The Cauchy problem in General Relativity

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Abstract. The paper revisits some of the classical and recent results on the Cauchy problem in General Relativity. Special emphasis is put on the problems concerning existence of a Cauchy development, break-down criteria and stability. The author would like to make a disclaimer that despite its general title the paper is not intended as a comprehensible survey. Due to the space-time constraints many remarkable results and developments are either mentioned briefly or not discussed at all. Most notably this concerns various work on the Einstein equations with matter and symmetry reduced problems.

Mathematics Subject Classification (2000). Primary 83C05; Secondary 35L15.

Keywords. Einstein equations, existence, break-down, stability.

1. Introduction

A mathematical description of General Relativity consists of a $3+1$-dimensional Lorentzian manifold M and a metric g verifying the Einstein equations

$$R_{\alpha\beta}(g) - \frac{1}{2} g_{\alpha\beta} R(g) = 8\pi\, T_{\alpha\beta},$$

where $R_{\alpha\beta}$ and R are respectively the Ricci tensor and scalar curvature of g and $T_{\alpha\beta}$ is the energy-momentum tensor of matter. Among the most popular mathematical matter models are[1]

1. the vacuum equations, where $T \equiv 0$, and the Einstein equations simply require that (M, g) is Ricci flat;
2. the Einstein-scalar field model, where $T_{\alpha\beta} = \partial_\alpha\phi\, \partial_\beta\phi - \frac{1}{2} g_{\alpha\beta}\, \partial^\mu\phi\, \partial_\mu\phi$ and ϕ is a real valued scalar field $\phi : M \to R$;
3. the Einstein–Maxwell equations, where $T_{\alpha\beta} = \frac{1}{4\pi}\left(F_\alpha{}^\mu F_{\beta\mu} - \frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu}\right)$, and $F_{\alpha\beta}$ is the electromagnetic tensor;
4. perfect fluid matter model, where $T_{\alpha\beta} = (\rho + p) u_\alpha\, u_\beta + p\, g_{\alpha\beta}$, and u^α is the four-velocity vector, p is the pressure and ρ is the proper energy density of the fluid.

*The paper was written while the author was visiting the Department of Mathematics at MIT.

[1] In what follows we will use the standard conventions of raising and lowering tensorial indices with the help of the metric g and summing over repeated indices.

Proceedings of the International Congress of Mathematicians, Madrid, Spain, 2006

The contracted Bianci identity $D^\alpha R_{\alpha\beta} = 2\partial_\beta R$ implies that the gravitational tensor $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$ is always divergence free, $D^\alpha G_{\alpha\beta} = 0$. As a consequence, evolution equations for the external fields in the models described above follow from the requirement that $D^\alpha T_{\alpha\beta} = 0$. In particular, in the scalar field model ϕ must satisfy the wave equation

$$\Box_g \phi = \frac{1}{\sqrt{|g|}} \partial_\alpha \big(g^{\alpha\beta} \sqrt{|g|} \partial_\beta \phi \big) = 0$$

on a curved background (M, g). For the Einstein–Maxwell problem the electromagnetic field obeys the Maxwell equations

$$D^\alpha F_{\alpha\beta} = 0, \quad D_\mu F_{\alpha\beta} + D_\beta F_{\mu\alpha} + D_\alpha F_{\beta\mu} = 0.$$

Finally, for the perfect fluid model,

$$u^\alpha D_\alpha \rho + (\rho + P) D^\alpha u_\alpha = 0, \quad (P + \rho) u^\alpha D_\alpha u_\beta + (g_{\alpha\beta} + u_\alpha u_\beta) D^\alpha P = 0.$$

Mathematical problems in Classical General Relativity can be loosely divided into the following categories:

1. Construction of special solutions (e.g. Minkowski, Schwarzschild, Kerr, Friedman–Robertson–Walker).
2. Mathematics of constraint equations (e.g. construction of solutions, positive mass theorem, Riemannian Penrose inequality).
3. Causality and global properties (e.g. singularity theorems, black hole uniqueness, splitting theorems).
4. Cauchy problem (e.g. existence and uniqueness of solutions, break-down, stability).

The evolution (Cauchy) problem in General Relativity consists of constructing a space-time (M, g) with the property that for a given data set [2] (Σ_0, g_0, k_0), a 3-dimensional Riemannian manifold Σ_0 with a Riemannian metric g_0 and a symmetric 2-tensor k_0, there exists an embedding $\Sigma_0 \subset M$ such that g_0 coincides with the restriction of g to Σ_0 and k_0 is the second fundamental form of the embedding. Since physically one should not be able to distinguish between different coordinate systems, i.e., the Einstein equations are covariant, a solution of the Cauchy problem can be unique only modulo a diffeomorphism.

The equations are overdetermined and the initial data has to satisfy the constraint equations

$$R_0 - |k_0|^2 + (\operatorname{tr} k_0)^2 = 16\pi T_{00}, \quad \nabla^j k_{0ij} - \nabla_i \operatorname{tr} k_0 = 8\pi T_{0i}, \tag{1}$$

where R_0 is the scalar curvature of g_0 and ∇ is its Levi-Civita covariant derivative.

[2] For simplicity we describe the Cauchy problem for the vacuum equations. In general, one also needs to add the data for the external fields on Σ_0.

A particular important class is the asymptotically flat initial data given by (Σ_0, g_0, k_0) with the properties[3] that Σ_0 minus a compact set is diffeomorphic to R^3 minus a ball and that there exists a system of coordinates "near infinity" such that

$$g_{0ij} = \left(1 + \frac{M}{r}\right)\delta_{ij} + O(r^{-1-\alpha}), \quad k_{0ij} = O(r^{-2-\alpha})$$

for some $\alpha > 0$. In particular, Minkowski, Schwarzschild and Kerr solutions belong to this class.

2. Existence and uniqueness of a maximal Cauchy development

Existence of a maximal Cauchy development for any sufficiently smooth initial data was established by Choquet-Bruhat in [CB1]. The proof exploited the covariance of the Einstein equations and a special choice of gauge (harmonic coordinate system) in which the Einstein equation can be cast as a system of quasilinear wave equations for the metric components.

Harmonic coordinates x^α, which appeared already in the work of A.Einstein, are determined by the conditions

$$\Box_g x^\alpha = 0, \quad \alpha = 0, \dots, 3. \tag{2}$$

This, in turn, is equivalent to requiring that the components of the metric $g_{\alpha\beta}$ relative to this particular system of coordinates satisfy

$$\partial_\alpha\big(g^{\alpha\beta}\sqrt{|g|}\big) = 0. \tag{3}$$

The Einstein equations then reduce to

$$\Box_g g_{\alpha\beta} - N_{\alpha\beta}(g, \partial g) = T_{\alpha\beta}. \tag{4}$$

In harmonic coordinates, $\Box_g = g^{\mu\nu}\partial^2_{\mu\nu}$ and the nonlinear term $N_{\alpha\beta}(u, v)$ depends quadratically on the variable v. Equation (4) is obtained by expressing the Ricci tensor $R_{\alpha\beta}$ of g in terms of the components of g and its (first and second) derivatives. To verify that a solution of (4) gives an Einstein metric one also has to satisfy the condition (2) or (3). However, as was observed by Choquet-Bruhat, these conditions are satisfied automatically for solutions of (4) provided that they are satisfied initially on Σ_0 and (Σ_0, g_0, k_0) obey the constraint equations (1). This fact is known as propagation of the harmonic gauge. Thus to complete the initial value problem set up in harmonic gauge we choose a local system of coordinates on Σ_0 in such a way that the harmonic coordinate condition is verified initially and express the initial values[4]

[3] The asymptotic flatness condition given here is more restrictive than necessary. In particular, it requires that Σ_0 has only one asymptotic end and that the linear momentum of initial data is equal to zero.

[4] We identify $t = x^0$ and assume that Σ_0 corresponds to $t = 0$.

for the components of the metric $g_{\mu\nu}|_{t=0}$ and their time derivatives $\partial_t g_{\mu\nu}|_{t=0}$ from the initial data (g_0, k_0)[5]. Restricting our analysis for simplicity to the case of the vacuum equations we obtain the following initial value problem

$$\begin{aligned} &\Box_g g_{\alpha\beta} = N_{\alpha\beta}(g, \partial g), \\ &g_{\alpha\beta}|_{t=0} = g^0_{\alpha\beta}, \quad \partial_t g_{\alpha\beta}|_{t=0} = g^1_{\alpha\beta}. \end{aligned} \tag{5}$$

The equations (5) constitute a system of quasilinear wave equations for the components $g_{\alpha\beta}$ on the background determined by the metric g. The problem (5) is then solved locally (using finite speed of propagation) on a small interval of time, the resulting metrics patched together to form a Cauchy development from given initial data. The above local solutions are constructed by iteration of the linear equations

$$\begin{aligned} &\Box_{g^n} g^{n+1}_{\alpha\beta} = N_{\alpha\beta}(g^n, \partial g^n), \\ &g^{n+1}_{\alpha\beta}|_{t=0} = g^0_{\alpha\beta}, \quad \partial_t g^{n+1}_{\alpha\beta}|_{t=0} = g^1_{\alpha\beta} \end{aligned} \tag{6}$$

with convergence guaranteed by estimates for (6) or alternatively a priori estimates for (5). The original approach of Choquet-Bruhat to (6) relied on the construction of a Kirchoff–Sobolev parametrix for an inhomogeneous scalar linear problem

$$\Box_g \phi = F, \quad \phi|_{t=0} = \phi^0, \quad \partial_t \phi|_{t=0} = \phi^1$$

but the method imposed high differentiability requirement on the initial data for (5). This was refined in the work of Dionne [Di], Fisher–Marsden [F-M] and Hughes–Kato–Marsden [H-K-M] via the energy method. The energy method, in which the equation (5) is multiplied by $\partial_t g$ and integrated by parts or differentiated required number of times, multiplied by the time derivative of the differentiated solution and then integrated by parts, applied to the problem (5) shows that the Sobolev norm H^s of the solution $g(t)$ at time t is controlled

$$\|g(t)\|_{H^s} + \|\partial_t g(t)\|_{H^{s-1}} \le C \exp\left(\int_0^t \|\partial g(\tau)\|_{L^\infty} d\tau\right) \left(\|g^0\|_{H^s} + \|g^1\|_{H^{s-1}}\right).$$

The desired a priori estimate for (5) follows from the Sobolev embedding $H^s \subset L^\infty$ provided that $s > 5/2$. This analysis essentially establishes *well-posedness*[6] of the system (5) in the scale of Sobolev spaces H^s for any $s > 5/2$.

An interesting phenomenon however occurs in passage from the system (5) to the original Einstein equations. The above construction leads to solutions (M, g) arising from arbitrary H^s initial data $(g_0, k_0) \in H^s \times H^{s-1}$, as long as $s > \frac{5}{2}$. These solutions remain in the space H^s relative to a system of coordinates (t, x) so that the metric components $g_{\alpha\beta} \in C([0, T]; H^s_x)$ and $\partial_t g_{\alpha\beta} \in C([0, T]; H^{s-1}_x)$ on a time interval $[0, T]$ with T dependent on the $H^s \times H^{s-1}$ norm of the data. However,

[5] Similar procedure is applied to the initial values for the external fields.

[6] Existence, uniqueness and continuous dependence on the initial data.

to show that two solutions (M, g) and (M', g') arising from the same initial data (Σ_0, g_0, k_0) are related by a diffeomorphism $\Phi : M \to M'$ so that $\Phi_* g' = g$, and thus geometrically and physically indistinguishable, actually requires one to consider data and thus solutions (M, g) and (M', g') from the Sobolev class H^σ with $\sigma > 7/2$. This means that while uniqueness for (5) holds in the same class H^s with $s > 5/2$ as the existence result, there is a potential loss of uniqueness for the Einstein equations unless more regular solutions are considered.

The existence result for the system (5) and consequently the (vacuum or scalar field) Einstein equations can be improved when the energy method is combined with the Strichartz estimates. This was first seen for scalar semilinear wave equations in [P-S],

$$\Box\phi = N(\phi, \partial\phi), \tag{7}$$

where the energy estimate

$$\|\partial\phi(t)\|_{H^s} \leq C \exp\left(\int_0^t \|\phi(\tau)\|_{L^\infty} d\tau\right)\big(\|\phi^0\|_{H^s} + \|\phi^1\|_{H^{s-1}}\big)$$

can be complemented by the Strichartz estimate

$$\|\partial\phi\|_{L^2_{[0,T]}L^\infty} \leq C\big(\|\phi^0\|_{H^s} + \|\phi^1\|_{H^{s-1}} + \|\Box\phi\|_{L^1_{[0,T]}H^{s-1}}\big), \tag{8}$$

which holds for any $s > 2$ and thus allows to establish the existence and uniqueness result for the equation (7) for solutions in the Sobolev space H^s with $s > 2$.

In the case of general quasilinear wave equations of the form (5), however, the situation is far more difficult. One can no longer rely on the Strichartz inequality (8) for the flat D'Alembertian; we need instead its extension to the operator $\Box_g$,

$$\|\partial\phi\|_{L^2_{[0,T]}L^\infty} \leq C\big(\|\phi^0\|_{H^s} + \|\phi^1\|_{H^{s-1}} + \|\Box_g\phi\|_{L^1_{[0,T]}H^{s-1}}\big). \tag{9}$$

To be able to apply such an estimate to the problem (5) and improve upon the energy method one needs to establish (9) for some $s \leq 5/2$ and with a constant C which itself depends on g only through its $\|\partial g\|_{L^\infty_{[0,T]}H^{s-1}}$ and $\|\partial g\|_{L^2_{[0,T]}L^\infty}$ norms. This means that we have to confront the issue of proving Strichartz estimates for wave operators $\Box_g$ on a *rough* background g.

This issue was first addressed in the work of Smith [Sm], Bahouri–Chemin [B-C1], [B-C2] and Tataru [Ta1], [Ta2].

In [Sm] a precise analog of (8) was established for the wave operator $\Box_g$ under the assumption that the metric g is at least C^2.

The results of Bahouri–Chemin and Tataru are based on establishing a Strichartz type inequality, *with a loss*, i.e. with $s > 2 + \sigma$, and are compatible with applications to the problem (5). The optimal result[7] in this regard, due to Tataru, see [Ta2], requires

[7] Recently Smith-Tataru [S-T1] have shown that the result of Tataru is indeed sharp.

a loss of $\sigma = \frac{1}{6}$. This led to a proof of local well posedness for systems of type (5)[8] with $s > 2 + \frac{1}{6}$.

To do better than that one needs to take into account the nonlinear structure of the equations. Both the classical work [CB1], [Di], [F-M], [H-K-M] and the Strichartz based results [B-C1], [B-C2], [Ta1], [Ta2] only used the fact that the background metric g is Lorentzian and obeys regularity conditions compatible with the final desired result. The additional important piece of information that g itself is a solution of (5) was not exploited.

In [K-R1] we were able to improve the result of Tataru by taking into account not only the expected regularity properties of the metric g but also the fact that they are themselves solutions to a similar system of equations. This allowed us to improve the exponent s, needed in the proof of well posedness of equations of type (5) to $s > 2 + \frac{2-\sqrt{3}}{2}$. Our approach was based on a combination of the paradifferential calculus ideas, initiated in [B-C1] and [Ta2], with a geometric treatment of the actual equations introduced in [K4]. The main improvement was due to a gain of conormal differentiability for solutions to the eikonal equations

$$g^{\alpha\beta}_{<\lambda}\partial_\alpha u\, \partial_\beta u = 0 \tag{11}$$

with $g_{<\lambda}$ a smoothed out version of the original metric g with the property that $|\nabla^k g_{<\lambda}| \le C_k \lambda^k$ for any spatial derivative ∇. Such smoothing can be constructed with the help of the standard Littlewood–Paley projections $P_{<\lambda}$ which smoothly remove Fourier frequencies $\ge \lambda$.

In [K-R2]–[K-R4] we developed the ideas of [K-R1] further in the context of the Einstein vacuum equations, i.e., equations (5) coupled with the condition that $R_{\alpha\beta}(g) = 0$. We make use of both the vanishing of the Ricci curvature of g and the harmonic gauge condition (3). The other important new features are the use of energy estimates along the null hypersurfaces generated by the optical function u and a deeper use of the conormal properties of the null structure equations.

Theorem 1 ([K-R2]). *Consider the reduced equation* (5) *with initial data* $g^0_{\alpha\beta}, g^1_{\alpha\beta} \in H^s \times H^{s-1}$ *for some* $s > 2$ *satisfying the constraint equations* (1) *with* $T \equiv 0$ *and the harmonic gauge condition* (3)*. Then there exists a time interval* $[0, T]$ *and unique* (*Lorentzian metric*) *solution* g *such that* $g_{\alpha\beta} \in C^0([0, T]; H^s)$ *with* T *depending only on the size of* $\|g^0_{\alpha\beta}\|_{H^s} + \|g^1_{\alpha\beta}\|_{H^{s-1}}$.

The results of Theorem 1 require that the initial data can be approximated by a smooth sequence of data satisfying the constraint equations. Using a conformal method a large class of such (Σ_0, g_0, k_0) was constructed in [CB3] and [Ma].

[8] The above results actually apply to more general quasilinear equations of the form

$$\Box_{g(\phi)}\phi = N(\phi, \partial\phi), \tag{10}$$

where g is a *given* metric smoothly dependent on a solution ϕ. However, there is no substantial difference between the equations (5) and (10) unless of course one also uses the fact that a solution of (5) with initial data in harmonic gauge is a solution of the vacuum Einstein equations.

In [S-T2] H. Smith and D. Tataru obtained the parallel H^s, $s > 2$ local well posedness result for general quasilinear equations, as well as the new improved results in other dimensions rather than $n = 3$. Their approach is based on the construction of a wave packet approximation of a solution. The geometry of wave packets controls the desired Strichartz estimate. The construction relies on the foliation by the null planes. It uses a gain of differentiablity along each plane, which can be traced to the decomposition of the tangential components of the curvature in the spirit [K-R1], but avoids references to the regularity of the foliation in the direction transversal to the leafs (i.e. torsion of the foliation).

It is very likely that the results of Theorem 1 are not sharp and the Einstein vacuum equations can be solved in even lower degree of regularity. A very satisfactory result both from the analytic and geometric point of view would a resolution of the L^2 *curvature conjecture*, see [K3], according to which the time of existence for solutions of the Einstein vacuum equations should depend only on the L^2 norms of the Riemann curvature tensor of g_0 and the gradient of the second fundamental form ∇k_0 and perhaps some other weaker geometric characteristics of Σ_0. Some geometric evidence in support of this conjecture is provided in the work [K-R5]–[K-R7] where it was shown that null hypersurfaces, level surfaces of the optical function solving the eikonal equation $g^{\alpha\beta}\partial_\alpha u\,\partial_\beta u = 0$, do not break down locally as long as the L^2 curvature flux along them is finite.

2.1. Existence results in other gauges. Existence results for the Einstein (vacuum) equations can be also established in other gauges than the harmonic coordinate gauge (2).

In [A-M1] application of the energy method yields a construction of the H^s with $s > 5/2$ vacuum space-times in the constant mean curvature spatially harmonic gauge. To describe the evolution equations in this particular gauge we write the metric g in the form

$$g = -N^2dt^2 + \gamma_{ij}(dx^i + X^i dt)(dx^j + X^j dt),$$

where N and X are the lapse and shift of the t-foliation. The Einstein vacuum equations are written as a system of evolution equations for the metric γ and the second fundamental form k of the t-foliation coupled to the constraint equations, while the gauge condition generates elliptic equations for N and X.

$$\partial_t\gamma_{ij} = -2Nk_{ij} + \mathcal{L}_X\gamma_{ij}, \tag{12}$$

$$\partial_t k_{ij} = -\nabla_i\nabla_j N + N({}^{(3)}R_{ij} + \operatorname{tr} k\, k_{ij} - 2k_{im}k_j^m) + \mathcal{L}_X k_{ij}. \tag{13}$$

Here $\mathcal{L}$ is the Lie derivative and ${}^{(3)}R_{ij}$ is the Ricci curvature of γ. The constant mean curvature condition is the requirement that on the hypersurface $t = \text{const}$ we have $\operatorname{tr} k = t$. Under this condition taking the trace in (13) and using the constraint equations we obtain an elliptic equation for the lapse N:

$$-\Delta_\gamma N + |k|^2 N = 1. \tag{14}$$

The constraint equations in this gauge also become

$$^{(3)}R = |k|^2 - t, \quad \nabla^j k_{ij} = 0. \tag{15}$$

We also fix the spatially harmonic gauge by requiring[9] that a system of coordinates x^i, $i = 1, 2, 3$ on each $t =$ constant is harmonic, i.e., satisfies the equation $\Delta_\gamma x^i = 0$. The Ricci curvature $^{(3)}R_{ij}$ can then be written on the form

$$^{(3)}R_{ij} = -\frac{1}{2}\Delta_\gamma \gamma_{ij} + N_{ij}(\gamma, \nabla\gamma),$$

where as before $N_{ij}(u, v)$ depends quadratically on v. Propagation of this gauge results in an elliptic equation for the shift so that (12), (13), (14), (15) form an elliptic-hyperbolic system.

A local existence result in the maximal gauge was also proved in [C-K]. This particular gauge corresponds to the choice $\operatorname{tr} k = X = 0$. The lapse and constraint equations take the form

$$-\Delta_\gamma N + |k|^2 N = 0,$$
$$^{(3)}R = |k|^2, \quad \nabla^j k_{ij} = 0$$

while the system (12), (13) describes the evolution of γ and k. To see the hyperbolic character of (12), (13) without imposing a spatially harmonic gauge one has to take an additional time derivative of (13) and express $\partial_t^{(3)}R$ in terms of γ and k.

Another interesting formulation of the Einstein (vacuum) equations arises by drawing an analogy between the Einstein equations and the Yang–Mills theory. In the Yang–Mills theory an electromagnetic field is represented by a Lie algebra valued 2-form $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]$ constructed from an electromagnetic potential A_α. The Yang–Mills equations on Minkowski space R^{3+1} for F take the form

$$D^\beta F_{\alpha\beta} = 0, \quad D_\mu F_{\alpha\beta} + D_\beta F_{\mu\alpha} + D_\alpha F_{\beta\mu} = 0,$$

where the covariant derivative $D_\alpha = \partial_\alpha + [A_\alpha, \cdot]$ and the second equation is the Bianci identity for the curvature form F. Differentiating the second equation we arrive at the second order hyperbolic problem for F

$$\Box_A F_{\alpha\beta} = 2F_\beta{}^\mu F_{\mu\alpha},$$

where $\Box_A = m^{\alpha\beta} D_\alpha D_\beta$.

In General Relativity the Riemann curvature tensor $R_{\alpha\beta\mu\nu}$ satisfies the Bianci identities

$$D_\sigma R_{\alpha\beta\mu\nu} + D_\beta R_{\sigma\alpha\mu\nu} + D_\alpha R_{\beta\sigma\mu\nu} = 0 \tag{16}$$

[9] The actual general harmonic gauge is only slightly more complicated.

and, if the Ricci curvature of g vanishes, i.e., (M, g) is a vacuum space-time, also a version of the contracted Bianci identities

$$D^{\alpha} R_{\alpha\beta\mu\nu} = 0. \tag{17}$$

Differentiating the Bianci identity (16) and also using (17) we easily obtain the wave equation for the Riemann curvature tensor

$$\Box_g R_{\alpha\beta\mu\nu} = (R \star R)_{\alpha\beta\mu\nu}\,, \tag{18}$$

where $\star$ denotes a combination of various contractions.

2.2. Large data problem in General Relativity. While the result of Choquet-Bruhat and its subsequent refinements guarantee the existence and uniqueness of a (maximal) Cauchy development, they provide no information about its geodesic completeness and thus, in the language of partial differential equations, constitutes a local existence result. Singularities could develop to the future (past) of the Cauchy hypersurface Σ_0 or the maximal Cauchy development could have a regular boundary, Cauchy horizon, beyond which the space-time could be continued thus losing its predictability from the initial data. Schwarzschild space-time is an example of a geodesically incomplete asymptotically flat space-time while the Reissner–Nordström solution of the Einstein–Maxwell equations possesses a Cauchy horizon.

More generally, there are a number of conditions that will guarantee that the space-time will be geodesically incomplete. The first such result was the Penrose singularity theorem:

Theorem 2 (Penrose). *A space-time*[10] (M, g) *is necessarily incomplete if it admits a non-compact Cauchy surface and a trapped*[11] 2*-dimensional compact surface.*

According to the result of [S-Y2] sufficient amount of matter placed in a region will create a trapped surface.

In the language of partial differential equations this means an impossibility of a large data global existence result for all initial data in General Relativity. In the absence of such a result a number of conjectures about the structure of space-times arising from generic data had been put forward in the '60s by Penrose. Among them is the *Weak Cosmic Censorship* which predicts that for generic asymptotically flat data null infinity will be affine complete or alternatively that singularities have to be hidden inside black holes. On the other hand the *Strong Cosmic Censorship* predicts a generic absence of the Cauchy horizons. At the moment not much progress has been made on either of these problems in the general case with remarkable exceptions in some cases of symmetry reduced Einstein equations: the proof of weak

[10] The energy-momentum tensor $T_{\alpha\beta}$ of matter is only required to satisfy what is called a null convergence energy condition.

[11] Infinitesimally deformations of such surface along both null outgoing and null incoming directions decrease its area.

cosmic censorship for the Einstein-scalar field equations in spherical symmetry by Christodoulou [C2],[C3], the work of Dafermos [D] on strong cosmic censorship and stability of the Cauchy horizons for the Einstein–Maxwell-scalar field equations in spherical symmetry, the proof of strong cosmic censorship in polarized Gowdy by Chruściel–Isenberg–Moncrief [C-I-M] and T^3-Gowdy by Ringström [Ri].

2.3. Break-down criteria in General Relativity. In the absence of a completeness result for general large data Cauchy problem in General Relativity and a very non-quantitative nature of the singularity theorems it is desirable to develop a better understanding of local or semi-local analytic mechanisms for break-down of solutions. Already the local existence results mentioned above provide such criteria. In particular an H^s local existence result in harmonic gauge guarantees that a solution can be extended as long as the H^s norms of the metric components in harmonic gauge remain finite. Such results however are not ideal as break-down criteria for the reasons that they are not geometric and strongly tied to a particular coordinate gauge and that they arise as a consequence of stronger local well-posedness statements.

The first *geometric* criterion for breakdown of solutions (M, g) of the vacuum Einstein equations

$$R_{\alpha\beta}(g) = 0 \tag{19}$$

appeared in the work of M. Anderson [A1]. To describe the problem we assume that a part of space-time $M_I \subset M$ is foliated by the level hypersurface Σ_t of a time function t, monotonically increasing towards future in the interval $I \subset R$, with lapse N and second fundamental form k so that

$$g = -N^2 dt^2 + \gamma_{ij} dx^i dx^j, \quad \partial_t \gamma_{ij} = -2N k_{ij}.$$

The surfaces Σ_t are compact, of Yamabe type -1, and of constant negative mean curvature, $\operatorname{tr} k = t$ with $t < 0$. Relative to a time foliation we can naturally associate a non-degenerate notion of a pointwise absolute value of a space-time tensor.

In [A1] it was shown that a break-down can be tied to the condition that

$$\lim \sup_{t \to t_*^-} \|R(t)\|_{L^\infty} = \infty,$$

where $R(t)$ denotes the Riemann curvature tensor of g and the norm is measured relative to the above described t-foliation.

A work in progress [K-R7] addresses the problem of break-down of solutions to the Einstein vacuum equations under the assumption that $T = N^{-1}\partial_t$ is an approximate Killing field. More precisely, the desirable break-down condition is

$$\lim \sup_{t \to t_*^-} \|\mathcal{L}_T g(t)\|_{L^\infty} = \infty, \tag{20}$$

where $\mathcal{L}_T g$ is the deformation tensor of T, equal to zero if T is Killing, and it can be expressed as

$$|\mathcal{L}_T g| = |k| + |\nabla \log N|.$$

This result would complement Anderson's criterion. It is clear however that the condition (20) is formally weaker as it refers only to the second fundamental form k and the lapse n and thus requires one degree less of differentiability than a condition on the Riemann curvature tensor. Moreover a condition on the boundedness of the L^∞ norm of $R(t)$ covers all the dynamical degrees of freedom of the equations. Indeed, once we know that $\|R(t)\|_{L^\infty}$ is finite, one can find bounds for n, ∇n and k on Σ_t purely by elliptic estimates. This is not true in our case.

A geometric criterion of the type (20) for the Einstein equations could be compared to the well known Beale–Kato–Majda [B-K-M] criterion for breakdown of solutions of the incompressible Euler equation

$$\partial_t v + (v \cdot \nabla) v = -\nabla p, \quad \operatorname{div} v = 0,$$

with smooth initial data at $t = t_0$. A routine application of the energy estimates shows that solution v blows up if and only if

$$\int_{t_0}^{t_*} \|\nabla v(t)\|_{L^\infty} \, dt = \infty. \tag{21}$$

The Beale–Kato–Majda work improves the blow up criterion by replacing it with the following condition on the vorticity $\omega = \operatorname{curl} v$:

$$\int_{t_0}^{t_*} \|\omega(t)\|_{L^\infty} \, dt = \infty. \tag{22}$$

The quantities ∇v and ω are related to each other via a singular integral operator, i.e., $\nabla v = P^0(\omega)$.

Although P^0 does not define a bounded map $L^\infty \to L^\infty$ it is sufficient to reduce the breakdown condition (21) to the more satisfying one (22), in terms of the vorticity alone.

Similarly, in the case of the Einstein equations energy estimates, expressed relative to a special system of coordinates (e.g. in harmonic gauge), show that break-down does not occur unless

$$\int_{t_0}^{t_*} \|\partial g(t)\|_{L^\infty} \, dt = \infty.$$

This condition however is not geometric as it depends on the choice of a full coordinate system. Observe that both the spatial derivatives of the lapse ∇n and the components of the second fundamental form, $k_{ij} = -\frac{1}{2} N^{-1} \partial_t g_{ij}$, can be interpreted as components of ∂g.

Note however that after prescribing k and ∇n we are still left with many more degrees of freedom in determining ∂g. The fundamental difficulty that one needs to overcome is that of deriving bounds for R using only bounds for $\|\nabla N(t)\|_{L^\infty} + \|k(t)\|_{L^\infty}$ and geometric informations on the initial hypersurface Σ_0. Clearly this cannot be done by elliptic estimates alone. Thus, as opposed to both the results of M.

Anderson and Beale–Kato–Majda, it is far less obvious that a condition such as (20) can cover all *dynamic* degrees of freedom of the Einstein equations.

The criterion (20) is motivated in part by the desire to adapt the Eardley–Moncrief argument [E-M1], [E-M2] for the large data global existence for the $3+1$ Yang–Mills equations to General Relativity, exploiting the analogy between the Einstein vacuum and the Yang–Mills equations.

The Eardley–Moncrief proof relies on two independent ingredients: conservation of energy and pointwise bounds on curvature, which are derived using the fundamental solution for $\Box$ in Minkowski space, and shown to depend only on the flux of curvature and initial data. Since the analog of the Yang–Mills energy in General Relativity (the Bel–Robinson energy) is not conserved one can only hope to reproduce the second part of the Eardley–Moncrief argument and prove a conditional regularity result which states, roughly, that smooth solutions of the Einstein equations, in vacuum, remain smooth, and can therefore be continued, as long as an integral quantity, we call the flux of curvature, remains bounded. The possibility of such a result was first pointed out by V. Moncrief.

However it is the fact that $T = \partial_t$ is a Killing field that is ultimately responsible for the conservation of energy in the Yang–Mills theory on Minkowski space. Similarly, in the extension [C-S] of the Eardley–Moncrief result to the Yang–Mills equations on a smooth globally hyperbolic background it is the fact that $T = \partial_t$ is an approximate Killing field that allows one to control the energy and the flux of curvature.

Thus in the context of General Relativity rather then imposing a direct condition on the finiteness of the Bel–Robinson energy and curvature flux we formulate conditions (perhaps more natural albeit more restrictive) which control the extent to which the energy is not conserved. These conditions, which form our breakdown criterion, involve uniform bounds on the second fundamental form k and the lapse N.

In what follows we describe how the main ideas of the proof of the Eardley–Moncrief result for Yang–Mills could be adapted to General Relativity.

The curvature tensor R of a $3+1$ dimensional vacuum spacetime (M, g), see (19), verifies a wave equation of the form,

$$\Box_g R = R \star R. \tag{23}$$

The Bel–Robinson energy-momentum tensor

$$\mathcal{Q}[R]_{\alpha\beta\gamma\delta} = R_{\alpha\lambda\gamma\mu} R_{\beta\ \delta}^{\ \lambda\ \mu} + \star R_{\alpha\lambda\gamma\mu} \star\ R_{\beta\ \delta}^{\ \lambda\ \mu}$$

verifies $D^\delta \mathcal{Q}[R]_{\alpha\beta\gamma\delta} = 0$ and can thus be used to derive energy and flux estimates for the curvature tensor R. The approximate Killing condition is sufficient to derive bounds for both energy and flux associated to the curvature tensor R. The flux is an integral of a square of the components of the Riemann curvature tensor tangent to a null hypersurface $N^-(p)$, boundary of the causal point of point p, generated, at least locally, as a level hypersurface $u = 0$ of an optical function u, solution of the eikonal equation $g^{\alpha\beta}\partial_\alpha u\, \partial_\beta u = 0$.

As in the case of the Yang–Mills equations it is precisely the boundedness of the flux of curvature that plays a crucial role in our analysis. In General Relativity the flux takes on even more fundamental role as it is also needed to control the geometry of the very object it is defined on, i.e. the boundary $N^-(p)$ of the causal past of p. This boundary, unlike in the case of Minkowski space, are not determined a-priori but depend in fact on the space-time we are trying to control.

The main idea is to show that if the condition (20) does not hold, i.e.,

$$\limsup_{t \to t_*^-} \|\mathcal{L}_T g(t)\|_{L^\infty} < \infty, \tag{24}$$

it implies a uniform curvature bound

$$\limsup_{t \to t_*^-} \|R(t)\|_{L^\infty} < \infty \tag{25}$$

and the solution can be continued beyond t_*.

The curvature bound (25) relies on the parametrix construction for the equation (23). In the construction of a parametrix for (23) we cannot, in any meaningful way, approximate $\square_g$ by the flat D'Alembertian $\square$. One could instead proceed via a geometric optics construction of parametrices for $\square_g$, as developed in [F]. Such an approach would require additional bounds on the background geometry, determined by the metric g, incompatible with the assumption (24) and the implied finiteness of the curvature flux.

We rely instead on a geometric version, which we develop in [K-R6], of the Kirchoff–Sobolev formula, similar to that used by Sobolev in [Sob] and Choquet-Bruhat in [CB1], see also [Mo]. Roughly, this can be obtained by applying to (23) the measure $A\delta(u)$, where u is an optical function whose level set $u = 0$ coincides with $N^-(p)$ and A is a 4-covariant 4-contravariant tensor defined as a solution of a transport equation along $N^-(p)$ with appropriate (blowing-up) initial data at the vertex p. After a careful integration by parts we arrive at the following analogue of the Kirchoff formula:

$$R(p) = -\int_{N^-(p;\delta_0)} A \cdot (R \star R) + R^0(p;\delta_0) + \int_{N^-(p;\delta_0)} \mathcal{E} \cdot R, \tag{26}$$

where $N^-(p;\delta_0)$ denotes the portion of the null boundary $N^-(p)$ in the time interval $[t(p) - \delta_0, t(p)]$ and the error term $\mathcal{E}$ depends only on the intrinsic geometry of $N^-(p;\delta_0)$. The term $R^0(p;\delta_0)$ is completely determined by the initial data on the hypersurface $\Sigma_{t(p)-\delta_0}$. As in the flat case[12], one can prove bounds for the sup-norm of $R^0(p;\delta_0)$ which depend only on uniform bounds for R and its first covariant derivatives at values of $t' \leq t(p) \leq t - \delta$.

As in the Yang–Mills setting the structure of the term $R \star R$ allows us to estimate one of the curvature terms by the flux of curvature.

[12]This is by no means obvious as we need to rely once more on the Kirchoff–Sobolev formula.

To control the error term in (26) one needs estimates for tangential derivatives of A and other geometric quantities associated to the null hypersurfaces $N^{-}(p)$. In particular, it requires showing that $N^{-}(p)$ remains a *smooth* (not merely Lipschitz) hypersurface in the time slab $(t(p) - \delta_0, t(p)]$ for some δ_0 dependent only on the initial data and (24). Thus to prove the desired theorem one would have to show that all geometric quantities, arising in the parametrix construction, can be estimated only in terms of the flux of the curvature along $N^{-}(p)$ and the bound in (24). Yet, to start with, it is not even clear that we can provide a lower bound for the radius of injectivity of $N^{-}(p)$. In other words the congruence of null geodesics, initiating at p, may not be controllable[13] only in terms of the curvature flux. Typically, in fact, lower bounds for the radius of conjugacy of a null hypersurface in a Lorentzian manifold are only available in terms of the sup-norm of the curvature tensor R along the hypersurface, while the problem of short, intersecting, null geodesics appears not to be fully understood even in that context. The situation is similar to that in Riemannian geometry, exemplified by the Cheeger's theorem, where pointwise bounds on sectional curvature are sufficient to control the radius of conjugacy but to prevent the occurrence of short geodesic loops one needs to assume in addition an upper bound on the diameter and a lower bound on the volume of the manifold.

In a sequence of papers, [K-R2]–[K-R4], we have been able to prove a lower bound, depending essentially only on the curvature flux, for the radius of conjugacy of null hypersurfaces[14] in a Lorentzian spacetime which verifies the Einstein vacuum equations. The methods used in these articles can be adapted to provide all the desired estimates, except a lower bound on the "size" of intersecting null geodesics which needs a separate argument. The lower bound on the radius of injectivity of the null hypersurfaces $N^{-}(p)$ has been established in [K-R5]

3. Stability problems in General Relativity

In the absence of a general "large data" result in General Relativity the problem of stability of special solutions becomes simultaneously more important and more tractable. Despite our best efforts however these stability questions appear to be quite difficult and still poorly understood. A singular achievement in this regard has been the proof of stability of Minkowski space-time, [C-K], [L-R2] and a semi-global version in [Fr1]. To this date this is the only global result in the category of the asymptotically flat space-times. In the realm of cosmological models stability of the de Sitter space has been shown in [Fr2], [A2]. Finally one should also mention the proof of stability in the expanding direction of a flat cone solution for spatially compact space-times [A-M2], [Re].

[13] Different null geodesics of the congruence may intersect, or the congruence itself may have conjugate points, arbitrarily close to p.

[14] together with many other estimates of various geometric quantities associated to $N^{-}(p)$.

3.1. Stability of the Minkowski space-time. The problem of stability of Minkowski space-time for the Einstein-vacuum equations can be described as follows:

Show existence of a causally geodesically complete vacuum space-time asymptotically "converging" to the Minkowski space-time for an arbitrary set of smooth asymptotically flat initial data $(\Sigma_0, g_{0ij}, k_{0ij})$ *with* $\Sigma_0 \approx R^3$,

$$g_{0ij} = \left(1+\frac{M}{r}\right)\delta_{ij} + o(r^{-1-\alpha}), \quad k_{0ij} = o(r^{-2-\alpha}), \quad r = |x| \to \infty,\ \alpha > 0 \tag{27}$$

where $(g_0 - \delta)$ *and* k_0 *satisfy global smallness assumptions* .

A positive parameter M in the asymptotic expansion for the metric g_0 is the ADM mass, positive according to [S-Y1], [W].

The stability of Minkowski space for the Einstein-vacuum equations was shown in a remarkable work of Christodouolou–Klainerman for strongly asymptotic initial data (the parameter $\alpha \geq 1/2$ in the asymptotic expansion (27)), [C-K] The approach taken in that work viewed the Einstein-vacuum equations as a system of equations

$$D^\alpha W_{\alpha\beta\gamma\delta} = 0, \quad D^\alpha * W_{\alpha\beta\gamma\delta} = 0$$

for the Weyl tensor $W_{\alpha\beta\gamma\delta}$ of the metric $g_{\alpha\beta}$ and used generalized energy inequalities associated with the Bel–Robinson energy-momentum tensor, constructed from components of W, and special geometrically constructed vector fields, designed to mimic the rotation and the conformal Morawetz vector fields of the Minkowski space-time, i.e., "almost conformally Killing" vector fields of the unknown metric g. The proof was manifestly invariant, in particular it did not use the harmonic coordinate gauge. This approach was later extended to the Einstein–Maxwell equations by N. Zipser [Z].

In [L-R2] we succeeded in developing a new relatively technically simple approach which allowed allowing us to prove stability of Minkowski space in harmonic coordinate gauge, for general asymptotically flat data, $\alpha > 0$, and simultaneously treat the case of the Einstein equations coupled to a scalar field,

$$R_{\alpha\beta}(g) = \partial_\alpha\phi\, \partial_\beta\phi, \quad \Box_g\phi = 0$$

where the scalar field requires a global smallness assumption on its initial data (ϕ_0, ϕ_1), which obey the asymptotic expansion

$$\phi_0 = o(r^{-1-\alpha}), \quad \phi_1 = o(r^{-2-\alpha}). \tag{28}$$

Theorem 3 ([L-R2]). *Let* $(\Sigma, g_0, k_0, \phi_0, \phi_1)$ *be initial data for the Einstein-scalar field equations. Assume that the initial time slice* Σ *is diffeomorphic to* R^3 *and admits a global coordinate chart relative to which the data is close to the initial data for the Minkowski space-time. More precisely, we assume that the data* $(g_0, k_0, \phi_0, \phi_1)$ *is smooth asymptotically flat in the sense of* (27)–(28) *with mass* M *and* $\alpha > 0$ *and satisfy a global smallness assumption as measured in the scale of weighted Sobolev spaces. Then the Einstein-scalar field equations possess a future causally geodesically* *complete solution* (g, ψ) *asymptotically converging to Minkowski space-time. In fact, there exists a global harmonic system coordinates relative to which the metric g remains close (and "converges") to the Minkowski metric.*

The appeal of the harmonic gauge for the proof of stability of Minkowski space-time lies in the fact that the latter can be simply viewed[15] as a *small data global existence result* for the quasilinear system (5) (for vacuum equations),

$$\Box_g g_{\alpha\beta} = N_{\alpha\beta}(g, \partial g), \quad g_{\alpha\beta}|_{t=0} = g^0_{\alpha\beta}, \ \partial_t g_{\alpha\beta}|_{t=0} = g^1_{\alpha\beta}. \tag{29}$$

However, usefulness of the harmonic gauge in this context was questioned earlier and it was suspected that harmonic coordinates are "unstable in the large", [CB1]. The conclusion is suggested from the analysis of the iteration scheme for the system (29), which resembles an iteration scheme for the semilinear equation $\Box\phi = (\partial_t \phi)^2$ shown to blow up in finite time for arbitrarily small initial data by F. John, [J].

To describe some of the difficulties in establishing a small data global existence result for the system (29) consider a generic quasilinear system of the form

$$\Box \phi_i = \sum b_i^{jk\alpha\beta}\, \partial_\alpha \phi_j\, \partial_\beta \phi_k + \sum c_i^{jk\alpha\beta}\, \phi_j\, \partial_\alpha \partial_\beta \phi_k + \text{cubic terms}. \tag{30}$$

The influence of cubic terms is negligible while the quadratic terms are of two types, the *semilinear terms* and the *quasilinear terms*, each of which present their own problems. D. Christodoulou [C1] and S. Klainerman [K2] showed global existence for systems of the form (30) if the semilinear terms satisfy the *null condition* and the quasilinear terms are absent. The null condition, first introduced by S. Klainerman in [K1], was designed to detect systems for which solutions are asymptotically free and decay like solutions of a linear equation. It requires special algebraic cancellations in the coefficients $b_i^{jk\alpha\beta}$, e.g. $\Box\,\phi = (\partial_t \phi)^2 - |\nabla_x \phi|^2$. However, the semilinear terms for the Einstein equations do not satisfy the null condition, see [CB2]. The quasilinear terms is another source of trouble. The only non-trivial example of a quasilinear equation of the type (30), for which the small data global existence result holds, is the model equation $\Box\,\phi = \phi\,\Delta\phi$, as shown in [L1] (radial case) and [Al] (general case), see also [L2].

In [L-R1] we identified a criteria under which it is more likely that a quasilinear system of the form (30) has global solutions[16]. We said that a system of the form (30) satisfy the *weak null condition* if the corresponding *asymptotic system* (c.f. [H]) has global solutions. We showed that the Einstein equations in harmonic coordinates satisfy the weak null condition. In addition an additional cancellation mechanism was found for the Einstein equations in harmonic coordinates that makes it better than a

[15] This statement requires additional care since a priori there is no guarantee that obtained "global in time" solution $g_{\mu\nu}$ defines a causally geodesically complete metric. However, the latter can be established provided one has good control on the difference between $g_{\mu\nu}$ and the Minkowski metric $m_{\mu\nu}$.

[16] At this point, it is unclear whether this criteria is sufficient for establishing a "small data global existence" result for a *general* system of quasilinear hyperbolic equations.

general system satisfying the weak null condition. The system decouples to leading order, when decomposed relative to the Minkowski *null frame*. An approximate model that describes the semilinear terms has the form

$$\Box\,\phi_2 = (\partial_t \phi_1)^2, \quad \Box\,\phi_1 = 0.$$

While every solution of this system is global in time, the system fails to satisfy the classical null condition and solutions are not asymptotically free: $\phi_2 \sim \varepsilon t^{-1} \ln |t|$. The semilinear terms in Einstein's equations can be shown to either satisfy the classical null condition or decouple in the above fashion when expressed in a null frame. The quasilinear terms also decouple but in a more subtle way. The influence of quasilinear terms can be detected via asymptotic behavior of the characteristic surfaces of metric g. It turns out that the main features of the characteristic surfaces at infinity are determined by a particular *null* component of the metric. The asymptotic flatness of the initial data and the harmonic coordinate condition (3)

$$\partial_\beta \big(g^{\alpha\beta} \sqrt{|g|}\, \big) = 0 \tag{31}$$

give good control of this particular component, i.e., $\sim M/r$, which in turn implies that the light cones associated with the metric g diverge only logarithmically $\sim M \ln t$ from the Minkowski cones. The main simplification in our approach comes from the fact the behavior of the system (29) coupled to the harmonic gauge (31) can be completely controlled by means of the generalized energy estimates exploiting only the *exact* symmetries of Minkowski space thus avoiding having to construct dynamically generators of the approximate symmetries of the space-time (M, g).

The asymptotic behavior of null components of the Riemann curvature tensor $R_{\alpha\beta\gamma\delta}$ of metric g – the so called "peeling estimates" – was discussed in the works of Bondi, Sachs and Penrose and becomes important in the framework of asymptotically simple space-times (roughly speaking, space-times which can be conformally compactified), see also the paper of Christodoulou [C4] for further discussion of such space-times. The work of [C-K] provided very precise, although not entirely consistent with peeling estimates, analysis of the asymptotic behavior of constructed global solutions. However, global solutions obtained by Klainerman–Nicolo [K-N1] in the problem of exterior[17] stability of Minkowski space were shown to possess peeling estimates for special initial data, [K-N2]. The work [L-R2] is less precise about the asymptotic behavior of the curvature components.

3.2. Beyond stability of Minkowski space-time. The simplest solutions of the Einstein vacuum equations of general relativity,

$$R_{\mu\nu} = 0, \tag{32}$$

[17] Outside of the domain of dependence of a compact set.

containing black holes are the one-parameter Schwarzschild family of solutions. In the exterior region ($r > 2M$) the Schwarzschild metric can be written in the form

$$g_s = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\sigma_{S^2}.$$

The Schwarzschild family is a sub-family of the two-parameter Kerr family which describe stationary rotating black holes. In its proper rigorous formulation, the problem of nonlinear stability of the Kerr family is one of the major open problems in general relativity. In particular, it is conjectured that perturbations of Schwarzschild initial data should evolve into a spacetime with complete null infinity whose past "suitably" approaches a nearby Kerr exterior. At the heuristic level, however, considerable progress has been made in the last 40 years towards an understanding of the issues involved. In particular, a very influential role was played by the work of R. Price [Pr] in 1972, who discovered a heuristic mechanism, known in the physics literature as the *red-shift effect*, allowing for the decay of scalar field linear perturbations on the Schwarzschild exterior, i.e., solutions of the linear wave equation

$$\Box_{g_s}\phi = 0.$$

Despite the abundance of heuristic and numerical arguments the nonlinear problem is still lacking proper mathematical understanding while some progress has been made recently on a problem of asymptotic behavior of the linear problem. The causal picture of the Schwarzschild space-time is very different from the one of the Minkowski space. In (a right quadrant of) Schwarzschild space-time in addition to the null infinity, parametrized by $(u, \omega) \in R \times S^2$ there is a special null hypersurface, the *event horizon*, parametrized by $(v, \omega) \in R \times S^2$, separating the exterior region from the black hole. It is the presence of the event horizon that is responsible for the red-shift effect in which the frequency of an observer leaving the exterior region gets shifted to the red as viewed by the second observer positioned to the future of the first one. The geometry of the Schwarzschild space-time also ultimately determines the behavior of linear waves. However, even uniform boundedness of solutions of linear scalar wave equations, almost trivial in Minkowski context, is by no means obvious and is termed as linear stability of Schwarzschild in the physics literature. This result was rigorously established in the work of Kay and Wald [K-W]. Decay for ϕ, without a rate, was first proven in [Tw].

Theorem 4 ([D-R2]). *Let ϕ be a sufficiently regular solution of the wave equation*

$$\Box_{g_s}\phi = 0 \tag{33}$$

on the (maximally extended) Schwarzschild spacetime $(\mathcal{M}, g)$, decaying suitably at spatial infinity on an arbitrary complete asymptotically flat Cauchy surface Σ. Fix retarded and advanced Eddington–Finkelstein coordinates u and v. We have the *following pointwise decay rates*

$$\begin{aligned} |\phi| &\le C \max(1, v)^{-1} \quad \text{in } r \ge 2M \\ |r\phi| &\le C_{\hat{R}}(1 + |u|)^{-\frac{1}{2}} \quad \text{in } \{r \ge \hat{R} > 2M\} \cap J^+(\Sigma). \end{aligned} \tag{34}$$

A variant of the problem considered here is also studied in [B-S].

In the spherically symmetric case, the above result follows from a very special case of [D-R1], where the so called Price law has been established. (See also [M-S].)

For the more general Kerr family, even uniform boundedness remains an open problem (see however [FKSY]).

The proof of Theorem (4) is based on the energy type estimates for (33) with vector fields adapted to different regions of space-times. An important role in this analysis is played by the "red-shift vector field", which has no equivalent in Minkowski space, constructed near the event horizon.

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