly regular on the set $\mathcal{U}_Y$, where $A(x, p) = -D_x^2 c(x, Y(x, p))$ and Y is given by* (1.10). *Then the potential $u \in C^3(\Omega)$.*

3.3. Global regularity. From the global second derivative estimate, Theorem 2.3, we obtain a global regularity result, corresponding to Theorem 3.1, which is proved in [31]. For its formulation we say that Ω (Ω^*) is *uniformly c-convex* (*c^*-convex*), with respect to Ω^*, (Ω), if the images $c_y(\cdot, y)(\Omega)$ ($c_x(x, \cdot)(\Omega^*)$) are uniformly convex with respect to $y \in \Omega^*$ ($x \in \Omega$). This agrees with our previous definitions in terms of the vector field Y and matrix A determined by c.

Theorem 3.2. *Let Ω and Ω^* be bounded C^4 domains in $\mathbb{R}^n$, $f \in C^2(\overline{\Omega})$, $g \in C^2(\overline{\Omega^*})$, $\inf f > 0$, $\inf g > 0$. Let $c \in C^4(\mathbb{R}^n \times \mathbb{R}^n)$ and let Ω, Ω^* be uniformly c-convex, c^*-convex with respect to each other. Suppose also that Ω is Y-bounded and A is regular on $\mathcal{U}_Y$. Then the potential function $u \in C^3(\overline{\Omega})$.*

3.4. Remarks

1. For the case of quadratic cost functions,

$$c(x, y) = x.y, \quad Y(x, p) = p, \quad A \equiv 0, \tag{3.9}$$

Theorem 3.1 is due to Caffarelli [1], Theorem 3.2 is due to Caffarelli [2] and Urbas [32]. Note that this case is excluded from Theorem 3.1 but embraced by Theorem 3.2. The interior estimate (2.2) is not valid when $A \equiv 0$.

2. By exploiting the geometric interpretation of strict regularity, Loeper [22] has shown that the potential $u \in C^{1,\alpha}(\Omega)$ for certain $\alpha > 0$, when the smoothness of the densities f, g is dropped. Moreover he has shown that the regularity of A is a necessary condition for $u \in C^1(\Omega)$ for arbitrary smooth positive densities.

3. As shown in [23], the c^*-convexity of Ω^* is also necessary for interior regularity for arbitrary smooth positive densities.

4. The condition of Y-boundedness may be dropped in Theorem 2.3 in the optimal transportation case [31].

5. Various examples of cost functions for which A is regular or strictly regular are presented in [23] and [31].

4. Conformal geometry

In recent years the Yamabe problem for the k-curvature of the Schouten tensor, or simply the k-Yamabe problem, has been extensively studied. Let $(\mathcal{M}, g_0)$ be a smooth compact manifold of dimension $n > 2$ and denote by Ric and R respectively the Ricci tensor and scalar curvature. The k-Yamabe problem is to prove the existence of a conformal metric $g = g_u = e^{-2u} g_0$ such that

$$\sigma_k(\lambda(S_g)) = 1 \quad \text{on } \mathcal{M}, \tag{4.1}$$

where $k = 1, \ldots, n$, $\lambda = (\lambda_1, \ldots, \lambda_n)$ denotes the eigenvalues of S_g with respect to the metric g, σ_k is the kth elementary symmetric function given by

$$\sigma_k(\lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}, \tag{4.2}$$

and S_g is the Schouten tensor of $(\mathcal{M}, g)$ given by

$$\begin{aligned} S_g &= \frac{1}{n-2}\left(\operatorname{Ric} g - \frac{R_g}{2(n-1)} g\right) \\ &= \nabla^2 u + \nabla u \otimes \nabla u - \tfrac{1}{2}|\nabla u|^2 g_0 + S_{g_0}. \end{aligned} \tag{4.3}$$

Accordingly we obtain the equation

$$\mathcal{F}_k[u] = F_k^{1/k}\left\{g_0^{-1}\left(\nabla^2 u + \nabla u \otimes \nabla u - \tfrac{1}{2}|\nabla u|^2 g_0 + S_{g_0}\right)\right\} = e^{-2u}, \tag{4.4}$$

where $F_k(r)$ denotes the sum of the $k \times k$ principal minors of the matrix $r \in \mathbb{S}^n$, which is elliptic for $\lambda(S_g) \in \Gamma_k$ where Γ_k is the cone in $\mathbb{R}^n$ given by

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \mid \sigma_j(\lambda) > 0,\ j = 1, \dots, k\}, \tag{4.5}$$

(see for example [3], [28]). When $k = 1$, we arrive at the well-known Yamabe problem [27], that was completely resolved by Schoen in [25]. Note that for Euclidean space $\mathbb{R}^n$, we have

$$S_g = D^2 u + Du \otimes Du - \tfrac{1}{2}|Du|^2 I, \tag{4.6}$$

in agreement with (1.13).

The operators $\mathcal{F}_k$ are *strictly regular* in the sense of (1.18) so that interior estimates corresponding to Theorem 2.1 are readily proven, [35], [13], [7], [14]. However crucial ingredients in the solution of the k-Yamabe problem are estimates in terms of $\inf u$ only. These were obtained by Guan and Wang [14] and recently simplified by Chen [8], who derived the gradient estimates directly from the second derivative estimates using $\sigma_1(\lambda) \geq 0$. The following theorem, due to Sheng, Trudinger and Wang [26] ($k \leq n/2$), Gursky and Viaclovsky [16] ($k > n/2$) concerns the solvability of the higher order Yamabe problem, $k > 1$.

Theorem 4.1. *Let $(\mathcal{M}, g_0)$ be a smooth compact manifold of dimension $n > 2$ and suppose there exists some metric g_1 conformal to g_0 for which $\lambda(S_{g_1}) \in \Gamma_k$. Then there exists a conformal metric g satisfying* (4.1) *if either $k > n/2$ or $k \leq n/2$ and* (4.1) *has variational structure, that is it is equivalent to an Euler equation of a functional.*

We remark that (4.1) is variational for $k = 1, 2$ and if $(\mathcal{M}, g_0)$ is locally conformally flat otherwise. The case of Theorem 4.1 when $k = 2$, $n = 4$ was proved in the pioneering work of Chang, Yang and Gursky [6], while the locally conformally flat case was proved by Guan and Wang [15] and Li and Li [19], [20]. The cases $k = 2$, $n > 8$ were obtained independently by Ge and Wang [11]. The reader is referred to the various papers cited above for further information. Also a more elaborate treatment of the case $k > n/2$ is presented in [30].

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