Correlation functions of the XXZ Heisenberg spin chain: Bethe ansatz approach

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Abstract. We review recent progress in the computation of correlation functions of the XXZ spin-1/2 chain. We describe both finite and infinite chain results. Long distance asymptotic behavior is discussed. Our method is based on the resolution of the quantum inverse scattering problem in the framework of the algebraic Bethe ansatz.

1. Introduction

The main challenging problem in the theory of quantum integrable models [1], [2], [3], [4], [5], [6] besides computing their spectrum is to obtain exact and manageable representations for their correlation functions. This issue is of great importance not only from theoretical and mathematical view points but also for applications to relevant physical situations. Although several important advances have been obtained over the years, we are still looking for a general method that could give a systematic solution to this problem. The purpose of this article is to give a review of an approach to this problem elaborated in [7], [8], [9], [10] and in [11], [12], [13], [14], together with a brief account of the more recent progress obtained in [15], [16], [17], [18], [19].

In our search for a general method to compute correlation functions of quantum integrable models our strategy was to consider a simple but representative model where it is possible to develop new tools to solve this problem. Such an archetype of quantum integrable lattice models is provided by the XXZ spin-$\frac{1}{2}$ Heisenberg [20] chain in a magnetic field. Indeed, Heisenberg spin chains play a prominent role in the theory of quantum integrable models: they were the first models for which Bethe ansatz [21], [22], [23], [24], [25] was invented and successfully applied to compute their spectrum. This method has been later used and generalized to solve a large variety of integrable models ranging from statistical mechanics to quantum field theories (see [1], [2], [3], [4], [5]).

The XXZ spin-$\frac{1}{2}$ Heisenberg chain in a magnetic field is a quantum interacting model defined on a one-dimensional lattice with Hamiltonian

$$H = H^{(0)} - h S_z; \tag{1}$$

$$H^{(0)} = \sum_{m=1}^{M} \left\{ \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta(\sigma_m^z \sigma_{m+1}^z - 1) \right\}, \tag{2}$$

$$S_z = \frac{1}{2} \sum_{m=1}^{M} \sigma_m^z, \quad [H^{(0)}, S_z] = 0. \tag{3}$$

Here Δ is the anisotropy parameter, h denotes the magnetic field, and $\sigma_m^{x,y,z}$ are the local spin operators (in the spin-$\frac{1}{2}$ representation) associated with each site m of the chain. The quantum space of states is $\mathcal{H} = \otimes_{m=1}^{M} \mathcal{H}_m$, where $\mathcal{H}_m \sim \mathbb{C}^2$ is called local quantum space, with dim $\mathcal{H} = 2^M$. The operators $\sigma_m^{x,y,z}$ act as the corresponding Pauli matrices in the space $\mathcal{H}_m$ and as the identity operator elsewhere. For simplicity, the length of the chain M is chosen to be even and we assume periodic boundary conditions. Since the simultaneous reversal of all spins is equivalent to a change of sign of the magnetic field, it is enough to consider the case $h \geq 0$.

The first task to solve such a model is to describe the spectrum of its Hamiltonian (1). The method to compute eigenvectors and associated energy levels of the Heisenberg spin chains goes back to H. Bethe in 1931 [21], [22], [23], [25] and is known as the Bethe ansatz. An algebraic version of it has been invented in the late 70s by Faddeev, Sklyanin and Taktajan [26], [27].

The second problem is to compute matrix elements of spin operators $\sigma_m^{x,y,z}$ between two eigenvectors of H and then all correlation functions of spin operators: at zero temperature they reduce to the average value of products of spin operators in the lowest energy level state (the ground state). Let us denote by $| \psi_g \rangle$ the normalized ground state vector. Let $E_m^{\epsilon'_m, \epsilon_m}$ be the elementary operators acting at site m as the 2×2 matrices $E_{lk}^{\epsilon', \epsilon} = \delta_{l,\epsilon'} \delta_{k,\epsilon}$. Any n-point correlation function can be reconstructed as a sum of the following elementary blocks:

$$F_m(\{\epsilon_j, \epsilon'_j\}, \ h) = \langle \psi_g | \prod_{j=1}^{m} E_j^{\epsilon'_j, \epsilon_j} | \psi_g \rangle. \tag{4}$$

The knowledge of such correlation functions was for a long time restricted to the free fermion point $\Delta = 0$, a case for which nevertheless tremendous works have been necessary to obtain full answers [28], [29], [30], [31], [32], [33]. For generic Δ, in the thermodynamic limit, at zero temperature, and for zero magnetic field, multiple integral representation of the above elementary blocks of the correlation functions have been obtained from the q-vertex operator approach (also using corner transfer matrix technique) in the massive regime $\Delta \geq 1$ in 1992 [34] and conjectured in 1996 [35] for the massless regime $-1 \leq \Delta \leq 1$ (see also [6]).

These results together with their extension to non-zero magnetic field have been obtained in 1999 [8], [9] using the algebraic Bethe ansatz framework [26], [27], [4] and the actual resolution of the so-called quantum inverse scattering problem [8], [10]. This method allows for the computation of the matrix elements of the local spin operators and the above elementary blocks of the correlation functions for the finite chain. Hence, thermodynamic limit can be considered separately. Moreover, time or temperature dependent correlation functions can also be computed [15], [16], [36] using such techniques. Let us mention also recent advances using q-KZ equations [37], [38].

This article is meant to be a rather brief review of the problem of correlation functions. More detailed account of the results sketched here together with their proofs can be found in the original articles [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17] and in [18], [19]. This article is organized as follows. The space of states of the Heisenberg spin chain will be described in the next section. It includes a brief introduction to the algebraic Bethe ansatz and to various tools of importance in the computation of correlation functions, like in particular the solution of the quantum inverse scattering problem and the determinant representations of the scalar products of states. Section 3 is devoted to the correlation functions of the finite chain. Correlation functions in the thermodynamic limit are studied in Section 4. In Section 5 we describe several exact and asymptotic results together with some open problems. Conclusions and some perspectives are given in the last section.

2. The space of states: algebraic Bethe ansatz

The space of states is of dimension 2^M. As can be observed from the definition of the Hamiltonian in (1), the construction of its eigenvectors is rather non trivial. The purpose of this section is to briefly explain the basics of the knowledge of the space of states in the framework of the algebraic Bethe ansatz, leading in particular to the determination of the spectrum of (1).

2.1. Algebraic Bethe ansatz. The algebraic Bethe ansatz originated from the fusion of the original (coordinate) Bethe ansatz and of the inverse scattering method in its Hamiltonian formulation [26], [27], [4]. At the root of the algebraic Bethe ansatz method is the construction of the quantum monodromy matrix. In the case of the XXZ chain (1) the monodromy matrix is a 2×2 matrix,

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \tag{5}$$

with operator-valued entries A, B, C and D which depend on a complex parameter λ (spectral parameter) and act in the quantum space of states $\mathcal{H}$ of the chain. One of the main property of these operators is that the trace of T, namely $A + D$, commutes with the Hamiltonian H, while operators B and C can be used as creation operators

of respectively eigenvectors and dual eigenvectors of $A + D$ and hence of H itself. The monodromy matrix is defined as the following ordered product:

$$T(\lambda) = L_M(\lambda) \dots L_2(\lambda) L_1(\lambda), \tag{6}$$

where $L_n(\lambda)$ denotes the quantum L-operator at the site n of the chain,

$$L_n(\lambda) = \begin{pmatrix} \sinh(\lambda + \frac{\eta}{2} \sigma_n^z) & \sinh \eta \, \sigma_n^- \\ \sinh \eta \, \sigma_n^+ & \sinh(\lambda - \frac{\eta}{2} \sigma_n^z) \end{pmatrix}. \tag{7}$$

The parameter η is related to the anisotropy parameter as $\Delta = \cosh \eta$. It follows from this definition that the monodromy matrix is an highly non local operator in terms of the local spin operators $\sigma_n^{x,y,z}$. However, the commutation relations between the operators A, B, C, D can be computed in a simple way. They are given by the quantum R-matrix

$$R(\lambda, \mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\lambda, \mu) & c(\lambda, \mu) & 0 \\ 0 & c(\lambda, \mu) & b(\lambda, \mu) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{8}$$

where

$$b(\lambda, \mu) = \frac{\sinh(\lambda - \mu)}{\sinh(\lambda - \mu + \eta)}, \quad c(\lambda, \mu) = \frac{\sinh(\eta)}{\sinh(\lambda - \mu + \eta)}. \tag{9}$$

The R-matrix is a linear operator in the tensor product $V_1 \otimes V_2$, where each V_i is isomorphic to $\mathbb{C}^2$, and depends generically on two spectral parameters λ_1 and λ_2 associated to these two vector spaces. It is denoted by $R_{12}(\lambda_1, \lambda_2)$. Such an R-matrix satisfies the Yang–Baxter equation,

$$R_{12}(\lambda_1, \lambda_2) \, R_{13}(\lambda_1, \lambda_3) \, R_{23}(\lambda_2, \lambda_3) = R_{23}(\lambda_2, \lambda_3) \, R_{13}(\lambda_1, \lambda_3) \, R_{12}(\lambda_1, \lambda_2). \tag{10}$$

It gives the following commutation relations among the operators entries of the monodromy matrix:

$$R_{12}(\lambda, \mu) \, T_1(\lambda) \, T_2(\mu) = T_2(\mu) \, T_1(\lambda) \, R_{12}(\lambda, \mu) \tag{11}$$

with the tensor notations $T_1(\lambda) = T(\lambda) \otimes \mathrm{Id}$ and $T_2(\mu) = \mathrm{Id} \otimes T(\mu)$. These commutation relations imply in particular that the transfer matrices, defined as

$$\mathcal{T}(\lambda) = \mathrm{tr}\, T(\lambda) = A(\lambda) + D(\lambda), \tag{12}$$

commute for different values of the spectral parameter $[\mathcal{T}(\lambda), \mathcal{T}(\mu)] = 0$ and also with S_z, $[\mathcal{T}(\lambda), S_z] = 0$. The Hamiltonian (2) at $h = 0$ is related to $\mathcal{T}(\lambda)$ by the 'trace identity'

$$H^{(0)} = 2 \sinh \eta \, \frac{d\mathcal{T}(\lambda)}{d\lambda} \mathcal{T}^{-1}(\lambda) \Big|_{\lambda = \frac{\eta}{2}} - 2M \cosh \eta. \tag{13}$$

Therefore, the spectrum of the Hamiltonian (1) is given by the common eigenvectors of the transfer matrices and of S_z.

For technical reasons, it is actually convenient to introduce a slightly more general object, the twisted transfer matrix

$$\mathcal{T}_\kappa(\lambda) = A(\lambda) + \kappa D(\lambda), \tag{14}$$

where κ is a complex parameter. The particular case of $\mathcal{T}_\kappa(\lambda)$ at $\kappa = 1$ corresponds to the usual (untwisted) transfer matrix $\mathcal{T}(\lambda)$. It will be also convenient to consider an inhomogeneous version of the *X X Z* chain for which

$$T_{1...M}(\lambda; \xi_1, \ldots, \xi_M) = L_M(\lambda - \xi_M + \eta/2) \ldots L_1(\lambda - \xi_1 + \eta/2). \tag{15}$$

Here, $\xi_1, \ldots, \xi_M$ are complex parameters (inhomogeneity parameters) attached to each site of the lattice. The homogeneous model (1) corresponds to the case where $\xi_j = \eta/2$ for $j = 1, \ldots, M$.

In the framework of algebraic Bethe ansatz, an arbitrary quantum state can be obtained from the vectors generated by multiple action of operators $B(\lambda)$ on the reference vector $|0\rangle$ with all spins up (respectively by multiple action of operators $C(\lambda)$ on the dual reference vector $\langle 0|$),

$$|\psi\rangle = \prod_{j=1}^{N} B(\lambda_j)|0\rangle, \quad \langle\psi| = \langle 0|\prod_{j=1}^{N} C(\lambda_j), \quad N = 0, 1, \ldots, M. \tag{16}$$

2.2. Description of the spectrum. Let us consider here the subspace $\mathcal{H}^{(M/2-N)}$ of the space of states $\mathcal{H}$ with a fixed number N of spins down. In this subspace, the eigenvectors $|\psi_\kappa(\{\lambda\})\rangle$ (respectively $\langle\psi_\kappa(\{\lambda\})|$) of the twisted transfer matrix $\mathcal{T}_\kappa(\mu)$ can be constructed in the form (16), where the parameters $\lambda_1, \ldots, \lambda_N$ satisfy the system of twisted Bethe equations

$$\mathcal{Y}_\kappa(\lambda_j|\{\lambda\}) = 0, \quad j = 1, \ldots, N. \tag{17}$$

Here, the function $\mathcal{Y}_\kappa$ is defined as

$$\mathcal{Y}_\kappa(\mu|\{\lambda\}) = a(\mu) \prod_{k=1}^{N} \sinh(\lambda_k - \mu + \eta) + \kappa \, d(\mu) \prod_{k=1}^{N} \sinh(\lambda_k - \mu - \eta), \tag{18}$$

and $a(\lambda)$, $d(\lambda)$ are the eigenvalues of the operators $A(\lambda)$ and $D(\lambda)$ on the reference state $|0\rangle$. In the normalization (7) and for the inhomogeneous model (15) we have

$$a(\lambda) = \prod_{a=1}^{M} \sinh(\lambda - \xi_a + \eta), \quad d(\lambda) = \prod_{a=1}^{M} \sinh(\lambda - \xi_a). \tag{19}$$

The corresponding eigenvalue of $\mathcal{T}_\kappa(\mu)$ on $|\psi_\kappa(\{\lambda\})\rangle$ (or on a dual eigenvector) is

$$\tau_\kappa(\mu|\{\lambda\}) = a(\mu) \prod_{k=1}^{N} \frac{\sinh(\lambda_k - \mu + \eta)}{\sinh(\lambda_k - \mu)} + \kappa \, d(\mu) \prod_{k=1}^{N} \frac{\sinh(\mu - \lambda_k + \eta)}{\sinh(\mu - \lambda_k)}. \quad (20)$$

The solutions of the system of twisted Bethe equations (17) have been analyzed in [39]. In general, not all of these solutions correspond to eigenvectors of $\mathcal{T}_\kappa(\mu)$.

Definition 2.1. A solution $\{\lambda\}$ of the system (17) is called *admissible* if

$$d(\lambda_j) \prod_{\substack{k=1 \\ k \neq j}}^{N} \sinh(\lambda_j - \lambda_k + \eta) \neq 0, \quad j = 1, \ldots, N, \quad (21)$$

and *un-admissible* otherwise. A solution is called *off-diagonal* if the corresponding parameters $\lambda_1, \ldots, \lambda_N$ are pairwise distinct, and *diagonal* otherwise.

One of the main result of [39] is that, for generic parameters κ and $\{\xi\}$, the set of the eigenvectors corresponding to the admissible off-diagonal solutions of the system of twisted Bethe equations (17) form a basis in the subspace $\mathcal{H}^{(M/2-N)}$. It has been proven in [16] that this result is still valid in the homogeneous case $\xi_j = \eta/2$, $j = 1, \ldots, N$, at least if κ is in a punctured vicinity of the origin (i.e. $0 < |\kappa| < \kappa_0$ for κ_0 small enough). Note however that, for specific values of κ and $\{\xi\}$, the basis of the eigenvectors in $\mathcal{H}^{(M/2-N)}$ may include some states corresponding to un-admissible solutions of (17) (in particular in the homogeneous limit at $\kappa = 1$).

At $\kappa = 1$, it follows from the trace identity (13) that the eigenvectors of the transfer matrix coincide, in the homogeneous limit, with the ones of the Hamiltonian (1). The corresponding eigenvalues in the case of zero magnetic field can be obtained from (13), (20):

$$H^{(0)} \, | \psi(\{\lambda\}) \rangle = \left(\sum_{j=1}^{N} E(\lambda_j) \right) \cdot | \psi(\{\lambda\}) \rangle, \quad (22)$$

where the (bare) one-particle energy $E(\lambda)$ is equal to

$$E(\lambda) = \frac{2 \sinh^2 \eta}{\sinh(\lambda + \frac{\eta}{2}) \sinh(\lambda - \frac{\eta}{2})}. \quad (23)$$

2.3. Drinfel'd twist and F-basis. As already noted, the operators A, B, C, D are highly non local in terms of local spin operators. There exists however an interesting description of these operators by means of a change of basis of the space of states. In particular, this basis will provide a direct access to the scalar products of states. The root of this new basis is provided by the notion of Drinfel'd twist [40] associated to the R-matrix of the XXZ chain. It leads to the notion of factorizing F-matrices. To be essentially self-contained we briefly recall here their main properties and refer to [7] for more details and proofs.

Definition 2.2. For inhomogeneity parameters ξ_j in generic positions and for any integer n one can associate to any element σ of the symmetric group S_n on n elements a unique R-matrix $R^\sigma_{1...n}(\xi_1, \ldots, \xi_n)$, denoted for simplicity $R^\sigma_{1...n}$, constructed as an ordered product (depending on σ) of the elementary R-matrices $R_{ij}(\xi_i, \xi_j)$.

We have the following property for an arbitrary integer n.

Proposition 2.1.

$$R^\sigma_{1...n}\, T_{1...n}(\lambda; \xi_1, \ldots, \xi_n) = T_{\sigma(1)...\sigma(n)}(\lambda; \xi_{\sigma(1)}, \ldots, \xi_{\sigma(n)})\, R^\sigma_{1...n}. \qquad (24)$$

We can now define the notion of a factorizing F-matrix:

Definition 2.3. A factorizing F-matrix associated to a given elementary R-matrix is an invertible matrix $F_{1...n}(\xi_1, \ldots, \xi_n)$, defined for an arbitrary integer n, satisfying the following relation for any element σ of S_n:

$$F_{\sigma(1)...\sigma(n)}(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(n)})\, R^\sigma_{1...n}(\xi_1, \ldots, \xi_n) = F_{1...n}(\xi_1, \ldots, \xi_n). \qquad (25)$$

In other words, such an F-matrix factorizes the corresponding R-matrix for arbitrary integers n. Taking into account the fact that the parameters ξ_n are in one to one correspondence with the vector spaces $\mathcal{H}_n$, we can adopt simplified notations such that

$$F_{1...n}(\xi_1, \ldots, \xi_n) = F_{1...n},$$
$$F_{\sigma(1)...\sigma(n)}(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(n)}) = F_{\sigma(1)...\sigma(n)}.$$

Theorem 2.1 ([7]). *For the XXZ model with inhomogeneity parameters ξ_n in generic positions there exists a factorizing, triangular F-matrix. It is constructed explicitly from the R-matrix.*

This matrix has two important properties:

Proposition 2.2 ([7]). *In the F-basis, the monodromy matrix $\widetilde{T}$,*

$$\widetilde{T}_{1...M}(\lambda; \xi_1, \ldots, \xi_M) \;=\; F_{1...M}T_{1...M}(\lambda; \xi_1, \ldots, \xi_M)\, F^{-1}_{1...M}, \qquad (26)$$

is totally symmetric under any simultaneous permutations of the lattice sites i and of the corresponding inhomogeneity parameters ξ_i.

The second property gives the explicit expressions of the monodromy matrix in the F-basis. For the XXZ-$\frac{1}{2}$ model, the quantum monodromy operator is a 2×2 matrix with entries A, B, C, D which are obtained as sums of 2^{M-1} operators, which themselves are products of M local spin operators on the quantum chain. As an example, the B operator is given as

$$B_{1...M}(\lambda) = \sum_{i=1}^{N} \sigma_i^- \, \Omega_i \; + \sum_{i \neq j \neq k} \sigma_i^- \, (\sigma_j^- \, \sigma_k^+) \, \Omega_{ijk} \; + \text{ higher terms}, \qquad (27)$$

where the matrices Ω_i, Ω_{ijk}, are diagonal operators acting respectively on all sites but i, on all sites but i, j, k, and the higher order terms involve more and more exchange spin terms like $\sigma_j^- \sigma_k^+$. It means that the B operator returns one spin somewhere on the chain, this operation being however dressed non-locally and with non-diagonal operators by multiple exchange terms of the type $\sigma_j^- \sigma_k^+$.

So, whereas these formulas in the original basis are quite involved, their expressions in the F-basis simplify drastically:

Proposition 2.3 ([7]). *The operators D, B and C in the F-basis are given by the formulas*

$$\widetilde{D}_{1\ldots M}(\lambda; \xi_1, \ldots, \xi_M) = \overset{M}{\underset{i=1}{\otimes}} \begin{pmatrix} b(\lambda, \xi_i) & 0 \\ 0 & 1 \end{pmatrix}_{[i]}, \tag{28}$$

$$\widetilde{B}_{1\ldots M}(\lambda) = \sum_{i=1}^{M} \sigma_i^- \, c(\lambda, \xi_i) \underset{j \neq i}{\otimes} \begin{pmatrix} b(\lambda, \xi_j) & 0 \\ 0 & b^{-1}(\xi_j, \xi_i) \end{pmatrix}_{[j]}, \tag{29}$$

$$\widetilde{C}_{1\ldots M}(\lambda) = \sum_{i=1}^{M} \sigma_i^+ \, c(\lambda, \xi_i) \underset{j \neq i}{\otimes} \begin{pmatrix} b(\lambda, \xi_j) \, b^{-1}(\xi_i, \xi_j) & 0 \\ 0 & 1 \end{pmatrix}_{[j]}, \tag{30}$$

and the operator $\widetilde{A}$ can be obtained from quantum determinant relations.

We wish first to stress that while the operators $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, $\widetilde{D}$ satisfy the same quadratic commutation relations as A, B, C, D, they are completely symmetric under simultaneous exchange of the inhomogeneity parameters and of the spaces $\mathcal{H}_n$. It really means that the factorizing F-matrices we have constructed solve the combinatorial problem induced by the non-trivial action of the permutation group S_M given by the R-matrix. In the F-basis the action of the permutation group on the operators $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, $\widetilde{D}$ is trivial.

Further, it can be shown that the pseudo-vacuum vector is left invariant, namely, it is an eigenvector of the total F-matrix with eigenvalue 1; in particular, the algebraic Bethe ansatz can be carried out also in the F-basis. Hence, a direct computation of Bethe eigenvectors and of their scalar products in this F-basis is made possible, while it was a priori very involved in the original basis. There, only commutation relations between the operators A, B, C, D can be used, leading (see [5]) to very intricate sums over partitions.

2.4. Solution of the quantum inverse problem. The very simple expressions of the monodromy matrix operators entries D, B, C in the F-basis suggests that any local operator $E_j^{\epsilon_j', \epsilon_j}$, acting in a local quantum space $\mathcal{H}_j$ at site j, can be expressed in terms of the entries of the monodromy matrix. This is the so-called quantum inverse scattering problem. The solution to this problem was found in [8], [10]:

Theorem 2.2.

$$E_j^{\epsilon_j', \epsilon_j} = \prod_{\alpha=1}^{j-1} \mathcal{T}(\xi_\alpha) \cdot T_{\epsilon_j, \epsilon_j'}(\xi_j) \cdot \prod_{\alpha=1}^{j} \mathcal{T}^{-1}(\xi_\alpha). \tag{31}$$

The proof of this theorem is elementary (see [8], [10]) and hence it can be obtained for a large class of lattice integrable models. It relies essentially on the property that the R-matrix $R(\lambda, \mu)$ reduces to the permutation operator for $\lambda = \mu$. An immediate consequence of this theorem is that the operators A, B, C, and D generate the space of all operators acting in $\mathcal{H}$.

2.5. Scalar products. We give here the expressions for the scalar product of an eigenvector of the twisted transfer matrix with any arbitrary state of the form (16). These scalar products can be expressed as determinant of rather simple matrices. The root of all these determinants is in fact the determinant representation for the partition function of the 6-vertex model with domain wall boundary conditions [41]. Let us first define, for arbitrary positive integers n, n' ($n \leq n'$) and arbitrary sets of variables $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n$ and $\nu_1, \ldots, \nu_{n'}$ such that $\{\lambda\} \subset \{\nu\}$, the $n \times n$ matrix $\Omega_\kappa(\{\lambda\}, \{\mu\}|\{\nu\})$ as

$$(\Omega_\kappa)_{jk}(\{\lambda\}, \{\mu\}|\{\nu\}) = a(\mu_k) \, t(\lambda_j, \mu_k) \prod_{a=1}^{n'} \sinh(\nu_a - \mu_k + \eta)$$

$$- \kappa \, d(\mu_k) \, t(\mu_k, \lambda_j) \prod_{a=1}^{n'} \sinh(\nu_a - \mu_k - \eta), \tag{32}$$

with

$$t(\lambda, \mu) = \frac{\sinh \eta}{\sinh(\lambda - \mu) \sinh(\lambda - \mu + \eta)}. \tag{33}$$

Proposition 2.4 ([42], [8], [15]). *Let $\{\lambda_1, \ldots, \lambda_N\}$ be a solution of the system of twisted Bethe equations (17), and let $\mu_1, \ldots, \mu_N$ be generic complex numbers. Then,*

$$\langle 0 | \prod_{j=1}^{N} C(\mu_j) | \psi_\kappa(\{\lambda\}) \rangle = \langle \psi_\kappa(\{\lambda\}) | \prod_{j=1}^{N} B(\mu_j) | 0 \rangle$$

$$= \frac{\prod\limits_{a=1}^{N} d(\lambda_a) \prod\limits_{a,b=1}^{N} \sinh(\mu_b - \lambda_a)}{\prod\limits_{a>b}^{N} \sinh(\lambda_a - \lambda_b) \sinh(\mu_b - \mu_a)} \cdot \det_N \left(\frac{\partial}{\partial \lambda_j} \tau_\kappa(\mu_k|\{\lambda\}) \right) \tag{34}$$

$$= \frac{\prod\limits_{a=1}^{N} d(\lambda_a)}{\prod\limits_{a>b}^{N} \sinh(\lambda_a - \lambda_b) \sinh(\mu_b - \mu_a)} \cdot \det_{N} \Omega_\kappa(\{\lambda\}, \{\mu\}|\{\lambda\}). \quad (35)$$

These equations are valid for any arbitrary complex parameter κ, in particular at $\kappa = 1$. In this case we may omit the subscript κ and denote $(\psi, \tau, \mathcal{Y}, \Omega) = (\psi_\kappa, \tau_\kappa, \mathcal{Y}_\kappa, \Omega_\kappa)|_{\kappa=1}$. If the sets $\{\lambda\}$ and $\{\mu\}$ are different, the eigenvector $|\psi_\kappa(\{\lambda\})\rangle$ is orthogonal to the dual eigenvector $\langle\psi_\kappa(\{\mu\})|$. Otherwise we obtain a formula for the norm of the corresponding vector [43], [44], [8],

$$\langle\psi_\kappa(\{\lambda\})|\psi_\kappa(\{\lambda\})\rangle = \frac{\prod\limits_{a=1}^{N} d(\lambda_a)}{\prod\limits_{\substack{a,b=1 \\ a\neq b}}^{N} \sinh(\lambda_a - \lambda_b)} \cdot \det_{N} \Omega_\kappa(\{\lambda\}, \{\lambda\}|\{\lambda\})$$

$$= (-1)^N \frac{\prod\limits_{a=1}^{N} d(\lambda_a)}{\prod\limits_{\substack{a,b=1 \\ a\neq b}}^{N} \sinh(\lambda_a - \lambda_b)} \cdot \det_{N}\left(\frac{\partial}{\partial\lambda_k}\mathcal{Y}_\kappa(\lambda_j|\{\lambda\})\right).$$

2.6. Action of operators A, B, C, D on a general state. An important step of the computation of the correlation function is to express the action of any product of local operators on any Bethe eigenvector. From the solution of the quantum inverse scattering problem, this is given by the successive action of A, B, C, D operators on a vector constructed by action of C operators on the reference vector. Action of A, B, C, D on such a vector are well known (see for example [5]). They can be written in the following form:

$$\langle 0|\prod\limits_{k=1}^{N} C(\lambda_k)\, A(\lambda_{N+1}) = \sum\limits_{a'=1}^{N+1} a(\lambda_{a'})\frac{\prod\limits_{k=1}^{N} \sinh(\lambda_k - \lambda_{a'} + \eta)}{\prod\limits_{\substack{k=1 \\ k\neq a'}}^{N+1} \sinh(\lambda_k - \lambda_{a'})}\langle 0|\prod\limits_{\substack{k=1 \\ k\neq a'}}^{N+1} C(\lambda_k); \quad (36)$$

$$\langle 0|\prod\limits_{k=1}^{N} C(\lambda_k)\, D(\lambda_{N+1}) = \sum\limits_{a=1}^{N+1} d(\lambda_a)\frac{\prod\limits_{k=1}^{N} \sinh(\lambda_a - \lambda_k + \eta)}{\prod\limits_{\substack{k=1 \\ k\neq a}}^{N+1} \sinh(\lambda_a - \lambda_k)}\langle 0|\prod\limits_{\substack{k=1 \\ k\neq a}}^{N+1} C(\lambda_k). \quad (37)$$

The action of the operator $B(\lambda)$ can be obtained similarly,

$$
\langle 0 | \prod_{k=1}^{N} C(\lambda_k) \, B(\lambda_{N+1}) = \sum_{a=1}^{N+1} d(\lambda_a) \frac{\prod\limits_{k=1}^{N} \sinh(\lambda_a - \lambda_k + \eta)}{\prod\limits_{\substack{k=1 \\ k \neq a}}^{N+1} \sinh(\lambda_a - \lambda_k)}
$$

$$
\times \sum_{\substack{a'=1 \\ a' \neq a}}^{N+1} \frac{a(\lambda_{a'})}{\sinh(\lambda_{N+1} - \lambda_{a'} + \eta)} \frac{\prod\limits_{\substack{j=1 \\ j \neq a}}^{N+1} \sinh(\lambda_j - \lambda_{a'} + \eta)}{\prod\limits_{\substack{j=1 \\ j \neq a,a'}}^{N+1} \sinh(\lambda_j - \lambda_{a'})} \langle 0 | \prod_{\substack{k=1 \\ k \neq a,a'}}^{N+1} C(\lambda_k),
$$

(38)

and the action of C is obvious.

3. Correlation functions: finite chain

To compute correlation functions of some product of local operators, the following successive problems have to be addressed: (i) determination of the ground state $\langle \psi_g |$, (ii) evaluation of the action of the product of the local operators on it, and (iii) computation of the scalar product of the resulting state with $| \psi_g \rangle$. Using the solution of the quantum inverse scattering problem together with the explicit determinant formulas for the scalar products and the norm of the Bethe state, one sees that matrix elements of local spin operators and correlation functions can be expressed as (multiple) sums of determinants [9]. It should be stressed that this result is purely algebraic and is valid for finite chains of arbitrary length M.

3.1. Matrix elements of local operators. We begin with the calculation of the one-point functions. These results follow directly from the solution of the quantum inverse scattering problem, the above action of operators A, B, C and D, and the determinant representation of the scalar products. We consider

$$
F_N^-(m, \{\mu_j\}, \{\lambda_k\}) = \langle 0 | \prod_{j=1}^{N+1} C(\mu_j) \, \sigma_m^- \prod_{k=1}^{N} B(\lambda_k) \, | 0 \rangle
$$

(39)

and

$$
F_N^+(m, \{\lambda_k\}, \{\mu_j\}) = \langle 0 | \prod_{k=1}^{N} C(\lambda_k) \, \sigma_m^+ \prod_{j=1}^{N+1} B(\mu_j) \, | 0 \rangle,
$$

(40)

where $\{\lambda_k\}_n$ and $\{\mu_j\}_{n+1}$ are solutions of Bethe equations.

Proposition 3.1. *For two Bethe states with spectral parameters $\{\lambda_k\}_N$ and $\{\mu_j\}_{N+1}$, the matrix element of the operator σ_m^- can be represented as a determinant,*

$$
F_N^-(m, \{\mu_j\}, \{\lambda_k\}) = \frac{\phi_{m-1}(\{\mu_j\})}{\phi_{m-1}(\{\lambda_k\})} \frac{\prod\limits_{j=1}^{N+1} \sinh(\mu_j - \xi_m + \eta)}{\prod\limits_{k=1}^{N} \sinh(\lambda_k - \xi_m + \eta)}
$$
$$
\cdot \frac{\det_{N+1} H^-(m, \{\mu_j\}, \{\lambda_k\})}{\prod\limits_{N+1 \geq k > j \geq 1} \sinh(\mu_k - \mu_j) \prod\limits_{1 \leq \beta < \alpha \leq N} \sinh(\lambda_\beta - \lambda_\alpha)}, \tag{41}
$$

$$
\phi_m(\{\lambda_k\}) = \prod_{k=1}^{N} \prod_{j=1}^{m} b^{-1}(\lambda_k, \xi_j), \tag{42}
$$

and the $(N+1) \times (N+1)$ matrix H^- is defined as

$$
H_{ab}^-(m) = \frac{\varphi(\eta)}{\varphi(\mu_a - \lambda_b)} \left(a(\lambda_b) \prod_{\substack{j=1 \\ j \neq a}}^{N+1} \varphi(\mu_j - \lambda_b + \eta) - d(\lambda_b) \prod_{\substack{j=1 \\ j \neq a}}^{N+1} \varphi(\mu_j - \lambda_b - \eta) \right) \tag{43}
$$

for $b < N+1$, and

$$
H_{aN+1}^-(m) = \frac{\varphi(\eta)}{\varphi(\mu_a - \xi_m + \eta)\varphi(\mu_a - \xi_m)}. \tag{44}
$$

For the matrix element $F_N^+(m, \{\lambda_k\}, \{\mu_j\})$ we get

$$
F_N^+(m, \{\lambda_k\}, \{\mu_j\}) = \frac{\phi_m(\lambda_k)\,\phi_{m-1}(\lambda_k)}{\phi_{m-1}(\mu_j)\,\phi_m(\mu_j)} F_N^-(m, \{\mu_j\}, \{\lambda_k\}). \tag{45}
$$

The matrix elements of the operator σ_m^z between two Bethe states have been obtained similarly [8].

3.2. Elementary blocks of correlation functions. In this section we consider a more general case of correlation functions: the ground state mean value of any product of the local elementary 2×2 matrices $E_{lk}^{\epsilon',\epsilon} = \delta_{l,\epsilon'}\delta_{k,\epsilon}$:

$$
F_m(\{\epsilon_j, \epsilon_j'\}) = \frac{\langle \psi_g | \prod\limits_{j=1}^{m} E_j^{\epsilon_j',\epsilon_j} | \psi_g \rangle}{\langle \psi_g | \psi_g \rangle}. \tag{46}
$$

An arbitrary n-point correlation function can be obtained as a sum of such mean values. Using the solution of the quantum inverse scattering problem, we reduce this

problem to the computation of the ground state mean value of an arbitrary ordered product of monodromy matrix elements,

$$F_m(\{\epsilon_j, \epsilon_j'\}) = \phi_m^{-1}(\{\lambda\}) \frac{\langle \psi_g | T_{\epsilon_1, \epsilon_1'}(\xi_1) \dots T_{\epsilon_m, \epsilon_m'}(\xi_m) | \psi_g \rangle}{\langle \psi_g | \psi_g \rangle}. \tag{47}$$

To calculate these mean values we first describe generically the product of the monodromy matrix elements. For that purpose, one should consider the two following sets of indices, $\alpha^+ = \{j : 1 \le j \le m, \epsilon_j = 1\}$, card$(\alpha^+) = s'$, max$_{j \in \alpha^+}(j) \equiv j'_{\max}$, min$_{j \in \alpha^+}(j) \equiv j'_{\min}$, and similarly $\alpha^- = \{j : 1 \le j \le m, \epsilon_j' = 2\}$, card$(\alpha^-) = s$, max$_{j \in \alpha^-}(j) \equiv j_{\max}$, min$_{j \in \alpha^-}(j) \equiv j_{\min}$. The intersection of these two sets is not empty and corresponds to the operators $B(\xi_j)$. Consider now the action, $\langle 0 | \prod_{k=1}^{N} C(\lambda_k) T_{\epsilon_1, \epsilon_1'}(\lambda_{N+1}) \dots T_{\epsilon_m, \epsilon_m'}(\lambda_{N+m})$, applying one by one the formulae (36)–(38). For all the indices j from the sets α^+ and α^- one obtains a summation on the corresponding indices a_j' (for $j \in \alpha^+$, corresponding to the action of the operators $A(\lambda)$ or $B(\lambda)$) or a_j (for $j \in \alpha^-$, corresponding to the action of the operators $D(\lambda)$ or $B(\lambda)$). As the product of the monodromy matrix elements is ordered these summations are also ordered and the corresponding indices should be taken from the following sets: $A_j = \{b : 1 \le b \le N + m, b \ne a_k, a_k', k < j\}$ and $A'_j = \{b : 1 \le b \le N + m, b \ne a_k', k < j, b \ne a_k, k \le j\}$. Thus,

$$\langle 0 | \prod_{k=1}^{N} C(\lambda_k) T_{\epsilon_1, \epsilon_1'}(\lambda_{N+1}) \dots T_{\epsilon_m, \epsilon_m'}(\lambda_{N+m})$$
$$= \sum_{\{a_j, a_j'\}} G_{\{a_j, a_j'\}}(\lambda_1, \dots, \lambda_{N+m}) \langle 0 | \prod_{b \in A_{m+1}} C(\lambda_b). \tag{48}$$

The summation is taken over the indices a_j for $j \in \alpha^-$ and a_j' for $j \in \alpha^+$ such that $1 \le a_j \le N + j$, $a_j \in A_j$, $1 \le a_j' \le N + j$, $a_j' \in A'_j$. The functions $G_{\{a_j, a_j'\}}(\lambda_1, \dots \lambda_{N+m})$ can then be easily obtained from the formulae (36)–(38) taking into acount that $\lambda_a = \xi_{a-N}$ for $a > N$:

$$G_{\{a_j, a_j'\}}(\lambda_1, \dots, \lambda_{N+m}) = \prod_{j \in \alpha^-} d(\lambda_{a_j}) \frac{\prod_{\substack{b=1 \\ b \in A_j}}^{N+j-1} \sinh(\lambda_{a_j} - \lambda_b + \eta)}{\prod_{\substack{b=1 \\ b \in A'_j}}^{N+j} \sinh(\lambda_{a_j} - \lambda_b)}$$

$$\times \prod_{j \in \alpha^+} a(\lambda_{a_j'}) \frac{\prod_{\substack{b=1 \\ b \in A'_j}}^{N+j-1} \sinh(\lambda_b - \lambda_{a_j'} + \eta)}{\prod_{\substack{b=1 \\ b \in A_{j+1}}}^{N+j} \sinh(\lambda_b - \lambda_{a_j'})}. \tag{49}$$

Now to calculate the normalized mean value (47) we apply the determinant representation for the scalar product. It should be mentioned that the number of operators $C(\lambda)$ has to be equal to the number of the operators $B(\lambda)$, as otherwise the mean value is zero, and hence the total number of elements in the sets $\boldsymbol{\alpha}^+$ and $\boldsymbol{\alpha}^-$ is $s + s' = m$. Taking into account that in (47), for $b > N$, $\lambda_b = \xi_{b-N}$ one has to consider the following scalar products:

$$
\frac{\langle 0 | \prod_{b \in A_{m+1}} C(\lambda_b) \prod_{k=1}^{N} B(\lambda_k) | 0 \rangle}{\langle 0 | \prod_{k=1}^{N} C(\lambda_k) \prod_{k=1}^{N} B(\lambda_k) | 0 \rangle},
$$

for all the permitted values of a_j, a'_j. Finally we obtain

$$
F_m(\{\epsilon_j, \epsilon'_j\}) = \frac{1}{\prod_{k<l} \sinh(\xi_k - \xi_l)} \sum_{\{a_j, a'_j\}} H_{\{a_j, a'_j\}}(\lambda_1, \ldots, \lambda_{N+m}), \qquad (50)
$$

the sum being taken on the same set of indices a_j, a'_j as in (48). The functions $H_{\{a_j, a'_j\}}(\{\lambda\})$ can be obtained using (49) and the determinant representations for the scalar products.

3.3. Two-point functions. The method presented in the last section is quite straightforward and gives formally the possibility to compute any correlation function. However, it has been developed for the computation of the expectation values of the monomials $T_{a_1 b_1}(\xi_1) \ldots T_{a_m b_m}(\xi_m)$, leading to the evaluation of elementary building blocks, whereas the study of the two-point functions involves big sums of such monomials. Indeed, let us consider for example the correlation function $\langle \sigma_1^z \sigma_{m+1}^z \rangle$. Then, according to the solution of the inverse scattering problem (31), we need to calculate the expectation value

$$
\langle \psi(\{\lambda\}) | (A - D)(\xi_1) \cdot \prod_{a=2}^{m} \mathcal{T}(\xi_a) \cdot (A - D)(\xi_{m+1}) \cdot \prod_{b=1}^{m+1} \mathcal{T}^{-1}(\xi_b) | \psi(\{\lambda\}) \rangle. \quad (51)
$$

Since $| \psi(\{\lambda\}) \rangle$ is an eigenvector, the action of $\prod_{b=1}^{m+1} \mathcal{T}^{-1}(\xi_b)$ on this state merely produces a numerical factor. However, it is much more complicated to evaluate the action of $\prod_{a=2}^{m} \mathcal{T}(\xi_a)$. Indeed, we have to act first with $(A - D)(\xi_1)$ on $\langle \psi(\{\lambda\}) |$ (or with $(A - D)(\xi_{m+1})$ on $| \psi(\{\lambda\}) \rangle$), which gives a sum of states which are no longer eigenvectors of the transfer matrix, and on which the multiple action of $\mathcal{T}$ is not simple. In fact, the product $\prod_{a=2}^{m} (A + D)(\xi_a)$ would lead to a sum of 2^{m-1} elementary blocks. This is not very convenient, in particular at large distance m. Therefore, to obtain manageable expressions for such correlation functions, it is of great importance to develop an alternative and compact way to express the multiple

action of the transfer matrix on arbitrary states or, in other words, to make an effective re-summation of the corresponding sum of the 2^{m-1} terms. This can be achieved in the following way:

Proposition 3.2. *Let κ, $x_1, \ldots, x_m$ and $\mu_1, \ldots, \mu_N$ be generic parameters. Then the action of $\prod_{a=1}^{m} \mathcal{T}_\kappa(x_a)$ on a state of the form $\langle 0 | \prod_{j=1}^{N} C(\mu_j)$ can be formally written as*

$$\langle 0 | \prod_{j=1}^{N} C(\mu_j) \prod_{a=1}^{m} \mathcal{T}_\kappa(x_a) = \frac{1}{N!} \oint_{\Gamma\{x\}\cup\Gamma\{\mu\}} \prod_{j=1}^{N} \frac{dz_j}{2\pi i} \cdot \prod_{a=1}^{m} \tau_\kappa(x_a|\{z\}) \cdot \prod_{a=1}^{N} \frac{1}{\mathcal{Y}_\kappa(z_a|\{z\})}$$

$$\times \prod_{\substack{j,k=1 \\ j<k}}^{N} \frac{\sinh(z_j - z_k)}{\sinh(\mu_j - \mu_k)} \cdot \det_N \Omega_\kappa(\{z\}, \{\mu\}|\{z\}) \cdot \langle 0 | \prod_{j=1}^{N} C(z_j), \quad (52)$$

where the integration contour $\Gamma\{x\} \cup \Gamma\{\mu\}$ surrounds the points[1] $x_1, \ldots, x_m$ and $\mu_1, \ldots, \mu_N$ and does not contain any other pole of the integrand.

One of the simplest applications concerns the generating function of the two-point correlation function of the third components of spin, which is defined as the normalized expectation value $\langle Q_{l,m}^\kappa \rangle$ of the operator

$$Q_{l,m}^\kappa = \prod_{n=l}^{m} \left(\frac{1+\kappa}{2} + \frac{1-\kappa}{2} \cdot \sigma_n^z \right) = \prod_{j=1}^{l-1} \mathcal{T}(\xi_j) \cdot \prod_{j=l}^{m} \mathcal{T}_\kappa(\xi_j) \cdot \prod_{j=1}^{m} \mathcal{T}^{-1}(\xi_j), \quad (53)$$

where $| \psi(\{\lambda\}) \rangle$ is an eigenvector of $\mathcal{T}(\mu)$ in the subspace $\mathcal{H}^{(M/2-N)}$. The two-point correlation function of the third components of local spins in the eigenvector $| \psi(\{\lambda\}) \rangle$ can be obtained in terms of the second 'lattice derivative' and the second derivative with respect to κ of the generating function $\langle Q_{l,m}^\kappa \rangle$ at $\kappa = 1$:

$$\langle \sigma_l^z \sigma_{l+m}^z \rangle = \langle \sigma_l^z \rangle + \langle \sigma_{l+m}^z \rangle - 1$$
$$+ 2 \frac{\partial^2}{\partial \kappa^2} \langle Q_{l,l+m}^\kappa - Q_{l,l+m-1}^\kappa - Q_{l+1,l+m}^\kappa + Q_{l+1,l+m-1}^\kappa \rangle \big|_{\kappa=1}. \quad (54)$$

Due to the translational invariance of the correlation functions in the homogeneous model, we will simply consider the expectation value $\langle Q_{1,m}^\kappa \rangle$. For any given eigenvector, we obtain the following result:

Theorem 3.1. *Let $\{\lambda\}$ be an admissible off-diagonal solution of the system of untwisted Bethe equations, and let us consider the corresponding expectation value $\langle Q_{1,m}^\kappa \rangle$ in*

---

[1]More precisely, for a set of complex variables $\{v_1, \ldots, v_l\}$, the notation $\Gamma\{v\}$ should be understood in the following way: $\Gamma\{v\}$ is the boundary of a set of poly-disks $\mathcal{D}_a(r)$ in $\mathbb{C}^N$, i.e. $\Gamma\{v\} = \cup_{a=1}^{l} \bar{\mathcal{D}}_a(r)$ with $\bar{\mathcal{D}}_a(r) = \{z \in \mathbb{C}^N : |z_k - v_a| = r, \quad k = 1, \ldots, N\}$.

the inhomogeneous finite XXZ chain. Then there exists $\kappa_0 > 0$ such that, for $|\kappa| < \kappa_0$, the following representations hold:

$$\langle\, Q_{1,m}^{\kappa}\,\rangle = \frac{1}{N!} \oint_{\Gamma\{\xi\}\cup\Gamma\{\lambda\}} \prod_{j=1}^{N} \frac{dz_j}{2\pi i} \cdot \prod_{a=1}^{m} \frac{\tau_\kappa(\xi_a|\{z\})}{\tau(\xi_a|\{\lambda\})} \cdot \prod_{a=1}^{N} \frac{1}{\mathcal{Y}_\kappa(z_a|\{z\})}$$

$$\times \det_N \Omega_\kappa(\{z\},\{\lambda\}|\{z\}) \cdot \frac{\det_N \Omega(\{\lambda\},\{z\}|\{\lambda\})}{\det_N \Omega(\{\lambda\},\{\lambda\}|\{\lambda\})}. \qquad (55)$$

The integration contours are such that the only singularities of the integrand which contribute to the integral are the points $\xi_1, \ldots, \xi_m$ and $\lambda_1 \ldots, \lambda_N$.

From this result we can extract a compact representation for the two-point function of σ^z [15]. Similar expressions exists for other correlation functions of the spin operators, and in particular for the time dependent case [15], [16]. Moreover, this multiple contour integral representation permits to relate two very different ways to compute two point correlation functions of the type, $g_{12} = \langle\omega|\theta_1\theta_2|\omega\rangle$, namely,

(i) to compute the action of local operators on the ground state $\theta_1\theta_2|\omega\rangle = |\tilde\omega\rangle$ and then to calculate the resulting scalar product $g_{12} = \langle\omega|\tilde\omega\rangle$ as was explained in the previous sections;

(ii) to insert a sum over a complete set of states $|\omega_i\rangle$ (for instance, a complete set of eigenvectors of the Hamiltonian) between the local operators θ_1 and θ_2 and to obtain the representation for the correlation function as a sum over matrix elements of local operators,

$$g_{12} = \sum_i \langle\omega|\theta_1|\omega_i\rangle \cdot \langle\omega_i|\theta_2|\omega\rangle. \qquad (56)$$

In fact the above representation as multiple contour integrals contains both expansions. Indeed there are two ways to evaluate the corresponding integrals: either to compute the residues in the poles inside Γ, or to compute the residues in the poles within strips of the width $i\pi$ outside Γ.

The first way leads to a representation of the correlation function $\langle\sigma_1^z\sigma_{m+1}^z\rangle$ in terms of the previously obtained [11] m-multiple sums. Evaluation of the above contour integral in terms of the poles outside the contour Γ gives us the expansion (ii) of the correlation function (i.e. an expansion in terms of matrix elements of σ^z between the ground state and all excited states). This relation holds also for the time dependent case [15], [16].

4. Correlation functions: infinite chain

In the thermodynamic limit $M \to \infty$ and at zero magnetic field, the model exhibits three different regimes depending on the value of Δ [1]. For $\Delta < -1$, the model is ferromagnetic, for $-1 < \Delta < 1$ the model has a non degenerated anti ferromagnetic

ground state, and no gap in the spectrum (massless regime), while for $\Delta > 1$ the ground state is twice degenerated with a gap in the spectrum (massive regime). In both cases, the ground state has spin zero. Hence the number of parameters λ in the ground state vectors is equal to half the size M of the chain. For $M \to \infty$, these parameters will be distributed in some continuous interval according to a density function ρ.

4.1. The thermodynamic limit. In this limit, the Bethe equations for the ground state, written in their logarithmic form, become a linear integral equation for the density distribution of these λ's,

$$\rho_{\text{tot}}(\alpha) + \int_{-\Lambda}^{\Lambda} K(\alpha - \beta) \rho_{\text{tot}}(\beta) \, d\beta = \frac{p'_{0_{\text{tot}}}(\alpha)}{2\pi}, \qquad (57)$$

where the new real variables α are defined in terms of general spectral parameters λ differently in the two domains. From now on, we only describe the massless regime (see [9] for the other case) $-1 < \Delta < 1$ where $\alpha = \lambda$. The density ρ is defined as the limit of the quantity $\frac{1}{M(\lambda_{j+1} - \lambda_j)}$, and the functions $K(\lambda)$ and $p'_{0_{\text{tot}}}(\lambda)$ are the derivatives with respect to λ of the functions $-\frac{\theta(\lambda)}{2\pi}$ and $p_{0_{\text{tot}}}(\lambda)$:

$$
\begin{aligned}
K(\alpha) &= \frac{\sin 2\zeta}{2\pi \, \sinh(\alpha + i\zeta) \sinh(\alpha - i\zeta)} \\
p'_0(\alpha) &= \frac{\sin \zeta}{\sinh\left(\alpha + i\frac{\zeta}{2}\right) \sinh\left(\alpha - i\frac{\zeta}{2}\right)}
\end{aligned}
\qquad \text{for } -1 < \Delta < 1, \ \zeta = i\eta, \qquad (58)
$$

with $p'_{0_{\text{tot}}}(\alpha) = \frac{1}{M} \sum_{i=1}^{M} p'_0\left(\alpha - \beta_k - i\frac{\zeta}{2}\right)$, where $\beta_k = \xi_k$. The integration limit Λ is equal to $+\infty$ for $-1 < \Delta < 1$. The solution for the equation (57) in the homogeneous model where all parameters ξ_k are equal to $\eta/2$, that is the density for the ground state of the Hamiltonian in the thermodynamic limit, is given by the following function [24]:

$$\rho(\alpha) = \frac{1}{2\zeta \cosh(\frac{\pi\alpha}{\zeta})}.$$

For technical convenience, we will also use the function

$$\rho_{\text{tot}}(\alpha) = \frac{1}{M} \sum_{i=1}^{M} \rho\left(\alpha - \beta_k - i\frac{\zeta}{2}\right).$$

It will be also convenient to consider, without any loss of generality, that the inhomogeneity parameters are contained in the region $-\zeta < \text{Im}\beta_j < 0$. Using these results, for any $\mathcal{C}^\infty$ function f (π-periodic in the domain $\Delta > 1$), sums over all the values of f at the point α_j, $1 \leq j \leq N$, parameterizing the ground state, can be replaced in the thermodynamic limit by an integral:

$$\frac{1}{M} \sum_{j=1}^{N} f(\alpha_j) = \int_{-\Lambda}^{\Lambda} f(\alpha) \rho_{\text{tot}}(\alpha) \, d\alpha + O(M^{-1}).$$

Thus, multiple sums obtained in correlation functions will become multiple integrals. Similarly, it is possible to evaluate the behavior of the determinant formulas for the scalar products and the norm of Bethe vectors (and in particular their ratios) in the limit $M \to \infty$.

4.2. Elementary blocks. From the representations as multiple sums of these elementary blocks in the finite chain we can obtain their multiple integral representations in the thermodynamic limit. Let us now consider separately the two regimes of the XXZ model. In the massless regime $\eta = -i\zeta$ is imaginary, the ground state parameters λ are real and the limit of integration is infinity $\Lambda = \infty$. In this case we consider the inhomogeneity parameters ξ_j such that $0 > \mathrm{Im}(\xi_j) > -\zeta$. For the correlation functions in the thermodynamic limit one obtains the following result in this regime.

Proposition 4.1.

$$
F_m(\{\epsilon_j, \epsilon_j'\}) = \prod_{k<l} \frac{\sinh \frac{\pi}{\zeta}(\xi_k - \xi_l)}{\sinh(\xi_k - \xi_l)} \prod_{j=1}^{s'} \int_{-\infty-i\zeta}^{\infty-i\zeta} \frac{d\lambda_j}{2i\zeta} \prod_{j=s'+1}^{m} \int_{-\infty}^{\infty} i\frac{d\lambda_j}{2\zeta}
$$

$$
\prod_{a=1}^{m} \prod_{k=1}^{m} \frac{1}{\sinh \frac{\pi}{\zeta}(\lambda_a - \xi_k)} \prod_{j\in\alpha^-} \left(\prod_{k=1}^{j-1} \sinh(\mu_j - \xi_k - i\zeta) \prod_{k=j+1}^{m} \sinh(\mu_j - \xi_k) \right)
$$

$$
\prod_{j\in\alpha^+} \left(\prod_{k=1}^{j-1} \sinh(\mu_j' - \xi_k + i\zeta) \prod_{k=j+1}^{m} \sinh(\mu_j' - \xi_k) \right) \prod_{a>b} \frac{\sinh \frac{\pi}{\zeta}(\lambda_a - \lambda_b)}{\sinh(\lambda_a - \lambda_b - i\zeta)},
$$

where the parameters of integration are ordered in the following way: $\{\lambda_1, \dots \lambda_m\} = \{\mu_{j'_{\max}}', \dots, \mu_{j'_{\min}}', \mu_{j_{\min}}, \dots, \mu_{j_{\max}}\}$.

The homogeneous limit ($\xi_j = -i\zeta/2$, for all j) of the correlation function $F_m(\{\epsilon_j, \epsilon_j'\})$ can then be taken in an obvious way. We have obtained similar representations for the massive regime, and also in the presence of a non-zero magnetic field [9]. For zero magnetic field, these results agree exactly with the ones obtained by Jimbo and Miwa in [35], using in particular q-KZ equations. It means that for zero magnetic field, the elementary blocks of correlation functions indeed satisfy q-KZ equations. Recently, more algebraic representations of solutions of the q-KZ equations have been obtained that correspond to the above correlation functions [37], [38]. From the finite chain representation for the two-point function it is also possible to obtain multiple integral representations for that case as well, in particular for their generating function [11], [13]. They correspond different huge re-summations and symmetrization of the corresponding elementary blocks, as in the finite chain situation [11]. Moreover, the case of time dependent correlation functions as also been obtained [15], [16]. Finally, let us note that at the free fermion point, all the results presented here lead, in a very elementary way, to already know results [12], [17], [19].

5. Exact and asymptotic results

5.1. Exact results at $\Delta = 1/2$. Up to now, two exact results have been obtained for the case of anisotropy $\Delta = 1/2$: the exact value of the emptiness formation probability for arbitrary distance m [13] and the two point function of the third component of spin [18]. These two results follow from the above multiple integral representations for which, due to the determinant structure of the integrand, the corresponding multiple integrals can be separated and hence explicitly computed for this special value of the anisotropy.

5.1.1. The emptiness formation probability. This correlation function $\tau(m)$ (the probability to find in the ground state a ferromagnetic string of length m) is defined as the following expectation value:

$$\tau(m) = \langle \psi_g | \prod_{k=1}^{m} \frac{1 - \sigma_k^z}{2} | \psi_g \rangle, \tag{59}$$

where $|\psi_g\rangle$ denotes the normalized ground state. In the thermodynamic limit $(M \to \infty)$, this quantity can be expressed as a multiple integral with m integrations [34], [35], [6], [8], [9].

Proposition 5.1. *For $\Delta = \cos \zeta$, $0 < \zeta < \pi$, $\tau(m) = \lim_{\xi_1, \dots \xi_m \to -\frac{i\zeta}{2}} \tau(m, \{\xi_j\})$, where*

$$\tau(m, \{\xi_j\}) = \frac{1}{m!} \int_{-\infty}^{\infty} \frac{Z_m(\{\lambda\}, \{\xi\})}{\prod\limits_{a<b}^{m} \sinh(\xi_a - \xi_b)} \det_m \left(\frac{i}{2\zeta \sinh \frac{\pi}{\zeta}(\lambda_j - \xi_k)} \right) d^m \lambda, \tag{60}$$

$$Z_m(\{\lambda\}, \{\xi\}) = \prod_{a=1}^{m} \prod_{b=1}^{m} \frac{\sinh(\lambda_a - \xi_b) \sinh(\lambda_a - \xi_b - i\zeta)}{\sinh(\lambda_a - \lambda_b - i\zeta)}$$

$$\cdot \frac{\det_m \left(\frac{-i \sin \zeta}{\sinh(\lambda_j - \xi_k) \sinh(\lambda_j - \xi_k - i\zeta)} \right)}{\prod\limits_{a>b}^{m} \sinh(\xi_a - \xi_b)}. \tag{61}$$

The proof is given in [11]. Due to the determinant structure of the integrand, the integrals can be separated and computed for the special case $\Delta = \frac{1}{2}$ ($\zeta = \pi/3$):

Proposition 5.2. *Let $\xi_k = \varepsilon_k - i\pi/6$ and $\varepsilon_{ab} = \varepsilon_a - \varepsilon_b$. Then we obtain*

$$\tau(m, \{\varepsilon_j\}) = \frac{(-1)^{\frac{m^2-m}{2}}}{2^{m^2}} \prod_{a>b}^{m} \frac{\sinh 3\varepsilon_{ba}}{\sinh \varepsilon_{ba}} \prod_{\substack{a,b=1 \\ a \neq b}}^{m} \frac{1}{\sinh \varepsilon_{ab}} \cdot \det_m \left(\frac{3 \sinh \frac{\varepsilon_{jk}}{2}}{\sinh \frac{3\varepsilon_{jk}}{2}} \right) \tag{62}$$

and

$$\tau(m) = \left(\frac{1}{2}\right)^{m^2} \prod_{k=0}^{m-1} \frac{(3k+1)!}{(m+k)!}. \tag{63}$$

Observe that the quantity $A_m = \prod_{k=0}^{m-1}(3k+1)!/(m+k)!$ is the number of alternating sign matrices of size m. This result was conjectured in [45].

5.1.2. The two point function of σ^z. The two point functions can be obtained, as in the finite chain situation, from a generating function $\langle Q_\kappa(m)\rangle$; in the thermodynamic limit, we use the following multiple integral representation [18]:

$$
\begin{aligned}
\langle Q_\kappa(m)\rangle = \sum_{n=0}^{m} \frac{\kappa^{m-n}}{n!(m-n)!} \oint_{\Gamma\{-i\zeta/2\}} \frac{d^m z}{(2\pi i)^m} \int_{\mathbb{R}-i\zeta} d^n\lambda \int_{\mathbb{R}} d^{m-n}\lambda \cdot \prod_{j=1}^{m} \frac{\varphi^m(z_j)}{\varphi^m(\lambda_j)} \\
\times \prod_{j=1}^{n} \left\{ t(z_j,\lambda_j) \prod_{k=1}^{m} \frac{\sinh(z_j-\lambda_k-i\zeta)}{\sinh(z_j-z_k-i\zeta)} \right\} \\
\times \prod_{j=n+1}^{m} \left\{ t(\lambda_j,z_j) \prod_{k=1}^{m} \frac{\sinh(\lambda_k-z_j-i\zeta)}{\sinh(z_k-z_j-i\zeta)} \right\} \\
\times \prod_{j=1}^{m}\prod_{k=1}^{m} \frac{\sinh(\lambda_k-z_j-i\zeta)}{\sinh(\lambda_k-\lambda_j-i\zeta)} \cdot \det_m \left(\frac{i}{2\zeta \sinh\frac{\pi}{\zeta}(\lambda-z)} \right).
\end{aligned}
\tag{64}
$$

Here,

$$\Delta = \cos\zeta, \quad t(z,\lambda) = \frac{-i\sin\zeta}{\sinh(z-\lambda)\sinh(z-\lambda-i\zeta)}, \quad \varphi(z) = \frac{\sinh(z-i\frac{\zeta}{2})}{\sinh(z+i\frac{\zeta}{2})}, \tag{65}$$

and the integrals over the variables z_j are taken with respect to a closed contour Γ which surrounds the point $-i\zeta/2$ and does not contain any other singularities of the integrand. The equation (64) is valid for the homogeneous XXZ chain with arbitrary $-1 < \Delta < 1$. If we consider the inhomogeneous XXZ model with inhomogeneities $\xi_1, \ldots, \xi_m$, then one should replace in the representation (64) the function φ^m in the following way:

$$\varphi^m(z) \to \prod_{b=1}^{m} \frac{\sinh(z-\xi_b-i\zeta)}{\sinh(z-\xi_b)}, \quad \varphi^{-m}(\lambda) \to \prod_{b=1}^{m} \frac{\sinh(\lambda-\xi_b)}{\sinh(\lambda-\xi_b-i\zeta)}. \tag{66}$$

In order to come back to the homogeneous case, one should set $\xi_k = -i\zeta/2$, $k = 1, \ldots, m$ in (66). In the inhomogeneous model, the integration contour Γ surrounds the points $\xi_1, \ldots, \xi_m$, and the integrals over z_j are therefore equal to the sum of the residues of the integrand in these simple poles. It turns out that again for the special case $\Delta = \frac{1}{2}$ integrals can be separated and computed to give:

Proposition 5.3.

$$\langle Q_\kappa(m) \rangle = \frac{3^m}{2^{m^2}} \prod_{a>b}^{m} \frac{\sinh 3(\xi_a - \xi_b)}{\sinh^3(\xi_a - \xi_b)} \sum_{n=0}^{m} \kappa^{m-n} \sum_{\substack{\{\xi\}=\{\xi_{\gamma_+}\}\cup\{\xi_{\gamma_-}\} \\ |\gamma_+|=n}} \det_m \hat\Phi^{(n)}$$

$$\times \prod_{a\in\gamma_+} \prod_{b\in\gamma_-} \frac{\sinh(\xi_b - \xi_a - \frac{i\pi}{3})\sinh(\xi_a - \xi_b)}{\sinh^2(\xi_b - \xi_a + \frac{i\pi}{3})},$$

$$\hat\Phi^{(n)}(\{\xi_{\gamma_+}\}, \{\xi_{\gamma_-}\}) = \left(\begin{array}{c|c} \Phi(\xi_j - \xi_k) & \Phi(\xi_j - \xi_k - \frac{i\pi}{3}) \\ \hline \\ \Phi(\xi_j - \xi_k + \frac{i\pi}{3}) & \Phi(\xi_j - \xi_k) \end{array} \right),$$

$$\Phi(x) = \frac{\sinh \frac{x}{2}}{\sinh \frac{3x}{2}}.$$

Here the sum is taken with respect to all partitions of the set $\{\xi\}$ into two disjoint subsets $\{\xi_{\gamma_+}\} \cup \{\xi_{\gamma_-}\}$ of cardinality n and $m - n$ respectively. The first n lines and columns of the matrix $\hat\Phi^{(n)}$ are associated with the parameters $\xi \in \{\xi_{\gamma_+}\}$. The remaining lines and columns are associated with $\xi \in \{\xi_{\gamma_-}\}$.

Thus, we have obtained an explicit answer for the generating function $\langle Q_\kappa(m) \rangle$ of the inhomogeneous *XXZ* model. It is also possible to check that the above sum over partitions remains indeed finite in the homogeneous limit $\xi_k \to 0$.

5.2. Asymptotic results. An important issue is the analysis of the multiple integral representations of correlation functions for large distances. There it means analyzing asymptotic behavior of m-fold integrals for m large. An interesting example to study in this respect is provided by the emptiness formation probability. This correlation function reduces to a single elementary block. Moreover, we already described its exact value for an anisotropy $\Delta = \frac{1}{2}$ in the previous section. In fact, it is possible to obtain the asymptotic behavior of $\tau(m)$ using the saddle-point method for arbitrary values of the anisotropy $\Delta > -1$. This was performed for the first time in [12] in the case of free fermions ($\Delta = 0$), but it can be applied to the general case as well. We present here the results in the massless and massive regimes [14], [19].

To apply the saddle-point method to the emptiness formation probability, it is convenient to express its integral representation in the following form:

$$\tau(m) = \int_{\mathcal{D}} d^m\lambda \, G_m(\{\lambda\}) \, e^{m^2 S_m(\{\lambda\})}, \tag{67}$$

with

$$S_m(\{\lambda\}) = -\frac{1}{m^2} \sum_{a>b}^{m} \log[\sinh(\lambda_a - \lambda_b + \eta) \, \sinh(\lambda_a - \lambda_b - \eta)]$$

$$+ \frac{1}{m} \sum_{a=1}^{m} \log[\sinh(\lambda_a + \eta/2) \, \sinh(\lambda_a - \eta/2)] \qquad (68)$$

$$+ \frac{1}{m^2} \lim_{\xi_1 \ldots \xi_m \to \eta/2} \log\left[\left(\frac{-2i\pi}{\sinh \eta}\right)^m \frac{\left(\det \rho(\lambda_j, \xi_k)\right)^2}{\prod_{a \neq b} \sinh(\xi_a - \xi_b)}\right]$$

and

$$G_m(\{\lambda\}) = \lim_{\xi_1 \ldots \xi_m \to \eta/2} \frac{\det_m \left[\frac{i}{2\pi} t(\lambda_j, \xi_k)\right]}{\det_m \rho(\lambda_j, \xi_k)}. \qquad (69)$$

In (67), the integration domain $\mathcal{D}$ is such that the variables of integration $\lambda_1, \ldots, \lambda_m$ are ordered in the interval $\mathcal{C} = [-\Lambda_h, \Lambda_h]$ (i.e. $-\Lambda_h < \lambda_1 < \cdots < \lambda_m < \Lambda_h$ in the massless case, and $-i\Lambda_h < i\lambda_1 < \cdots < i\lambda_m < i\Lambda_h$ in the massive case).

The main problem in the saddle point analysis is that, a priori, we do not know any asymptotic equivalent of the quantity $G_m(\lambda)$ when $m \to \infty$. Nevertheless, in the case of zero magnetic field, it is still possible to compute the asymptotic behavior of (67) in the leading order, provided we make the following hypothesis: we assume that the integrand of (67) admits a maximum for a certain value $\lambda'_1, \ldots, \lambda'_m$ of the integration variables $\lambda_1, \ldots, \lambda_m$ such that, for large m, the distribution of these parameters $\lambda'_1, \ldots, \lambda'_m$ can be described by a density function $\rho_s(\lambda')$ of the form

$$\rho_s(\lambda'_j) = \lim_{m \to \infty} \frac{1}{m(\lambda'_{j+1} - \lambda'_j)} \qquad (70)$$

on the symmetric interval $[-\Lambda, \Lambda]$, and that, at the leading order in m, we can replace the sums over the set of parameters $\{\lambda'\}$ by integrals weighted with the density $\rho_s(\lambda')$.

First, it is easy to determine the maximum of the function $S_m(\{\lambda\})$. Indeed, let $\{\tilde{\lambda}\}$ be solution of the system

$$\partial_{\lambda_j} S_m(\{\tilde{\lambda}\}) = 0, \quad 1 \leq j \leq m. \qquad (71)$$

In the limit $m \to \infty$, if we suppose again that the parameters $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_m$ become distributed according to a certain density $\tilde{\rho}_s(\lambda)$ and that sums over the $\tilde{\lambda}_j$ become integrals over this density, the system (71) turns again into a single integral equation for $\tilde{\rho}_s$, that can be solved explicitly in the case of zero magnetic field. It gives the maximum of $S_m(\{\lambda\})$ when $m \to \infty^2$.

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[2]At this main order in m, there exists a unique solution of the integral equation for $\tilde{\rho}_s$, and we know it corresponds to a maximum because $S_m(\{\lambda\}) \to -\infty$ on the boundary of $\mathcal{D}$.

The second step is to show that the factor $G_m(\{\lambda\})$ gives always a negligible contribution compared to $S_m(\{\tilde{\lambda}\})$ at this order in m, at least for any distribution of the variables λ_j satisfying the previous hypothesis of regularity. We obtain

$$\lim_{m \to \infty} \frac{1}{m^2} \log G_m(\{\lambda\}) = 0 \tag{72}$$

for any distribution of $\{\lambda\}$ with good properties of regularity, in particular for the saddle point. This means that, at the main order in m, the factor $G_m(\{\lambda\})$ does not contribute to the value of the maximum of the integrand.

Finally we obtain the following result concerning the asymptotic behaviour of $\tau(m)$ for $m \to \infty$ (see [14], [19]):

$$S^{(0)}(\Delta) = \lim_{m \to \infty} \frac{\log \tau(m)}{m^2}, \tag{73}$$

$$= -\frac{\zeta}{2} - \sum_{n=1}^{\infty} \frac{e^{-n\zeta}}{n} \frac{\sinh(n\zeta)}{\cosh(2n\zeta)}, \qquad (\Delta = \cosh \zeta > 1), \tag{74}$$

$$= \log \frac{\pi}{\zeta} + \frac{1}{2} \int_{\mathbb{R}-i0} \frac{d\omega}{\omega} \frac{\sinh \frac{\omega}{2}(\pi - \zeta) \cosh^2 \frac{\omega \zeta}{2}}{\sinh \frac{\pi \omega}{2} \sinh \frac{\omega \zeta}{2} \cosh \omega \zeta}, \quad (-1 < \Delta = \cos \zeta < 1).$$
$$\tag{75}$$

It coincides with the exact known results obtained in [46], [12] at the free fermion point and in [45], [13] at $\Delta = 1/2$, and is in agreement with the expected (infinite) value in the Ising limit. Similar techniques can be applied to the two point function. However, the result that has been extracted so far is only the absence of the gaussian term. Unfortunately, we do not know up to now how to extract the expected power law corrections to the gaussian behavior from this saddle point analysis. More powerful methods will certainly be needed to go further.

Conclusion and perspectives

In this article, we have reviewed recent results concerning the computation of correlation functions in the *X X Z* chain by the methods of the inverse scattering problem and the algebraic Bethe ansatz. In conclusion, we would like to discuss some perspectives and problems to be solved.

One of the most interesting open problems is to prove the conformal field theory predictions [47], [48], [49], [50] concerning the asymptotic behavior of the correlation functions. This is certainly a very important issue not only for physical applications but also from a theoretical view point. Moreover, it also would open the route towards the generalization of the methods presented here to quantum integrable models of field theory. We have seen that in particular cases, the multiple integral representations

enable for a preliminary asymptotic analysis. Nevertheless, this problem remains one of the main challenges in the topics that have been described in this article.

A possible way to solve this problem would be to find the thermodynamic limit of the master equations (like the one obtained for the two point correlation functions). It is natural to expect that, in this limit, one should obtain a representation for these correlation functions in terms of a functional integral, which could eventually be estimated for large time and distance.

Note that the master equation shows a direct analytic relation between the multiple integral representations and the form factor expansions for the correlation functions. It seems likely that similar representations exist for other models solvable by algebraic Bethe ansatz. It would be in particular very interesting to obtain an analogue of this master equation in the case of the field theory models, which could provide an analytic link between short distance and long distance expansions of their correlation functions.

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