Matrix ansatz and large deviations of the density in exclusion processes

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Abstract. Exclusion processes describe a gas of particles on a lattice with hard core repulsion. When such a lattice gas is maintained in contact with two reservoirs at unequal densities, or driven by an external field, it exhibits a non-equilibrium steady state. In one dimension, a number of properties of this steady state can be calculated exactly using a matrix ansatz. This talk gives a short review on results obtained recently by this matrix ansatz approach.

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1. Introduction

Exclusion processes have been studied for a long time as microscopic models of fluids which satisfy at large scale hydrodynamic equations [2], [13], [27], [28], [36], [39], [47]. They give also some of the simplest examples of non-equilibrium steady state [38], [12], [15], [16], [44], [29]. Here I will try to review a number of recent results on exclusion processes which have been obtained using an exact matrix representation of the weights of microscopic configurations in the non-equilibrium steady state.

One of the simplest cases for which this can be done is the symmetric simple exclusion process defined in Section 2. The matrix ansatz is discussed in Section 3 and the large deviation function of the density is obtained in Section 5 (using an additivity property given in Section 4). Section 6 gives a short review of an alternative approach to calculate this large deviation function, the macroscopic fluctuation theory [3], [4], [5]. Section 7 gives the extension of the matrix ansatz to the asymmetric exclusion process from which one can calculate the phase diagram (Section 8) and the fluctuations of density (Section 9).

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2. The symmetric simple exclusion process

The symmetric simple exclusion process (SSEP) describes a lattice gas of particles diffusing on a lattice with an exclusion rule which prevents a particle to move to a site already occupied by another particle. Here we consider the one dimensional version with open boundaries. The lattice consists of L sites, each site being either occupied by a single particle or empty. During every infinitesimal time interval dt, each particle has a probability dt of jumping to the left if the neighboring site on its left is empty, dt of jumping to the right if the neighboring site on its right is empty. At the two boundaries the dynamics is modified to mimic the coupling with reservoirs of particles: at the left boundary, during each time interval dt, a particle is injected on site 1 with probability αdt (if this site is empty) and a particle is removed from site 1 with probability γdt (if this site is occupied). Similarly on site L, particles are injected at rate δ and removed at rate β.

From the very definition of the SSEP, if $\tau_i = 0$ or 1 is a binary variable indicating whether site i is occupied or empty, one can write the time evolution of the average occupation $\langle \tau_i \rangle$:

$$\begin{aligned} \frac{d\langle \tau_1 \rangle}{dt} &= \alpha - (\alpha + \gamma + 1)\langle \tau_1 \rangle + \langle \tau_2 \rangle, \\ \frac{d\langle \tau_i \rangle}{dt} &= \langle \tau_{i-1} \rangle - 2\langle \tau_i \rangle + \langle \tau_{i+1} \rangle \qquad \text{for } 2 \leq i \leq L-1, \\ \frac{d\langle \tau_L \rangle}{dt} &= \langle \tau_{L-1} \rangle - (1 + \beta + \delta)\langle \tau_L \rangle + \delta. \end{aligned} \tag{1}$$

The steady state density profile (obtained by writing that $\frac{d\langle \tau_i \rangle}{dt} = 0$) is [20]

$$\langle \tau_i \rangle = \frac{\rho_a\big(L + \frac{1}{\beta+\delta} - i\big) + \rho_b\big(i - 1 + \frac{1}{\alpha+\gamma}\big)}{L + \frac{1}{\alpha+\gamma} + \frac{1}{\beta+\delta} - 1} \tag{2}$$

where ρ_a and ρ_b are defined by

$$\rho_a = \frac{\alpha}{\alpha + \gamma}, \quad \rho_b = \frac{\delta}{\beta + \delta}. \tag{3}$$

For a large system size ($L \to \infty$) one can notice that $\langle \tau_1 \rangle \to \rho_a$ and $\langle \tau_L \rangle \to \rho_b$ indicating that ρ_a and ρ_b defined by (3) represent the densities of the left and right reservoirs. One can in fact show [19], [20] that the rates $\alpha, \gamma, \beta, \delta$ do correspond to the left and right boundaries being connected respectively to reservoirs at densities ρ_a and ρ_b.

In a similar way one can write down the equations which govern the time evolution of the two point function or higher correlations. For example one finds [46], [23] in the steady state for $1 \leq i < j \leq L$

$$\langle \tau_i \tau_i \rangle_c \equiv \langle \tau_i \tau_j \rangle - \langle \tau_i \rangle \langle \tau_j \rangle$$
$$= -\frac{\left(\frac{1}{\alpha+\gamma} + i - 1\right)\left(\frac{1}{\beta+\delta} + L - j\right)}{\left(\frac{1}{\alpha+\gamma} + \frac{1}{\beta+\delta} + L - 1\right)^2 \left(\frac{1}{\alpha+\gamma} + \frac{1}{\beta+\delta} + L - 2\right)} (\rho_a - \rho_b)^2.$$

One can notice that for large L, if one introduces macroscopic coordinates $i = Lx$ and $j = Ly$, this becomes

$$\langle \tau_{Lx} \tau_{Ly} \rangle_c = -\frac{x(1-y)}{L} (\rho_a - \rho_b)^2$$

for $x < y$. These weak, but long range, correlations are characteristic of the steady state of non equilibrium systems [47], [23], [41].

The average current in the steady state is given by

$$\bar{j} = \langle \tau_i (1 - \tau_{i+1}) - \tau_{i+1}(1 - \tau_i) \rangle = \langle \tau_i - \tau_{i+1} \rangle = \frac{\rho_a - \rho_b}{L + \frac{1}{\alpha+\gamma} + \frac{1}{\beta+\delta} - 1}. \quad (4)$$

This shows that for large L, the current $\bar{j} \simeq \frac{\rho_a - \rho_b}{L}$ is proportional to the gradient of the density (with a coefficient of proportionality which is here simply 1) and therefore follows Fick's law.

3. The matrix ansatz for the SSEP

For the SSEP, one can write down the steady state equations satisfied by higher and higher correlation functions, but solving these equations becomes quickly inextricable.

The matrix ansatz gives an algebraic way of calculating exactly the weights of all the configurations in the steady state: in [16] it was shown that the probability of a microscopic configuration $\{\tau_1, \tau_2, \ldots, \tau_L\}$ can be written as the matrix element of a product of L matrices

$$\text{Pro}(\{\tau_1, \tau_2, \ldots, \tau_L\}) = \frac{\langle W | X_1 X_2 \ldots X_L | V \rangle}{\langle W | (D + E)^L | V \rangle} \quad (5)$$

where the matrix X_i depends on the occupation τ_i of site i,

$$X_i = \tau_i D + (1 - \tau_i) E, \quad (6)$$

and the matrices D and E satisfy the following algebraic rules:

$$\begin{aligned} DE - ED &= D + E, \\ \langle W | (\alpha E - \gamma D) &= \langle W |, \\ (\beta D - \delta E) | V \rangle &= | V \rangle. \end{aligned} \quad (7)$$

Let us check on a simple example that expression (5) does give the steady state weights: if one chooses the configuration where the first p sites on the left are occupied and the remaining $L - p$ sites on the right are empty, the weight of this configuration is given by

$$\frac{\langle W|D^p E^{L-p}|V\rangle}{\langle W|(D+E)^L|V\rangle}. \tag{8}$$

For (5) to be the weights of all configurations in the steady state, one needs that the rate at which the system enters each configuration and the rate at which the system leaves it should be equal. In the case of the configuration we consider in (8), this means that the following steady state identity should be satisfied:

$$\begin{aligned}(\gamma+1+\delta)\frac{\langle W|D^p E^{L-p}|V\rangle}{\langle W|(D+E)^L|V\rangle} = \alpha &\frac{\langle W|ED^{p-1}E^{L-p}|V\rangle}{\langle W|(D+E)^L|V\rangle} \\ &+ \frac{\langle W|D^{p-1}EDE^{L-p-1}|V\rangle}{\langle W|(D+E)^L|V\rangle} \\ &+ \beta\frac{\langle W|D^p E^{L-p-1}D|V\rangle}{\langle W|(D+E)^L|V\rangle}.\end{aligned} \tag{9}$$

This equality is easy to check by rewriting (9) as

$$\begin{aligned}\frac{\langle W|(\alpha E-\gamma D)D^{p-1}E^{L-p}|V\rangle}{\langle W|(D+E)^L|V\rangle} - &\frac{\langle W|D^{p-1}(DE-ED)E^{L-p-1}|V\rangle}{\langle W|(D+E)^L|V\rangle} \\ &+\frac{\langle W|D^p E^{L-p-1}(\beta D-\delta E)|V\rangle}{\langle W|(D+E)^L|V\rangle} = 0\end{aligned} \tag{10}$$

and by using (7). A similar reasoning allows one to prove that the corresponding steady state identity holds for any other configuration.

A priori one should construct the matrices D and E (which might be infinite-dimensional) and the vectors $\langle W|$ and $|V\rangle$ satisfying (7) to calculate the weights of the microscopic configurations. However these weights do not depend on the particular representation chosen and can be calculated directly from (7).

This can be easily seen by using the two matrices A and B defined by

$$\begin{aligned} A &= \beta D - \delta E, \\ B &= \alpha E - \gamma D, \end{aligned} \tag{11}$$

which satisfy

$$AB - BA = (\alpha\beta - \gamma\delta)(D+E) = (\alpha+\gamma)A + (\beta+\delta)B. \tag{12}$$

Each product of D's and E's can be written as a sum of products of A's and B's which can be ordered using (12) by pushing all the A's to the right and all the B's to the left. One gets that way a sum of terms of the form $B^p A^q$, the matrix elements of which can be evaluated easily ($\langle W|B^p A^q|V\rangle = \langle W|V\rangle$) from (7) and (11).

One can calculate that way the average density profile

$$\langle \tau_i \rangle = \frac{\langle W|(D+E)^{i-1} D (D+E)^{L-i}|V\rangle}{\langle W|(D+E)^L|V\rangle}$$

as well as all the correlation functions and one can recover that way (2).

One can also show that (equation (3.11) of [20])

$$\frac{\langle W|(D+E)^L|V\rangle}{\langle W|V\rangle} = \frac{1}{(\rho_a-\rho_b)^L} \frac{\Gamma(L+\frac{1}{\alpha+\gamma}+\frac{1}{\beta+\delta})}{\Gamma(\frac{1}{\alpha+\gamma}+\frac{1}{\beta+\delta})} \tag{13}$$

and using the fact that the average current between sites i and $i+1$ is given by

$$\bar{j} = \frac{\langle W|(D+E)^{i-1}(DE-ED)(D+E)^{L-i-1}|V\rangle}{\langle W|(D+E)^L|V\rangle} = \frac{\langle W|(D+E)^{L-1}|V\rangle}{\langle W|(D+E)^L|V\rangle}$$

one recovers (4) (of course in the steady state the current does not depend on i).

Remark. When $\rho_a = \rho_b = r$, i.e. for $\alpha\delta = \beta\gamma$ (see (3)), the two reservoirs are at the same density and the steady state becomes the equilibrium (Gibbs state) of the lattice gas at this density r. In this case, the weights of the configurations are those of a Bernoulli measure at density r, that is

$$\text{Pro}(\{\tau_1, \tau_2, \ldots, \tau_L\}) = \prod_{i=1}^{L} [r\tau_i + (1-r)(1-\tau_i)] \tag{14}$$

as steady state identities such as (9) can be checked directly for $r = \alpha/(\alpha+\gamma) = \delta/(\beta+\delta)$. All steady state properties can also be recovered by making all the calculations with the matrices (5), (7) for $\rho_a \neq \rho_b$ and by taking the limit $\rho_a \to \rho_b$ in the final expressions, as all the expectations, for a lattice of finite size L, are rational functions of ρ_a and ρ_b.

4. Additivity

As in (5) the weight of each configuration is written as the matrix element of a product of L matrices, one can try to insert at a position L_1 a complete basis in order to relate the properties of a lattice of L sites to those of two subsystems of sizes L_1 and $L-L_1$.

To do so let us define the following left and right eigenvectors $\langle \rho_a, a|$ and $|\rho_b, b\rangle$ of the operators $\rho_a E - (1-\rho_a)D$ and $(1-\rho_b)D - \rho_b E$:

$$\begin{aligned} \langle \rho_a, a|\,[\rho_a E - (1-\rho_a)D] &= a\langle \rho_a, a|, \\ [(1-\rho_b)D - \rho_b E]\,|\rho_b, b\rangle &= b|\rho_b, b\rangle. \end{aligned} \tag{15}$$

It is easy to see, using the definition (3), that the vectors $\langle W|$ and $|V\rangle$ which appear in (7) are given by

$$\begin{aligned} \langle W| &= \langle \rho_a, (\alpha+\gamma)^{-1}|, \\ |V\rangle &= |\rho_b, (\beta+\delta)^{-1}\rangle. \end{aligned} \tag{16}$$

It is then possible to show, using simply the fact (7) that $DE - ED = D + E$ and the definition of the eigenvectors (15), that (for $\rho_b < \rho_a$)

$$\begin{aligned} &\frac{\langle \rho_a, a|Y_1 Y_2|\rho_b, b\rangle}{\langle \rho_a, a|\rho_b, b\rangle} \\ &= \oint_{\rho_b<|\rho|<\rho_a} \frac{d\rho}{2i\pi} \frac{(\rho_a-\rho_b)^{a+b}}{(\rho_a-\rho)^{a+b}(\rho-\rho_b)} \frac{\langle \rho_a, a|Y_1|\rho, b\rangle}{\langle \rho_a, a|\rho, b\rangle} \frac{\langle \rho, 1-b|Y_2|\rho_b, b\rangle}{\langle \rho, 1-b|\rho_b, b\rangle} \end{aligned} \tag{17}$$

where Y_1 and Y_2 are arbitrary polynomials of matrices D and E. (To prove (17) it is sufficient to establish it when Y_1 and Y_2 are both of the form $E^n D^{n'}$ as any polynomial can be reduced to a sum of such terms by the relation $DE - ED = D + E$. One can also, and this is easier, prove (17) for Y_1 of the form $[\rho_a E - (1-\rho_a)D]^n [D+E]^{n'}$ and Y_2 of the form $[D+E]^{n''}[(1-\rho_b)D - \rho_b E]^{n'''}$ and show using $DE-ED = D+E$ that any polynomial Y_1 or Y_2 can be reduced to a finite sum of such terms).

5. Large deviation function of density profiles

If one divides a chain of L sites into n boxes of linear size l (one has of course $n = L/l$ such boxes), one can try to determine the probability of finding a certain density profile $\{\rho_1, \rho_2, \ldots, \rho_n\}$, i.e. the probability of seeing $l\rho_1$ particles in the first box, $l\rho_2$ particles in the second box, $\ldots l\rho_n$ in the nth box. For large L one expects the following L dependence of this probability

$$\text{Pro}_L(\rho_1, \ldots, \rho_n|\rho_a, \rho_b) \sim \exp[-L\mathcal{F}_n(\rho_1, \rho_2, \ldots, \rho_n|\rho_a, \rho_b)] \tag{18}$$

where $\mathcal{F}_n$ is a large deviation function. If one defines a reduced coordinate x by

$$i = Lx \tag{19}$$

and if one takes the limit $l \to \infty$ with $l \ll L$ so that the number of boxes becomes infinite, one can define a functional $\mathcal{F}$ for an arbitrary density profile $\rho(x)$

$$\text{Pro}_L(\{\rho(x)\}) \sim \exp[-L\mathcal{F}(\{\rho(x)\}|\rho_a, \rho_b)]. \tag{20}$$

For the SSEP (in one dimension), the functional $\mathcal{F}(\rho(x)|\rho_a, \rho_b)$ is given by the following exact expressions:

At equilibrium, i.e. for $\rho_a = \rho_b = r$

$$\mathcal{F}(\{\rho(x)\}|r, r) = \int_0^1 B(\rho(x), r)dx \tag{21}$$

where

$$B(\rho, r) = (1-\rho)\log\frac{1-\rho}{1-r} + \rho\log\frac{\rho}{r}. \tag{22}$$

This can be derived easily. When $\rho_a = \rho_b = r$, the steady state is a Bernoulli measure (14) where all the sites are occupied independently with probability r. Therefore if one divides a chain of length L into L/l intervals of length l, one has

$$\mathrm{Pro}_L(\rho_1, \ldots, \rho_n|r, r) = \prod_i^{L/l} \frac{l!}{[l\rho_i]!\ [l(1-\rho_i)]!}\, r^{l\rho_i}\ (1-r)^{l(1-\rho_i)} \tag{23}$$

and using Stirling's formula one gets (21), (22).

For the non-equilibrium case, i.e. for $\rho_a \neq \rho_b$, it was shown in [19], [4], [20] that

$$\mathcal{F}(\{\rho(x)\}|\rho_a, \rho_b) = \int_0^1 dx\ \left[B(\rho(x), F(x)) + \log\frac{F'(x)}{\rho_b - \rho_a}\right] \tag{24}$$

where the function $F(x)$ is the monotone solution of the differential equation

$$\rho(x) = F + \frac{F(1-F)F''}{F'^2} \tag{25}$$

satisfying the boundary conditions $F(0) = \rho_a$ and $F(1) = \rho_b$.

This expression shows that $\mathcal{F}$ is a *non-local* functional of the density profile $\rho(x)$ as $F(x)$ depends (in a non-linear way) on the profile $\rho(y)$ at all points y. For example if the difference $\rho_a - \rho_b$ is small, one can expand $\mathcal{F}$ and obtain an expression where the non-local character of the functional is clearly visible

$$\begin{aligned}
&\mathcal{F}(\{\rho(x)\}|\rho_a, \rho_b) \\
&\quad = \int_0^1 dx\ B(\rho(x), \bar{\rho}(x)) \\
&\qquad + \frac{(\rho_a - \rho_b)^2}{[\rho_a(1-\rho_a)]^2}\int_0^1 dx \int_x^1 dy\, x(1-y)(\rho(x) - \bar{\rho}(x))(\rho(y) - \bar{\rho}(y)) \\
&\qquad + O(\rho_a - \rho_b)^3.
\end{aligned}$$

Here $\bar{\rho}(x)$ is the most likely profile given by

$$\bar{\rho}(x) = (1-x)\rho_a + x\rho_b. \tag{26}$$

It would be too long to reproduce here the full derivation of (24), (25) from the matrix ansatz [19], [20]. The idea is to decompose the chain into L/l boxes of l sites and to sum the weights given by the matrix ansatz (5), (7) over all the microscopic configurations for which the number of particles is $l\rho_1$ in the first box, $l\rho_2$ in the second box,…, $l\rho_n$ in the nth box.

A rather easy way to derive (24), (25) is to write (we do it here in the particular case where $a + b = 1$, i.e. $\frac{1}{\alpha+\gamma} + \frac{1}{\beta+\delta} = 1$, and $\rho_a < \rho_b$) from (17) and (13)

$$P_{nl}(\rho_1, \rho_2, \dots, \rho_n|\rho_a \rho_b) = \frac{(kl)!((n-k)l)!}{(nl)!} \oint_{\rho_b<|\rho|<\rho_a} \frac{d\rho}{2i\pi} \times \frac{(\rho_a - \rho_b)^{nl+1}}{(\rho_a-\rho)^{kl+1}(\rho-\rho_b)^{(n-k)l+1}} \times P_{kl}(\rho_1, \dots, \rho_n|\rho_a, \rho) P_{(n-k)l}(\rho_{k+1}, \dots, \rho_n|\rho, \rho_b). \tag{27}$$

Note that in (27) the density ρ has become a complex variable. This is not a difficulty as all the weighs (and therefore the probabilities which appear in (27)) are rational functions of ρ_a and ρ_b.

For large nl, if one writes $k = nx$, by evaluating (27) at the saddle point one gets

$$\begin{aligned} &\mathcal{F}_n(\rho_1, \rho_2, \dots, \rho_n|\rho_a, \rho_b) \\ &\quad = \max_{\rho_b<F<\rho_a} x\mathcal{F}_k(\rho_1, \dots, \rho_k|\rho_a, F) \\ &\qquad + (1-x)\mathcal{F}_{n-k}(\rho_{k+1}, \dots, \rho_n|F, \rho_b) \\ &\qquad + x \log\left(\frac{\rho_a - F}{x}\right) + (1-x)\log\left(\frac{F-\rho_b}{1-x}\right) - \log(\rho_a - \rho_b). \end{aligned} \tag{28}$$

(Note that to estimate (27) by a saddle point method, one should find the value of ρ which maximizes the integrand over the contour. As the contour is perpendicular to the real axis at their crossing point, this becomes a minimum when ρ varies along the real axis).

If one repeats the same procedure n times, one gets

$$\begin{aligned} &\mathcal{F}_n(\rho_1, \rho_2, \dots, \rho_n|\rho_a, \rho_b) \\ &\quad = \max_{\rho_b=F_0<F_1<\cdots<F_n=\rho_a} \frac{1}{n}\sum_{i=1}^{n} \mathcal{F}_1(\rho_i|F_{i-1}, F_i) + \log\left(\frac{(F_{i-1}-F_i)n}{\rho_a-\rho_b}\right). \end{aligned} \tag{29}$$

For large n, as F_i is monotone, the difference $F_{i-1} - F_i$ is small for almost all i and one can replace $\mathcal{F}_1(\rho_i|F_{i-1}, F_i)$ by its equilibrium value $\mathcal{F}_1(\rho_i|F_i, F_i) = B(\rho_i, F_i)$. Therefore (29) becomes (24) in the limit $n \to \infty$, with (25) being the equation satisfied by the optimal $F(x)$.

6. The macroscopic fluctuation theory

Bertini, De Sole, Gabrielli, Jona-Lasinio and Landim [3], [4], [5] have developped a different and more general theory to calculate this large deviation functional which can be summarized as follows: one starts from the expression of the probability $\mathrm{Q}(\{\rho(x,s),\, j(x,s)\})$ of observing a certain time dependent macroscopic density profile $\rho(x,s)$ and current profile $j(x,t)$ over a time interval $0 \le s \le L^2 t$

$$\mathrm{Q}(\{\rho(x,s),\, j(x,s)\}) \sim \max_{\rho(x,s)} \exp\left\{-L\int_{-\infty}^{t} ds \int_0^1 dx \frac{[j + D(\rho)\frac{d\rho}{dx}]^2}{2\sigma(\rho)}\right\} \tag{30}$$

where the current $j(x,s)$ is related to the density profile $\rho(x,s)$ by the conservation law

$$\frac{d\rho(x,s)}{ds} = -\frac{dj(x,s)}{dx} \tag{31}$$

and the functions $D(\rho)$ and $\sigma(\rho)$ are characteristic of the diffusive system studied [9], [10].

Then to calculate the probability of observing a certain density profile $\rho(x)$ in the steady state, one has to find out how this fluctuation is produced. For large L, one has to find the optimal path $\rho(x,s)$ for $-\infty < s < t$ in the space of profiles which goes from the typical profile $\bar\rho(x)$ to the desired profile $\rho(x)$ and

$$\mathrm{Pro}_L(\{\rho(x)\}) \sim \max_{\rho(x,s)} \mathrm{Q}(\{\rho(x,s),\, j(x,s)\}) \tag{32}$$

where the optimal path $\rho(x,s)$ satisfies

$$\begin{aligned} \rho(x,-\infty) &= \bar\rho(x), \\ \rho(x,t) &= \rho(x). \end{aligned}$$

Finding this optimal path is usually a hard problem, and so far it has not been possible to find the explicit expression of the functional $\mathcal{F}$ for general $D(\rho)$ and $\sigma(\rho)$. For the SSEP [4], where $D(\rho) = 1$ and $\sigma(\rho) = 2\rho(1-\rho)$, this approach allows one nevertheless to derive (24), (25). It also leads to the same expression of $\mathcal{F}$ as found by the matrix approach [24] in the weakly asymmetric exclusion process and allowed one to calculate the large deviation function $\mathcal{F}$ for the KPM model [35], [7] for which no matrix approach or alternative derivation has been used so far.

The macroscopic fluctuation theory has also been successfully used recently to calculate the fluctuations and the large deviations of the current through diffusive systems [6], [9], [10], [33].

7. The matrix approach for the asymmetric exclusion process

The matrix ansatz of Section 3 (which gives the weights of the microscopic configurations in the steady state) has been generalized to describe the steady state of several other systems [1], [8], [11], [14], [18], [25], [30], [31], [32], [37], [40], [42], [43], [45], with of course modified algebraic rules for the matrices the vectors $\langle W|$ and $|V\rangle$.

For example for the asymmetric exclusion process (ASEP), for which the definition is the same as the SSEP of Section 2, except that particles jump at rate 1 to their right and at rate $q \neq 1$ to their left (it the target site is empty), one can show [16], [8], [42], [43] that in this case too, the weights are still given by (5) with the algebra (7) replaced by

$$DE - qED = D + E, \tag{33}$$

$$\langle W|(\alpha E - \gamma D) = \langle W|, \tag{34}$$

$$(\beta D - \delta E)|V\rangle = |V\rangle. \tag{35}$$

One should notice that for the ASEP, the direct approach of calculating the steady state properties by writing the time evolution does not work. Indeed (1) becomes

$$\begin{aligned}
\frac{d\langle\tau_1\rangle}{dt} &= \alpha - (\alpha + \gamma + 1)\langle\tau_1\rangle + q\langle\tau_2\rangle + (1-q)\langle\tau_1\tau_2\rangle,\\
\frac{d\langle\tau_i\rangle}{dt} &= \langle\tau_{i-1}\rangle - (1+q)\langle\tau_i\rangle + q\langle\tau_{i+1}\rangle - (1-q)(\langle\tau_{i-1}\tau_i\rangle - \langle\tau_i\tau_{i+1}\rangle),\\
\frac{d\langle\tau_L\rangle}{dt} &= \langle\tau_{L-1}\rangle - (q + \beta + \delta)\langle\tau_L\rangle + \delta - (1-q)\langle\tau_{L-1}\tau_L\rangle,
\end{aligned} \tag{36}$$

and the equations which determine the one-point functions are no longer closed. Therefore all the correlation functions have to be determined at the same time and this is what the matrix ansatz does.

The large deviation function $\mathcal{F}$ of the density defined by (20) has been calculated for the ASEP [21], [22], [24] by an extension of the approach sketched in Sections 4 and 5.

8. The phase diagram of the totally asymmetric exclusion process

The last two sections (8 and 9) present two results which can be obtained in the totally asymmetric case (TASEP), i.e. for $q = 0$ (in the particular case where particles are injected only at the left boundary and removed only at the right boundary, i.e. when the input rates $\gamma = \delta = 0$). In this case the algebra (33) becomes

$$DE = D + E, \tag{37}$$

$$\langle W|\alpha E = \langle W|, \tag{38}$$

$$\beta D|V\rangle = |V\rangle. \tag{39}$$

As for the SSEP the average current is still given in terms of the vectors $\langle W|$, $V\rangle$ and of the matrices D and E by

$$\bar{j} = \frac{\langle W|(D+E)^{L-1}|V\rangle}{\langle W|(D+E)^L|V\rangle} \tag{40}$$

However as the algebraic rules have changed, the expression of the current is different for the SSEP and the ASEP. From the relation $DE = D + E$ it is easy to prove by recurrence that

$$DF(E) = F(1) + E\,\frac{F(E) - F(1)}{E - 1}$$

for any polynomial $F(E)$ and

$$(D+E)^N = \sum_{p=1}^{N} \frac{p(2N-1-p)!}{N!(N-p)!}(E^p + E^{p-1}D + \cdots + D^p).$$

Using the fact that

$$\frac{\langle W|E^m D^n|V\rangle}{\langle W|V\rangle} = \frac{1}{\alpha^m}\frac{1}{\beta^n},$$

one gets [16]

$$\frac{\langle W|(D+E)^N|V\rangle}{\langle W|V\rangle} = \sum_{p=1}^{N} \frac{p(2N-1-p)!}{N!(N-p)!}\,\frac{\dfrac{1}{\alpha^{p+1}} - \dfrac{1}{\beta^{p+1}}}{\dfrac{1}{\alpha} - \dfrac{1}{\beta}}. \tag{41}$$

For large N this sum is dominated either by $p \sim 1$, or $p \sim N$ depending one the values of α and β and one obtains

$$\frac{\langle W|(D+E)^N|V\rangle}{\langle W|V\rangle} \sim \begin{cases} 4^N & \text{if } \alpha > \frac{1}{2} \text{ and } \beta > \frac{1}{2}, \\ [\beta(1-\beta)]^{-N} & \text{if } \beta < \alpha \text{ and } \beta < \frac{1}{2}, \\ [\alpha(1-\alpha)]^{-N} & \text{if } \beta > \alpha \text{ and } \alpha < \frac{1}{2}. \end{cases} \tag{42}$$

This leads to three different expressions of the current (40) for large L corresponding to the three different phases:

- the low density phase ($\beta > \alpha$ and $\alpha < \frac{1}{2}$) where $\bar{j} = \alpha(1-\alpha)$
- the high density phase ($\alpha > \beta$ and $\beta < \frac{1}{2}$) where $\bar{j} = \beta(1-\beta)$
- the maximal current phase ($\alpha > \frac{1}{2}$ and $\beta > \frac{1}{2}$) where $\bar{j} = \frac{1}{4}$

which is the exact phase diagram of the TASEP [38], [15], [16], [44]. The existence of phase transitions [26], [34] in these driven lattice gases is one of the most striking properties of non-equilibrium systems, as it is well known that one dimensional systems at equilibrium with short range interactions cannot exhibit phase transitions.

9. Correlation functions in the TASEP and Brownian excursions

For the TASEP, in the maximal current phase ($\alpha > \frac{1}{2}$ and $\beta > \frac{1}{2}$) one can show [17], using the matrix ansatz, that the correlation function of the occupations of k sites at positions $i_1 = Lx_1, i_2 = Lx_2, \dots, i_k = Lx_k$ with $(Lx_1 < Lx_2 < \cdots < Lx_k)$ are given for large L by

$$\left\langle \left(\tau_{Lx_1} - \frac{1}{2}\right) \dots \left(\tau_{Lx_k} - \frac{1}{2}\right) \right\rangle = \frac{1}{2^k} \frac{1}{L^{k/2}} \frac{d^k}{dx_1 \dots dx_k} \langle y_1 \dots y_k \rangle, \tag{43}$$

where $y(x)$ is a Brownian excursion between 0 and 1 (a Brownian excursion is a Brownian path constrained to $y(x) > 0$ for $0 < x < 1$ with the boundaries $y(0) = y(1)$). The probability $P(y_1 \dots y_k; x_1 \dots x_k)$ of finding the Brownian excursion at positions $y_1 \dots y_k$ for $0 < x_1 < \cdots < x_k < 1$ is

$$P(y_1 \dots y_k; x_1 \dots x_k) = \frac{h_{x_1}(y_1)\, g_{x_2 - x_1}(y_1, y_2) \dots g_{x_k - x_{k-1}}(y_{k-1}, y_k)\, h_{1-x_k}(y_k)}{\sqrt{\pi}},$$

where h_x and g_x are defined by

$$\begin{cases} h_x(y) = \frac{2y}{x^{3/2}} e^{-y^2/x}, \\ g_x(y, y') = \frac{1}{\sqrt{\pi x}} (e^{-(y-y')^2/x} - e^{-(y+y')^2/x}). \end{cases}$$

One can derive easily (43) in the particular case $\alpha = \beta = 1$ using a representation of (37) which consists of two infinite dimensional bidiagonal matrices

$$D = \sum_{n \geq 1} |n\rangle\langle n| + |n\rangle\langle n+1| = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & 0 & \cdots \\ & & & & \ddots & \ddots & \end{pmatrix}$$

and

$$E = \sum_{n \geq 1} |n\rangle\langle n+1| + |n\rangle\langle n| = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & 0 & \cdots \\ & & & \ddots & \ddots & & \end{pmatrix}$$

with

$$\langle W| = \langle 1| = (1, 0, 0 \dots),$$

$$\langle V| = \langle 1| = (1, 0, 0 \dots).$$

With this representation one can write $\langle W|(D+E)^L|V\rangle$ as a sum over a set $\mathcal{M}_L$ of one dimensional random walks w of L steps which never come back to the origin. Each walk w is defined by a sequence $(n_i(w))$ of $L-1$ heights $(n_i(w) \geq 1)$ (with $n_0(w) = n_L(w) = 1$ at the boundaries and the constraint $|n_{i+1} - n_i| \leq 1$). One then has

$$\langle W|(D+E)^L|V\rangle = \sum_{w \in \mathcal{M}_L} \Omega(w),$$

where

$$\Omega(w) = \prod_{i=1}^{L} v(n_{i-1}, n_i) \quad \text{with } v(n, n') = \begin{cases} 2 & \text{if } |n - n'| = 0, \\ 1 & \text{if } |n - n'| = 1, \end{cases}$$

using the fact that $v(n, n') = \langle n|D+E|n'\rangle$ since $D+E$ can be written as

$$D + E = \begin{pmatrix} 2 & 1 & & (0) \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ (0) & & 1 & 2 \end{pmatrix}.$$

Then from the matrix expression one gets $\langle \tau_i \rangle$ and $\langle \tau_i \tau_j \rangle$:

$$\left\langle \left(\tau_{i_1} - \frac{1}{2}\right) \dots \left(\tau_{i_k} - \frac{1}{2}\right) \right\rangle = \frac{1}{2^k} \sum_w \nu(w)(n_{i_1} - n_{i_1 - 1}) \dots (n_{i_k} - n_{i_k - 1}), \quad (44)$$

where $\nu(w)$ is the probability of the walk w induced by the weights Ω:

$$\nu(w) = \frac{\Omega(w)}{\sum_w' \Omega(w')}.$$

The expression (44) is the discrete version of (43). The result (43) can be extended [17] to arbitrary values of α and β in the maximal current phase (i.e. for $\alpha > 1/2$ and $\beta > 1/2$).

From this link between the density fluctuations and Brownian excursions, one can show that, for a TASEP of L sites, the number N of particles between sites Lx_1 and Lx_2, has non-Gaussian fluctuations in the maximal current phase: if one defines the reduced density

$$\mu = \frac{N - L(x_2 - x_1)/2}{\sqrt{L}}$$

one can show [17] that for large L

$$P(\mu) = \int_0^\infty dy_1 \int_0^\infty dy_2 \, \frac{1}{\sqrt{2\pi(x_2 - x_1)}} \, \exp\left(-\frac{(2\mu + y_1 - y_2)^2}{x_2 - x_1}\right). \quad (45)$$

According to numerical simulations [17] this distribution (properly rescaled) of the fluctuations of the density remains valid for more general driven systems in their maximal current phase. Of course proving it in a more general case is an interesting open question.

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