Gromov–Witten invariants and topological strings: a progress report

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Abstract. In this talk I summarize recent progress in the theory of Gromov–Witten invariants from topological string theory and string dualities. On the one hand, large N dualities have led to the theory of the topological vertex, which solves Gromov–Witten theory to all genera on toric, noncompact Calabi–Yau threefolds. On the other hand, heterotic/type II duality and the holomorphic anomaly equations can be used to analyze Gromov–Witten theory in some simple compact examples. I sketch the physical ideas behind these results and connect the results obtained in physics with the results obtained in algebraic geometry.

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1. Introduction

The theory of Gromov–Witten invariants was largely motivated by the study of string theory on Calabi–Yau manifolds, and has now developed into one of the most dynamic fields of algebraic geometry. During the last years there has been enormous progress in the development of the theory and of its computational techniques. Roughly speaking, and restricting ourselves to Calabi–Yau threefolds, we have the following mathematical approaches to the computation of Gromov–Witten invariants:

1. *Localization*. This was first proposed by Kontsevich, and requires torus actions in the Calabi–Yau in order to work. Localization provides *a priori* a complete solution of the theory on toric (hence non-compact) Calabi–Yau manifolds, and reduces the computation of Gromov–Witten invariants to the calculation of Hodge integrals in Deligne–Mumford moduli space. Localization techniques make also possible to solve the theory at genus zero on a wide class of compact manifolds, see [8], [14] for a review.

2. *Deformation and topological approach*. This has been developed more recently and relies on *relative* Gromov–Witten invariants [11], [24]. It provides a cut-and-paste approach to the calculation of the invariants and seems to be the most

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powerful approach to higher genus Gromov–Witten invariants in the compact case.

3. *D-brane moduli spaces*. Gromov–Witten invariants can be reformulated in terms of the so-called Gopakumar–Vafa invariants (see [14], [21] for a summary of these). Heuristic techniques to compute these have been developed in [16], as Euler characteristics of moduli space of embedded surfaces, and one can recover to a large extent the original information of Gromov–Witten theory. The equivalence between these two invariants remains however conjectural, and a general, rigorous definition of the Gopakumar–Vafa invariants in terms of appropriate moduli spaces is still not known.

4. *Equivalence to Donaldson–Thomas invariants*. In [23] it was proposed that Gromov–Witten invariants are equivalent to Donaldson–Thomas invariants, which are associated to moduli spaces of sheaves. This equivalence remains largely conjectural and so far it has led to little computational progress, although it is currently an area of active research.

Gromov–Witten invariants are closely related to string theory. It turns out that type IIA theory on a Calabi–Yau manifold X leads to a four-dimensional supersymmetric theory whose Lagrangian contains moduli-dependent couplings $F_g(t)$, where t denotes the Kähler moduli of the Calabi–Yau. When these couplings are expanded in the large radius limit, they are of the form

$$F_g(t) = \sum_{Q \in H_2(X)} N_{g,Q}\, \mathrm{e}^{-Q\cdot t}, \tag{1}$$

where $N_{g,Q}$ are the Gromov–Witten invariants for the class Q at genus g. It turns out that there is a simplified version of string theory, called topological string theory, which captures precisely the information contained in these couplings. Topological string theory comes in two versions, called the A and the B model (see [21], [14] for a review). Type A topological string theory is related to Gromov–Witten theory, and its free energy at genus g is precisely given by (1). Type B topological string theory is related to the deformation theory of complex structures of the Calabi–Yau manifold. In the last years, various dualities of string theory have led to powerful techniques to compute these couplings, hence Gromov–Witten invariants:

1. *Mirror symmetry*. Mirror symmetry relates type A theory on a Calabi–Yau manifold X to type B theory on the mirror manifold $\widetilde{X}$. When the mirror of the Calabi–Yau X is known, this leads to a complete solution at genus zero in terms of variation of the complex structures of $\widetilde{X}$. For genus $g \geq 1$, mirror symmetry can be combined with the holomorphic anomaly equations of [5] to obtain $F_g(t)$. However, this does not provide the full solution to the model due to the so-called holomorphic ambiguity. On the other hand, mirror symmetry and the holomorphic anomaly equation are very general and work for both compact and non-compact Calabi–Yau manifolds.

2. *Large N dualities*. Large N dualities lead to a computation of the $F_g(t)$ couplings in terms of correlation functions and partition functions in Chern–Simons theory. Although this was formulated originally only for the resolved conifold, one ends up with a general theory – the theory of the topological vertex, introduced in [2] – which leads to a complete solution on toric Calabi–Yau manifolds. The theory of the topological vertex is closely related to localization and to Hodge integrals, and it can be formulated in a rigorous mathematical way [19], [23].

3. *Heterotic duality*. When the Calabi–Yau manifold has the structure of a K3 fibration, type IIA theory often has a heterotic dual, and the evaluation of $F_g(t)$ restricted to the K3 fiber can be reduced to a one-loop integral in heterotic string theory [4], [22]. This leads to explicit, conjectural formulae for Gromov–Witten invariants in terms of modular forms.

In this progress report I will concentrate on two results: (1) I will summarize how large N dualities lead to a complete solution of topological string theory on toric Calabi–Yau manifolds. (2) I will discuss what is probably the simplest, non-trivial compact Calabi–Yau manifold, the so-called Enriques Calabi–Yau manifold, which is very tractable both mathematically and physically, and might be the natural starting point to understand the compact case.

2. The toric case

In this section we give a rather brief summary of the results obtained in the context of topological string theory to compute Gromov–Witten invariants of toric geometries. This subject has been extensively reviewed in [20], [21], to which we refer for further information and/or references.

2.1. The Gopakumar–Vafa duality. The Gopakumar–Vafa duality [12] is an example of the string/gauge theory dualities which have been discovered in the last years. It relates a particular gauge theory – $U(N)$ Chern–Simons theory on the three-sphere with coupling k – to a particular string theory – the type A topological string on the small resolution of the conifold singularity. This is a toric (hence non-compact) Calabi–Yau manifold which can be regarded as the total space of the bundle

$$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1. \tag{2}$$

It has a single Kähler parameter t which gives the complexified area of the $\mathbb{P}^1$. The identification is such that the effective coupling constant of the gauge theory

$$g = \frac{2\pi \mathrm{i}}{k+N} \tag{3}$$

is identified to the string coupling constant g_s, while the 't Hooft parameter gN is identified with the Kähler parameter t:

$$t = \frac{2\pi \mathrm{i} N}{k+N}. \tag{4}$$

In particular, the duality asserts that the free energy of Chern–Simons theory on $\mathbb{S}^3$ equals the total free energy of the topological string, which is defined by summing the topological string amplitudes to all genera,

$$F = \sum_{g=0}^{\infty} F_g(t) g_s^{2g-2}, \tag{5}$$

This can be checked explicitly since both quantities are known. The free energies $F_g(t)$ of topological string theory on the resolved conifold have been computed in various ways (see for example [7]), and the free energy of Chern–Simons theory on the sphere was computed by Witten and is given by

$$F = \log Z = \log\, S_{00}, \tag{6}$$

where

$$S_{00} = \prod_{j=1}^{N} \left(2 \sin \frac{2\pi \mathrm{i} j}{k+N} \right)^{N-j} \tag{7}$$

is obtained from the theory of affine Lie algebras.

The duality of Gopakumar and Vafa gives some important information on Gromov–Witten theory, but it only deals with one particular Calabi–Yau manifold: the resolved conifold. From the point of view of topological string theory it would be extremely useful to have generalizations of the duality which cover other situations, and express the amplitude $F_g(t)$ in terms of gauge theoretic quantities. It turns out that this can be done in two different ways, which I consider in the next two subsections.

2.2. Extensions to other geometries. The first possibility to generalize the Gopakumar–Vafa duality consists on taking Chern–Simons theory on more general three-manifolds and search for Calabi–Yau duals. One obvious way to achieve this is to do a quotient of both sides of the duality by $\mathbb{Z}_p$ symmetry. On the gauge theory side one obtains Chern–Simons theory on the lens space $L(p, 1)$. The quotient of the resolved conifold leads to a toric geometry, the A_{p-1} fibration over $\mathbb{P}^1$, which has p Kähler parameters. For example, for $p = 2$ this leads to a duality between Chern–Simons theory on $\mathbb{RP}^3$ and topological string theory on the Calabi–Yau manifold given by the anticanonical bundle of $\mathbb{P}^1 \times \mathbb{P}^1$.

This generalization of the Gopakumar–Vafa duality was proposed in [1], where it was tested in detail for $p = 2$. One interesting aspect of it is that one has to consider the Chern–Simons theory around an arbitrary reducible flat connection which breaks the gauge group

$$U(N) \to \prod_{i=1}^{p} U(N_i). \tag{8}$$

The p Kähler parameters of the Calabi–Yau manifold, t_i, are identified with the partial 't Hooft parameters of the Chern–Simons theory with broken gauge symmetry:

$$t_i = g_s N_i, \quad i = 1, \cdots, p. \tag{9}$$

Unfortunately, it is not known if there are further generalizations of these results. However, it seems natural to state the following

Conjecture 2.1. Given a rational homology sphere M, there exists a Calabi–Yau manifold X_M such that the free energy of Chern–Simons theory on M, expanded around a generic reducible flat connection, equals the total free energy of topological string theory on X_M.

2.3. The cut-and-paste approach: the topological vertex. The basic idea of the topological vertex is to regard a generic toric geometry as made out of pieces where one can use the duality of Gopakumar and Vafa between the resolved conifold and Chern–Simons theory. A first approach is then to cut the manifold into pieces that are locally like resolved conifolds, to associate a topological string amplitude to each of the pieces, and then to glue the results together. This program was developed in [3], [9]. It turns out that there is a natural way to cut the original manifold into pieces, and this is by introducing D-branes around Lagrangian submanifolds. The amplitude associated to each of the pieces is then, due to the presence of D-branes, an *open* topological string amplitude. Fortunately, the duality of Gopakumar and Vafa also holds in the open setting [26], and the open amplitudes are closely related to *Chern–Simons invariants of knots and links* in $\mathbb{S}^3$. One then finds a surprising relation between these invariants and Gromov–Witten invariants of toric Calabi–Yau threefolds.

This procedure was refined and generalized in [2]. In the approach of [3], [9], one has to divide the geometry into pieces which are like the resolved conifold, which in terms of toric diagrams can be regarded as a four-valent graph. However, the basic building block is in fact a *trivalent* vertex that corresponds to $\mathbb{C}^3$ with three sets of D-branes wrapping Lagrangian submanifolds. The open topological string amplitude associated to this graph is called the topological vertex, and it is denoted by

$$C_{R_1 R_2 R_3}, \tag{10}$$

where R_i are representations of $U(\infty)$ (or, equivalently, Young tableaux) which correspond roughly to the Chan–Paton factors associated to the open strings ending on the branes. The topological vertex depends only on the string coupling g_s, and from its power series expansion in g_s one can extract open Gromov–Witten invariants of $\mathbb{C}^3$ with specified Lagrangian boundary conditions. These invariants do not have a rigorous mathematical definition, but they can be re-interpreted as *relative* Gromov–Witten invariants and computed by localization [19]. As shown in [2], one can use a subtle variant of the Gopakumar–Vafa duality to obtain an explicit expression for the topological vertex in terms of known quantum group invariants for arbitrary representations R_i. The final result is

$$C_{R_1R_2R_3} = q^{\frac{\kappa_{R_2}+\kappa_{R_3}}{2}} \sum_{Q_1,Q_3,Q} N^{R_1}_{QQ_1} N^{R_3^t}_{QQ_3} \frac{W_{R_2^tQ_1}W_{R_2Q_3}}{W_{R_2}}. \tag{11}$$

In this equation, $N^R_{R_1R_2}$ are Littlewood–Richardson coefficients, and R^t denotes the transpose of the representation R. κ_R is related to the second Casimir of R as a representation of $U(N)$ and can be written as

$$\kappa_R = \ell(R) + \sum_i l_i^R(l_i^R - 2i), \tag{12}$$

where $\ell(R)$ is the number of boxes in the Young tableau of R and l_i^R is the number of boxes in the i-th row of R. Finally, $W_{R_1R_2}$ are related to quantum group invariants of the Hopf link and can be written in terms of Schur polynomials as

$$W_{R_1R_2}(q) = s_{R_2}\big(x_i = q^{-i+\frac{1}{2}}\big)s_{R_1}\big(x_i = q^{l_i^{R_2}-i+\frac{1}{2}}\big), \tag{13}$$

where $q = \mathrm{e}^{g_s}$.

In [2] it is shown on a physical basis that the all genus Gromov–Witten invariants of any toric Calabi–Yau manifold can be computed from (11) and some simple gluing rules. Essentially, one takes the toric diagram of the Calabi–Yau and decomposes it into trivalent vertices. To each of these vertices one assigns the amplitude (11), and then one glues them together according to simple instructions encoded in the diagram. The result is the partition function $Z = \exp F$, where F is the free energy (5). One simple example is the so-called local $\mathbb{P}^2$ manifold, namely the total space of the bundle $\mathcal{O}(-3) \to \mathbb{P}^2$. The rules of [2] give

$$Z_{\mathbb{P}^2} = \sum_{R_1,R_2,R_3} (-1)^{\sum_i \ell(R_i)} e^{-\sum_i \ell(R_i)t} q^{-\sum_i \kappa_{R_i}} C_{0R_2^tR_3} C_{0R_1^tR_2} C_{0R_3^tR_1}. \tag{14}$$

The theory of the topological vertex developed in [2] and largely based on the ideas of string/gauge theory duality has been re-derived to a large extent on a rigorous mathematical basis in [19], by using relative Gromov–Witten invariants. This theory gives a full solution of topological string theory in the toric case. Let us now consider the compact case.

3. The compact case

3.1. Heterotic duality. As we already explained in the introduction, the main tool to compute Gromov–Witten invariants in the compact case is the combination of mirror symmetry and the holomorphic anomaly equation of [5]. This approach is very general, but it does not give a complete solution to the problem of computing the topological string amplitudes due to the holomorphic ambiguity. It turns out that if the Calabi–Yau manifold has the structure of a K3 fibration over $\mathbb{P}^1$, one can do better. The reason is that type IIA theory on such manifolds has a heterotic dual [15], and for Kähler classes in the K3 fiber, one can compute the $F_g(t)$ couplings by doing a one-loop computation in the heterotic string. This leads to close expressions for the topological string amplitudes in terms of modular forms [13], [22].

Let us consider for example the STU model first studied in [15]. This is a K3 fibration where the K3 fiber has two complexified Kähler parameters, $t = (t^+, t^-)$ with $\mathrm{Re}\, t^{\pm} > 0$. The Kähler classes are labelled by two integers $r = (n, m)$. These classes form a lattice $\Gamma^{1,1}$ with intersection form

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{15}$$

which defines an inner product such that $r \cdot t = mt^+ + nt^-$ and $r^2 = 2nm$. An involved heterotic computation [22] leads to a topological string coupling $F_g(t)$ of the form

$$F_g(t) = \sum_{r>0} c_g(r^2/2)\mathrm{Li}_{3-2g}(\mathrm{e}^{-r\cdot t}). \tag{16}$$

In this formula, $r > 0$ means the following possibilities: $(n, m) = (1, -1)$, $n > 0$, $m > 0$, $n = 0$, $m > 0$, or $n > 0$. The coefficients $c(n)$ are defined by

$$\sum_n c_g(n)q^n = -\frac{2E_4(q)E_6(q)}{\eta^{24}(q)}\mathcal{P}_g(q), \tag{17}$$

where E_4, E_6 are Eisenstein series, η is the Dedekind eta function, and the polynomials $\mathcal{P}_g(q)$ are given by

$$\left(\frac{2\pi\eta^3\lambda}{\vartheta_1(\lambda|\tau)}\right)^2 = \sum_{g=0}^{\infty}(2\pi\lambda)^{2g}\mathcal{P}_g(q), \tag{18}$$

and can be written as polynomials in the Eisenstein series E_2, E_4 and E_6 (see [22], [18] for more details on the modular forms involved).

Other examples of heterotic computations of topological string amplitudes can be found in [17], which also tested the predictions by using the holomorphic anomaly equations and extended them up to genus two by including the "missing" Kähler parameter on the base of the fibration.

3.2. A very special example: the Enriques Calabi–Yau. Although many techniques have been developed in order to compute topological string amplitudes in the compact case, typically the theory becomes intractable at high genus and/or degree. In this sense, it would be important to identify *the* compact CY manifold where topological string theory is most tractable.

There is a compact example where topological string theory is exactly solvable, namely $\mathrm{K3}\times\mathbb{T}^2$. The topological string amplitudes are however rather trivial in this case, and in particular they vanish for $g \geq 2$. Hence this example is too simple, and this is due to the extended $\mathcal{N} = 4$ supersymmetry of the corresponding type II theory, related in turn to the $\mathrm{SU}(2)$ holonomy. It is then natural to consider Calabi–Yau manifolds with holonomy H which is intermediate between the $\mathrm{SU}(2)$ and the generic one $\mathrm{SU}(3)$. A Calabi–Yau manifold with intermediate holonomy $\mathrm{SU}(2)\times\mathbb{Z}_2$ has been constructed in [6], [27], [10] as an orbifold w.r.t. a free $\mathbb{Z}_2$ involution of $\mathrm{K3}\times\mathbb{T}^2$. The resulting space exhibits a K3 fibration with four fibers of multiplicity two over the four fixed points of the involution in the base. These fibers are Enriques surfaces. A good deal of the nontrivial geometry of this CY comes from the geometry of the Enriques fibers, and therefore this example has been called the *Enriques CY manifold*. The string model obtained by compactifying type II theory on the Enriques CY has $\mathcal{N} = 2$ supersymmetry and is known as the FHSV model. The Enriques CY seems to be the simplest CY compactification with nontrivial topological string amplitudes. Moreover it has a dual description as an asymmetric orbifold of the heterotic string [10].

A first step in order to determine the topological string amplitudes is then to compute the amplitudes on the fiber, by using the heterotic dual and techniques similar to those of [22]. In order to write down the result, we notice that the Enriques fiber has ten Kähler parameters $t = (t^+, t^-, \vec{t}\,)$. The Kähler classes belong to the cone $\Gamma_E = \Gamma^{1,1} \oplus E_8(-1)$, and will be parametrized by a vector of integer numbers $r = (n, m, \vec{q})$. The topological string amplitudes on the Kähler cone are given by

$$F_g^E(t) = \sum_{r>0} c_g(r^2)\big\{2^{3-2g}\mathrm{Li}_{3-2g}(e^{-r\cdot t}) - \mathrm{Li}_{3-2g}(e^{-2r\cdot t})\big\}, \tag{19}$$

where

$$\sum_n c_g(n)q^n = -\frac{2}{q}\prod_{n=1}^{\infty}(1-q^{2n})^{-12}\mathcal{P}_g(q), \tag{20}$$

and $r^2 = 2mn - \vec{q}^{\,2}$. The restriction $r > 0$ means now that $n > 0$, or $n = 0, m > 0$, or $n = m = 0, \vec{q} > 0$. The superscript E refers to the Enriques fiber.

The Enriques Calabi–Yau manifold has an extra Kähler class corresponding to the $\mathbb{P}^1$ in the base of the fibration. The perturbative heterotic string theory does not give information on the dependence of $F_g(t)$ on this parameter, and the only available technique to do that for the moment being is the holomorphic anomaly equation. It turns out, however, that the Enriques Calabi–Yau is particularly simple in that respect, and one can solve the holomorphic anomaly equation at low genera. In this way one finds explicit expressions for F_1 and F_2 on the *total* Calabi–Yau. If we denote by S the Kähler parameter of the base, one finds

$$\begin{aligned} F_1(t,S) &= F_1^E(t) - 12\log\,\eta(q_S), \\ F_2(t,S) &= E_2(q_S)F_2^E(t), \end{aligned} \tag{21}$$

where $q_S = \mathrm{e}^{-S}$. Notice that, as $S \to \infty$, one recovers the results in the fiber. These results are surprisingly simple, and they have been verified in [25] by using the topological techniques in Gromov–Witten theory. This is the only compact manifold where the topological string amplitudes have been computed up to genus two *both* in the context of topological string theory and in the context of algebraic geometry. It seems to be the most accessible compact example and it provides a fruitful interaction between physical and mathematical techniques. Results for genus 3 and 4 including the Kähler parameter of the base have also been obtained in [18]. One important remaining question is: is the Enriques Calabi–Yau an exactly solvable example?

4. Conclusions

One obvious and general conclusion is that the interaction between topological string theory and Gromov–Witten theory and algebraic geometry has been extremely rich and rewarding for both fields. String dualities have led to the solution of many models and to sometimes surprising mathematical predictions, while rigorous mathematical techniques have confirmed many of the physical ideas underlying the predictions. This interplay has led so far to a rather complete solution of the problem in the toric case. Progress in the compact case has been also significant, although the problem seems to be much harder and we are lacking an effective strategy to address general models. For this reason, we have proposed in [18] to focus on relatively simple examples where the theory might have some simplifying features yet be rich enough to display the essential complexities of the problem. So far, the simplest nontrivial valley in the landscape of compact Calabi–Yau's seems to be the Enriques Calabi–Yau, which has been solved at low genera in [18], [25]. The main open problem is of course to find an organizing principle that makes possible to address the general compact case in an effective way.

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