Categorification and correlation functions in conformal field theory

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Abstract. A modular tensor category provides the appropriate data for the construction of a three-dimensional topological field theory. We describe the following analogue for two-dimensional conformal field theories: a 2-category whose objects are symmetric special Frobenius algebras in a modular tensor category and whose morphisms are categories of bimodules. This 2-category provides sufficient ingredients for constructing all correlation functions of a two-dimensional rational conformal field theory. The bimodules have the physical interpretation of chiral data, boundary conditions, and topological defect lines of this theory.

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1. Quantum field theories as functors

In approaches to quantum field theory that are based on the concepts of fields and states, the utility of categories and functors is by now well-established. The following pattern has been recognized: There is a geometric category $\mathcal{G}$ which, for every concrete model, must be suitably "decorated". The decoration is achieved with the help of objects and morphisms from another category $\mathcal{C}$. For known classes of quantum field theories, the decoration category $\mathcal{C}$ typically has a representation-theoretic origin – the reader is encouraged to think of it as the representation category of some algebraic object, like a quantum group, a loop group, a vertex algebra, a net of observable algebras, etc. This way one obtains a decorated geometric category $\mathcal{G}_{\mathcal{C}}$. The quantum field theory can then be formulated as a (tensor) functor $qft_{\mathcal{C}}$ from $\mathcal{G}_{\mathcal{C}}$ to some category of vector spaces. In this contribution, we mainly consider cases for which this latter category is the tensor category of finite-dimensional complex vector spaces.

A prototypical example for this pattern is provided by topological quantum field theories (TFTs). For such theories, the geometric category $\mathcal{G}$ is based on a cobordism category: its objects are $d-1$-dimensional topological manifolds without boundary. It is convenient to include two types of morphisms [42]: homeomorphisms of $d-1$-dimensional manifolds, and cobordisms. A cobordism $\mathrm{M}\colon \mathrm{Y}_1 \to \mathrm{Y}_2$ is a d-dimen-

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sional topological manifold M together with a parametrization of its boundary given by a homeomorphism $\partial \mathrm{M} \xrightarrow{\cong} \overline{\mathrm{Y}}_1 \sqcup \mathrm{Y}_2$, where $\overline{\mathrm{Y}}_1$ has the orientation opposite to the one of Y_1. The composition of morphisms is by concatenation, by gluing, and by changing the parametrization of the boundary, respectively. Cobordisms that coincide by a homeomorphism of the d-manifold M restricting to the identity on ∂M must be identified.

In the simplest case, a topological field theory thus associates to a closed $d-1$-dimensional manifold X a vector space $\mathit{qft}_{\mathcal{C}}(\mathrm{X})$, and to a homeomorphism or a cobordism $\mathrm{M}\colon \mathrm{Y}_1 \to \mathrm{Y}_2$ a linear map

$$\mathit{qft}_{\mathcal{C}}(\mathrm{M})\colon \mathit{qft}_{\mathcal{C}}(\mathrm{Y}_1) \to \mathit{qft}_{\mathcal{C}}(\mathrm{Y}_2).$$

The assignment $\mathit{qft}_{\mathcal{C}}$ is required to be a (strict) tensor functor. This requirement implies the usual axioms (cf. e.g. [43]) of naturality, multiplicativity, functoriality and normalization.

There also exists a path-integral approach to certain classes of topological field theories. Its relation to the categorical framework described above is as follows: One can think of the vector space $\mathit{qft}_{\mathcal{C}}(\partial_-\mathrm{M})$ as the space of (equivalence classes of) possible initial data for "fields" in the path integral, and of $\mathit{qft}_{\mathcal{C}}(\partial_+\mathrm{M})$ as the possible final data. The matrix elements of the linear map $\mathit{qft}_{\mathcal{C}}(\mathrm{M})$ are then the transition amplitudes for fixed initial and final values of the fields.

This picture is still oversimplified. In particular, it turns out that it is natural to enrich also the geometric category $\mathcal{G}$ over the category of complex vector spaces. As a consequence, when studying functors on $\mathcal{G}$, one should then consider also projective functors. These issues, which are closely related to anomalies in quantum field theory will, however, be suppressed in this note.

A prominent class of examples of 3-dimensional topological field theories arises from Chern–Simons field theories. For G a simple connected and simply-connected complex Lie group, consider holomorphic G-bundles on a closed two-manifold X of genus g with complex structure. Pick a generator $\mathcal{L}$ for the Picard group of the moduli space $\mathcal{M}_{\mathrm{X}}^G$ of such bundles. Upon changing the complex structure of X, the vector spaces $H^0(\mathcal{M}_{\mathrm{X}}^G, \mathcal{L}^{\otimes k})$ fit together into a vector bundle with projectively flat connection over the moduli space $\mathcal{M}_g^G$ of curves of genus g. The complex modular functor [3] associates these bundles to X; these bundles and their monodromies provide a formalization of all aspects of the chiral level-k Wess–Zumino–Witten (WZW) conformal field theory for G that are needed for the discussions in the subsequent sections.

As a next step, it is natural to extend the formalism by allowing for marked points with additional structure on the two-manifolds. From a field theoretical point of view, this is motivated by the desire to account for insertions of fields. In the case of Chern–Simons theory, the additional structure amounts to specifying parabolic structures at the marked points. The marked points thus have to carry labels, which we will identify in a moment as objects of a decoration category $\mathcal{C}$.

This structure must be extended to the geometric morphisms: Maps of 2-dimensional manifolds are required to preserve marked points and the decoration in $\mathcal{C}$. The decoration of the 2-dimensional manifolds is extended to the 3-dimensional manifolds M underlying cobordisms by supplying them with oriented (ribbon) graphs ending on (arcs through) the marked points on $\partial_\pm$M. The ribbon graph is allowed to have vertices with a finite number of ingoing and outgoing ribbons. From the construction of invariants of knots and links, it is known that this enforces a categorification of the set of labels: $\mathcal{C}$ must be a ribbon category, i.e. a braided sovereign tensor category. In particular, the vertices of the graph are to be labeled by morphisms in the decoration category $\mathcal{C}$.

This approach has been very fruitful and has, in particular, made a rigorous construction of Chern–Simons theory possible [38], [43]. The extension from invariants of links in $\mathbb{R}^3$ to link invariants in arbitrary oriented three-manifolds has revealed an important subclass of tensor categories: modular tensor categories.

For the purposes of the present contribution, we adopt the following definition of a modular tensor category: it is an abelian, $\mathbb{C}$-linear, semi-simple ribbon category with a finite number of isomorphism classes of simple objects. The tensor unit **1** is required to be simple, and the braiding must be nondegenerate in the sense that the natural transformations of the identity functor on $\mathcal{C}$ are controlled by the fusion ring $K_0(\mathcal{C})$:

$$\mathrm{End}(\mathrm{Id}_{\mathcal{C}}) \cong K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}.$$

The relation between Chern–Simons theory and chiral Wess–Zumino–Witten theory [44] was a first indication that modular tensor categories also constitute the appropriate mathematical formalization of the chiral data [31], [16] of a conformal field theory. Recent progress in representation theory has made this idea much more precise; for the following classes of representation categories it has been established that they carry the structure of a modular tensor category:

- The representation category of a connected ribbon factorizable weak Hopf algebra over $\mathbb{C}$ (or, more generally, over an algebraically closed field $\Bbbk$) with a Haar integral [33].

- The category of unitary representations of the double of a connected C^* weak Hopf algebra [33].

- The category of local sectors of a finite-index net of von Neumann algebras on the real line, if the net is strongly additive (which for conformal nets is equivalent to Haag duality) and has the split property [27].
  (In this example and in the previous one one obtains unitary modular tensor categories.)

- The representation category of a self-dual vertex algebra that obeys Zhu's C_2 cofiniteness condition and certain conditions on its homogeneous subspaces, provided that this category is semisimple [25].

The last two entries in this list correspond to two different mathematical formalizations of chiral conformal field theories. The results of [27] and [25] therefore justify the point of view that modular tensor categories constitute an œcumenic formalization of the chiral data of a conformal field theory.

2. Two-dimensional conformal field theories

Three-dimensional topological field theory will indeed appear as a tool in constructions below. Our main interest here is, however, in a different class of quantum field theories: full (and in particular local) two-dimensional conformal field theories, or CFTs, for short.

For these theories, the geometric category of interest is the category of two-dimensional conformal manifolds, possibly with non-empty boundary. This fact already indicates that full conformal field theories are different from the chiral conformal field theories that we encountered in the last section for the case of WZW theories. Morphisms in the category of conformal manifolds are maps respecting the conformal structure. Actually, there are two different types of full conformal field theories, corresponding to two different geometric categories: One considers either oriented conformal manifolds, leading to a category $\mathcal{G}^{\mathrm{or}}$, or unoriented manifolds, leading to a different geometric category $\mathcal{G}^{\mathrm{unor}}$. As morphisms, we admit maps that preserve the respective structure. (In the application of conformal field theory to string theory, $\mathcal{G}^{\mathrm{or}}$ plays a role in superstrings of "type II", while $\mathcal{G}^{\mathrm{unor}}$ appears in superstring theories of "type I".)

As in the case of topological field theories, the geometric category needs to be decorated. To find the appropriate decoration data, we first discuss a physical structure that is known to be present in specific classes of models and that our approach to conformal field theory should take into account:

- Whenever a two-manifold X has a boundary, one expects that it is necessary to specify boundary conditions. In a path integral approach, a boundary condition is a prescription for the boundary values of fields that appear in the Lagrangian. Here, a more abstract approach is adequate: we take the possible boundary conditions to be the objects of a decoration category $\mathcal{M}$. This constitutes again a categorification of the decoration data. It can be motivated further by the observation that insertions of boundary fields can change the boundary condition; they will be related to morphisms of the category $\mathcal{M}$.

A second structure, which in the literature has received much less attention than boundary conditions, turns out to provide crucial clues for the construction of full conformal field theories:

- Conformal field theories can have topological defect lines. Such defect lines have e.g. been known for the Ising model for a long time: This CFT describes the continuum limit of a lattice model with $\mathbb{Z}_2$-valued variables at the vertices of a two-dimensional lattice and with ferromagnetic couplings along its edges. A defect line is obtained when one changes the couplings on all edges that intersect a given line in the lattice from ferromagnetic to antiferromagnetic.

In the continuum limit, such a defect line can be described by a condition on the values of bulk fields at the defect line. In particular, when crossing a defect line, the correlation function of a bulk field can acquire a branch cut. Indeed, the defect lines we are interested in behave very much like branch cuts: they are topological in the sense that their precise location does not matter. In a field theoretic language, this is attributed to the fact that the stress-energy tensor of the theory is required to be smooth across defect lines. As in the case of boundary conditions, in our framework it is not desirable to express defect lines through conditions on the values of fields. Instead, we anticipate again a categorification of the decoration data and label the possible types of defect lines by objects in yet another decoration category $\mathcal{D}$.

There is a natural notion of fusion of defect lines, see e.g. [36], [15]. Accordingly, $\mathcal{D}$ will be a tensor category. Also, to take into account the topological nature of defect lines, we assume that the tensor category $\mathcal{D}$ has dualities and that it is even sovereign. In contrast, there is no natural notion of a braiding of defect lines, so $\mathcal{D}$ is, in general, not a ribbon category.

The two decoration categories – $\mathcal{D}$ for the defects and $\mathcal{M}$ for the boundary conditions – are themselves related. One can fuse a defect line to a boundary condition, thereby obtaining another boundary condition; see e.g. [24]. This endows the category $\mathcal{M}$ of boundary conditions with the structure of a module category over $\mathcal{D}$, i.e. there is a bifunctor

$$\otimes\colon \mathcal{D} \times \mathcal{M} \to \mathcal{M}$$

which has (mixed) associativity constraints obeying a mixed pentagon identity.

The structure just unraveled – a tensor category $\mathcal{D}$ together with a module category $\mathcal{M}$ over $\mathcal{D}$ – calls for the following natural extension: One should also consider the category of module functors, i.e. the category $\mathcal{C}$ whose objects are endofunctors of $\mathcal{M}$ that are compatible with the structure of a module category over $\mathcal{D}$ and whose morphisms are natural transformations between such functors. The concatenation of functors naturally endows $\mathcal{C}$ with a product so that $\mathcal{C}$ is a tensor category.

A recent insight is the following: In the application to two-dimensional conformal field theory, the category $\mathcal{C}$ obtained this way is equivalent to the category of chiral data that we discussed in Section 1! There is a particularly amenable subclass of conformal field theories, called *rational* conformal field theories (*R*CFTs), which can be rigorously discussed on the basis of this idea. For these theories, the category $\mathcal{C}$ of chiral data has the structure of a modular tensor category. In this case the idea can be exploited to arrive at a construction of correlation functions (see Section 4) of rational conformal field theories that is based on three-dimensional topological field theory. This TFT approach to RCFT correlators will be presented in Section 5 below.

3. The 2-category of Frobenius algebras

In practice, one frequently takes an opposite point of view: Instead of obtaining $\mathcal{C}$ as a functor category, one starts from some knowledge about the chiral symmetries of a conformal field theory. This allows one to use the representation-theoretic results mentioned in Section 1 so as to get a modular tensor category $\mathcal{C}$ describing the chiral data of the theory. Afterwards one realizes that the category $\mathcal{M}$ of boundary conditions is also a (right-) module category over $\mathcal{C}$. General arguments involving internal Hom's [35] together with specific properties relevant to rational conformal field theories then imply that in the tensor category $\mathcal{C}$ there exists an associative algebra A such that $\mathcal{M}$ is equivalent to the category $\mathcal{C}_A$ of left A-modules in $\mathcal{C}$. By similar arguments one concludes that the category $\mathcal{D}$ is equivalent to the category of A-bimodules. Additional constraints, in particular the non-degeneracy of the two-point functions of boundary fields on a disk, lead to further conditions on this algebra [17]: A must be a symmetric special Frobenius algebra. Owing to these insights we are able to use a generalization[1] of the theory of Frobenius algebras to braided tensor categories as a powerful tool to analyze (rational) conformal field theory. The algebraic theory in the braided setting is, however, genuinely richer; see [14] for a discussion of some new phenomena.

A *Frobenius* algebra $A = (A, m, \eta, \Delta, \varepsilon)$ in $\mathcal{C}$ is, by definition, an object of $\mathcal{C}$ carrying the structures of a unital associative algebra (A, m, η) and of a counital coassociative coalgebra (A, Δ, ε) in $\mathcal{C}$, with the algebra and coalgebra structures satisfying the compatibility requirement that the coproduct $\Delta\colon A \to A \otimes A$ is a morphism of A-bimodules (or, equivalently, that the product $m\colon A \otimes A \to A$ is a morphism of A-bi-comodules). A Frobenius algebra is called *special* iff the coproduct is a right-inverse to the product – this means in particular that the algebra is separable – and a nonvanishing multiple of the unit $\eta\colon \mathbf{1} \to A$ is a right-inverse to the counit $\varepsilon\colon A \to \mathbf{1}$. There are two isomorphisms $A \to A^\vee$ that are naturally induced by product, counit and duality; A is called *symmetric* iff these two isomorphisms coincide.

Two algebras in a tensor category $\mathcal{C}$ are called Morita equivalent iff their representation categories are equivalent as module categories over $\mathcal{C}$. Since the algebra A is characterized by the requirement that $\mathcal{C}_A$ is equivalent to the given decoration category $\mathcal{M}$, it is clear that only the Morita class of the algebra should matter. It is a non-trivial internal consistency check on the constructions to be presented in Section 5 that this is indeed the case.

Taking the modular tensor category $\mathcal{C}$ as the starting point, the following further generalization of the setup (compare also [32], [46], [28]) is now natural:[2] Consider the set of *all* (symmetric special) Frobenius algebras in $\mathcal{C}$. This gives rise to a *family* of full conformal field theories that are based on the same chiral data. And

[1] When $\mathcal{C}$ is the modular tensor category of finite-dimensional complex vector spaces, the CFT is a topological CFT. In particular, A is then an ordinary Frobenius algebra. This case has served as a toy model for conformal field theories, see e.g. [40], [30].

[2] We are grateful to Urs Schreiber for discussions on this point.

again we categorify the structure: we introduce a 2-category $\mathcal{F}rob_{\mathcal{C}}$ whose objects are symmetric special Frobenius algebras in $\mathcal{C}$. The 1-morphisms $\mathcal{H}om(A, A')$ are given by the category of A-A'-bimodules. The 2-category $\mathcal{F}rob_{\mathcal{C}}$ has a distinguished object I: as an object of $\mathcal{C}$, I is just the tensor unit, which is naturally a symmetric special Frobenius algebra. Because of the considerations in [8], the full conformal field theory corresponding to I is often referred to as the "Cardy case"; for this case a construction of the correlators in the spirit of Section 4 was given in [11].

We are now in a position to attribute a physical interpretation to the morphisms of $\mathcal{F}rob_{\mathcal{C}}$. $\mathcal{H}om(I, I)$ is naturally identified with the tensor category $\mathcal{C}$ of chiral data. Further, for any A the tensor category $\mathcal{H}om(A, A)$ describes topological defects in the full conformal field theory associated to A; more generally, the category $\mathcal{H}om(A, A')$ accounts for topological defect lines that separate two different conformal field theories which share the same chiral data. Finally, the category $\mathcal{H}om(I, A)$ also describes boundary conditions for the full conformal field theory labeled by A.

We have thus learned that the decoration data of a family of full rational conformal field theories based on the same chiral data are described by a 2-category. This nicely fits with insight gained in other contexts:

- 2-categories appear in recent approaches to elliptic objects [2], [42].
- Hermitian bundle gerbes, which appear naturally in a semi-classical description of WZW conformal field theories [23], form a 2-category [41].

Unfortunately, at the time of writing, a unified approach to conformal field theories based on 2-categories has not been established yet. For this reason, in the sequel we will not be able to use this language systematically.

We close this section with two further comments. First, so far we have discussed the decoration data relevant to the oriented geometric category $\mathcal{G}^{\mathrm{or}}$. For the unoriented geometric category $\mathcal{G}^{\mathrm{unor}}$, additional structure on the relevant Frobenius algebra is needed: A must then be a *Jandl* algebra, that is, a symmetric special Frobenius algebra coming with an algebra isomorphism $A \xrightarrow{\cong} A^{\mathrm{opp}}$ that squares to the twist on A. This turns out to be the appropriate generalization of the notion of an algebra with involution to braided tensor categories. We refrain from discussing this issue in the present contribution, but rather refer to [18], [21] for details.

Second, the general situation encountered above – a module category $\mathcal{M}$ over a tensor category $\mathcal{C}$ – naturally appears in various other contexts as well:

- The left modules over a weak Hopf algebra H form a tensor category $\mathcal{C} = H$-Mod. In a weak Hopf algebra, one can identify two subalgebras H_s and H_t that are each other's opposed algebras [6]. Forgetful functors thus endow any H-module with the structure of an H_t-bimodule; one even obtains a tensor functor from H-Mod to H_t-Bimod. The usual tensor product of a H_t-bimodule and an H_t-left module endows the category of H_t-modules with the structure of a module category over H_t-Bimod and thus over H-Mod.

Weak Hopf algebras have indeed been proposed [37] as a framework for rational conformal field theories. Unfortunately, such a description must cope with two problems: First, to account for a braiding on $\mathcal{C}$, one must work with an R-matrix on H; not too surprisingly, this is technically involved, and indeed not very much is known about R-matrices on weak Hopf algebras. Secondly, given a tensor category $\mathcal{C}$ (describing the chiral data), there does not exist a canonical weak Hopf algebra such that H-Mod is equivalent to $\mathcal{C}$. Rather, as an additional datum, a fiber functor to H_t-bimodules needs to be chosen. A physical interpretation of this datum is unclear. On the other hand, Hopf algebras are still useful in the analysis of full rational CFT: Their Hochschild cohomology was used [10] to compute the Davydov–Yetter cohomology of the pair $(\mathcal{C}, \mathcal{M})$; from the vanishing of this cohomology, rigidity properties of rational conformal field theories follow.

- Weak Hopf algebras also appear in the study of inclusions of subfactors. For a review and further references, we refer to Sections 8 and 9 of [34].
- The same category-theoretic structures have been recovered in the theory of vertex algebras from so-called open-string vertex algebras [26] which are, in particular, extensions of ordinary vertex algebras.

Not surprisingly, some of the structures that will be encountered in the rest of this paper also have counterparts in the context of weak Hopf algebras, of nets of subfactors, or of open-string vertex algebras.

4. Correlation functions

The observations made in the preceding section raise the question whether one can construct a full rational conformal field theory by using a modular tensor category $\mathcal{C}$ and the 2-category $\mathcal{F}rob_{\mathcal{C}}$ as an input. These data should then in particular encode information about the correlation functions of the conformal field theory.

To decide this question, it is helpful to reformulate first the geometric categories $\mathcal{G}^{\mathrm{or}}$ and $\mathcal{G}^{\mathrm{unor}}$. This is achieved with the help of a crucial aspect of complex geometry in *two* dimensions: a complex structure on a two-dimensional manifold is equivalent to a conformal structure and the choice of an orientation. The complex double $\widehat{\mathrm{X}}$ of a conformal manifold X is a two-sheeted cover of X whose points are pairs consisting of a point $p \in \mathrm{X}$ and a local orientation at p. In view of the previous comment, it is clear that $\widehat{\mathrm{X}}$ is a complex curve. We have thus associated to any object X of the geometric categories $\mathcal{G}^{\mathrm{unor}}$ and $\mathcal{G}^{\mathrm{or}}$ a complex curve $\widehat{\mathrm{X}}$ that comes with an orientation reversing involution σ such that the quotient $\widehat{\mathrm{X}}/\langle\sigma\rangle$ is naturally isomorphic to X, and we have a canonical projection

$$\pi : \widehat{\mathrm{X}} \mapsto \mathrm{X} \cong \widehat{\mathrm{X}}/\langle\sigma\rangle.$$

The set of fixed points of σ is just the preimage under π of the boundary $\partial\mathrm{X}$.

To be able to use the tools of complex geometry, we therefore reformulate our geometric categories as follows: in the case of $\mathcal{G}^{\text{unor}}$, the objects are pairs $(\widehat{\mathrm{X}}, \sigma)$ consisting of a complex curve $\widehat{\mathrm{X}}$ and an anticonformal involution σ that implements the action of the Galois group of $\mathbb{C}/\mathbb{R}$. In the case of $\mathcal{G}^{\text{or}}$, we fix a global section of π as an additional datum.

Next, since we are interested in correlation functions depending on insertion points, it is natural to consider simultaneously the family $\mathcal{M}_{g,m}$ of all complex curves with marked points that have the same topological type (i.e., genus g and number m of marked points) as $\widehat{\mathrm{X}}$. It is convenient to treat the positions of the insertion points and the moduli of the complex structure on the same footing. The curves that admit an involution σ of the same type as $\widehat{\mathrm{X}}$ and for which the marked points are related by σ form a submanifold $\mathcal{M}^{\sigma}_{g,m}$ of $\mathcal{M}_{g,m}$. (To be precise, one obtains [7] such a relation for Teichmüller spaces, rather than for moduli spaces.)

Given the modular tensor category $\mathcal{C}$, the complex modular functor [3] provides us with a vector bundle $\mathcal{V}$ with projectively flat connection on $\mathcal{M}_{g,m}$. We can now formulate the 'principle of holomorphic factorization' (which for certain classes of conformal field theories follows from chiral Ward identities that can formally be derived from an action functional [45]). It states that, first of all, the conformal surface X should be decorated in such a way that the double $\widehat{\mathrm{X}}$ has the structure of an object in the decorated cobordism category for the topological field theory based on $\mathcal{C}$. It then makes sense to require, secondly, that the correlation function is a certain global section of the restriction of $\mathcal{V}$ to $\mathcal{M}^{\sigma}_{g,m}$.

At this point, it proves to be convenient to use the equivalence of the complex modular functor and the topological modular functor $\mathit{tft}_{\mathcal{C}}$ based on the modular tensor category $\mathcal{C}$ [3] so as to work in a topological (rather than complex-analytic) category. We are thereby lead to the description of a correlation function on X as a specific vector Cor(X) in the vector space $\mathit{tft}_{\mathcal{C}}(\widehat{\mathrm{X}})$ that is assigned to the double $\widehat{\mathrm{X}}$ by the topological modular functor $\mathit{tft}_{\mathcal{C}}$. These vectors must obey two additional axioms:

- *Covariance*: Given any morphism $f : \mathrm{X} \to \mathrm{Y}$ in the relevant decorated geometric category $\mathcal{G}_{\mathcal{C}}$, we demand

$$\mathrm{Cor}(\mathrm{Y}) = \mathit{tft}_{\mathcal{C}}(f)(\mathrm{Cor}(\mathrm{X})).$$

- *Factorization*: Certain factorization properties must be fulfilled.

  We refer to [12], [13] for a precise formulation of these constraints.

The covariance axiom implies in particular that the vector Cor(X) is invariant under the action of the mapping class group $\mathrm{Map}(\mathrm{X}) \cong \mathrm{Map}(\widehat{\mathrm{X}})^{\sigma}$. This group, also called the relative modular group [4], acts genuinely on $\mathit{tft}_{\mathcal{C}}(\widehat{\mathrm{X}})$.

5. Surface holonomy

To find solutions to the covariance and factorization constraints on the vectors $\mathrm{Cor}(\mathrm{X}) \in \mathit{tft}_{\mathcal{C}}(\widehat{\mathrm{X}})$ we use the three-dimensional topological field theory associated to the modular tensor category $\mathcal{C}$. Thus we look for a (decorated) cobordism $(\mathrm{M_X}, \emptyset, \widehat{\mathrm{X}})$ such that the vector $\mathit{tft}_{\mathcal{C}}(\mathrm{M_X}, \emptyset, \widehat{\mathrm{X}})1 \in \mathit{tft}_{\mathcal{C}}(\widehat{\mathrm{X}})$ is the correlator $\mathrm{Cor}(\mathrm{X})$.

The three-manifold $\mathrm{M_X}$ should better not introduce any topological information that is not already contained in X. This leads to the idea to use an interval bundle as a "fattening" of the world sheet. It turns out that the following quotient of the interval bundle on $\widehat{\mathrm{X}}$, called the connecting (three-) manifold, is appropriate [11]:

$$\mathrm{M_X} = (\,\widehat{\mathrm{X}} \times [-1, 1]\,)/\langle(\sigma, t \mapsto -t)\rangle.$$

This three-manifold is oriented, has boundary $\partial \mathrm{M_X} \cong \widehat{\mathrm{X}}$, and it contains X as a retract: the embedding ι of X is to the fiber $t = 0$, the retracting map contracts along the intervals.

The connecting manifold $\mathrm{M_X}$ must now be decorated with the help of the decoration categories $\mathcal{H}om(A, A')$. We will describe this procedure for the oriented case only. The conformal surface X is decomposed by defect lines (which are allowed to end on $\partial\mathrm{X}$) into various two-dimensional regions. There are two types of one-dimensional structures: boundary components of X and defect lines. Defect lines, in general, form a network; they can be closed or have end points, and in the latter case they can end either on the boundary or in the interior of X. Both one-dimensional structures are partitioned into segments by marked "insertion" points. The end points of defect lines carry insertions, too. Finally, we also allow for insertion points in the interior of two-dimensional regions.

To these geometric structures, data of the 2-category $\mathcal{F}rob_{\mathcal{C}}$ are now assigned as follows. First, we attach to each two-dimensional region a symmetric special Frobenius algebra, i.e. an object of $\mathcal{F}rob_{\mathcal{C}}$. To a segment of a defect line that separates regions with label A and A', respectively, we associate a 1-morphism in $\mathcal{H}om(A, A')$, i.e. an A-A'-bimodule. Similarly, to a boundary segment adjacent to a region labeled by A, we assign an object in $\mathcal{H}om(I, A)$, i.e. a left A-module. Finally, zero-dimensional geometric objects are labeled with 2-morphisms of $\mathcal{F}rob_{\mathcal{C}}$; in particular, junctions of defect lines with each other or with a boundary segment are labeled by 2-morphisms from the obvious spaces.

Two types of points, however, still deserve more comments: those separating boundary segments on the one hand, and those separating or creating segments of defect lines or appearing in the interior of two-dimensional regions on the other. These are the *insertion points* that were mentioned above. An insertion point $p \in \partial\mathrm{X}$ that separates two boundary segments labeled by objects $M_1, M_2 \in \mathcal{H}om(I, A)$ has a single preimage under the canonical projection π from $\widehat{\mathrm{X}}$ to X; to the interval in $\mathrm{M_X}$ that joins this preimage to the image $\iota(p)$ of p under the embedding ι of X into $\mathrm{M_X}$, we assign an object U of the category $\mathcal{C} = \mathcal{H}om(I, I)$ of chiral data. To the insertion point itself, we then attach a 2-morphism in the morphism space $\mathrm{Hom}(M_1 \otimes U, M_2)$ in $\mathcal{H}om(I, A)$.

An insertion point in the interior of X has two preimages on $\widehat{\mathrm{X}}$; these two points are connected to $\iota(p)$ by two intervals. To each of these two intervals we assign to each of these two intervals an object U and V, respectively, of the category $\mathcal{C}$ of chiral data. In the oriented case, the global section of π is used to attribute the two objects U, V to the two preimages. (For the unoriented case, the situation is more involved; in particular, the Jandl structure on the relevant Frobenius algebra enters the prescription.) We now first consider an insertion point separating a segment of a defect line labeled by an object $B_1 \in \mathcal{H}om(A, A')$ from a segment labeled by $B_2 \in \mathcal{H}om(A, A')$. We then use the left action ρ_l of A and the right action ρ_r of A' on the bimodule B_1 to define a bimodule structure on the object $U \otimes B_1 \otimes V$ of $\mathcal{C}$ by taking the morphisms $(\mathrm{id}_U \otimes \rho_l \otimes \mathrm{id}_V) \circ (c^{-1}_{U,A} \otimes \mathrm{id}_{B_1} \otimes \mathrm{id}_V)$ and $(\mathrm{id}_U \otimes \rho_r \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes \mathrm{id}_{B_1} \otimes c^{-1}_{A',V})$ as the action of A and A', respectively, where c denotes the braiding isomorphisms of $\mathcal{C}$. The insertion point separating the defect lines is now labeled by a 2-morphism in $\mathrm{Hom}(U \otimes B_1 \otimes V, B_2)$, i.e. by a morphism of A-A'-bimodules.

To deal with insertion points in the interior of a two-dimensional region labeled by a Frobenius algebra A, we need to invoke one further idea: such a region has to be endowed with (the dual of) a triangulation Γ. To each edge of Γ we attach the morphism $\Delta \circ \eta \in \mathrm{Hom}(\mathbf{1}, A \otimes A)$, and to each trivalent vertex of Γ the morphism $\varepsilon \circ m \circ (m \otimes \mathrm{id}_A) \in \mathrm{Hom}(A \otimes A \otimes A), \mathbf{1})$. This pattern is characteristic for notions of surface holonomy. It has appeared in lattice topological field theories [22] and shows up in the surface holonomy of bundle gerbes as well. (For more details, references, and the relation to the Wess–Zumino term of WZW conformal field theories in a Lagrangian description, see [23].)

Now each of the insertion points p that we still need to discuss is located inside a two-dimensional region labeled by some Frobenius algebra A or creates a defect line. For the first type of points, we choose the triangulation such that an A-ribbon passes through p; to p we then attach a bimodule morphism in $\mathrm{Hom}(U \otimes A \otimes V, A)$, with U and V objects of $\mathcal{C}$ as above. To a point p at which a defect line of type B starts or ends, we attach a bimodule morphism in $\mathrm{Hom}(U \otimes A \otimes V, B)$ and in $\mathrm{Hom}(U \otimes B \otimes V, A)$, respectively.

We have now obtained a complete labelling of a ribbon graph in the connecting manifold $\mathrm{M_X}$ with objects and morphisms of the modular tensor category $\mathcal{C}$; in other words, a cobordism from $\emptyset$ to $\widehat{\mathrm{X}}$ in the decorated geometric category $\mathcal{G}_{\mathcal{C}}$. Applying the modular functor for the tensor category $\mathcal{C}$ to this cobordism, we obtain a vector

$$\mathrm{Cor}(\mathrm{X}) = \mathit{tft}_{\mathcal{C}}(\mathrm{M_X})\, 1 \in \mathit{tft}_{\mathcal{C}}(\widehat{\mathrm{X}}).$$

This is the prescription for RCFT correlation functions in the TFT approach. It follows from the defining properties of a symmetric special Frobenius algebra that Cor(X) does not depend on the choice of triangulation; for details see [12].

6. Results

On the basis of this construction one can establish many further results. Let us list some of them, without indicating any of their proofs:

[17] Of particular interest are the correlators for X being the torus or the annulus without field insertions, but possibly with defect lines. From these "one-loop amplitudes" one can derive concrete expressions for partition functions of boundary, bulk and defect fields.

The coefficients of these partition functions in the distinguished basis of the zero-point blocks on the torus that is given by characters can be shown to be equal to the dimensions of certain spaces of 2-morphisms of the 2-category $\mathcal{F}rob_{\mathcal{C}}$. Thus in particular they are non-negative integers.

In fact, one recovers expressions that had also been obtained in an approach based on subfactors [29], [5]. Moreover, these coefficients can be shown to satisfy other consistency requirements like forming so-called NIMreps of the fusion rules.

[18] To extend these results to unoriented (in particular, to unorientable) surfaces one must specify as additional datum a Jandl structure on the relevant Frobenius algebra. One can then e.g. compute the partition functions for the Möbius strip and Klein bottle. Their coefficients in the distinguished basis of zero-point torus blocks are integers, and for CFT models which serve as building blocks of type I string theories, these partition functions combine with the torus and annulus amplitudes in a way consistent with an interpretation in terms of state spaces of the string theory.

[19] The expressions for correlation functions can be made particularly explicit for conformal field theories of simple current type [39], which correspond to Frobenius algebras for which every simple subobject is invertible. Eilenberg–Mac Lane's [9] abelian group cohomology turns out to provide a crucial tool for analyzing this case.

It should be stressed, though, that the TFT approach to RCFT correlators treats the simple current case and other conformal field theories (i.e. those having an 'exceptional modular invariant' as their torus partition function) on an equal footing.

[20] By expressing some specific correlation functions for the sphere, the disk, and the real projective plane through the appropriate (two- or three-point) conformal blocks, one can derive explicit expressions for the coefficients of operator product expansions of bulk, boundary, and defect fields.

[12] For arbitrary topology of the surface X the correlators obtained in the TFT construction can be shown to satisfy the covariance and factorization axioms that were stated at the end of Section 4.

[15] The Picard group of the tensor category $\mathcal{H}om(A, A)$ describes symmetries of the full conformal field theory that is associated to A. The fusion ring $K_0(\mathcal{H}om(A, A))$ of that category contains information about Kramers–Wannier-like dualities as well.

7. Conclusions

The TFT approach to the construction of CFT correlation functions, which represents CFT quantities as invariants of knots and links in three-manifolds, relates a general paradigm of quantum field theory to the theory of (symmetric special) Frobenius algebras in (modular) tensor categories. It thereby constitutes a powerful algebraization of many questions that arise in the study of conformal field theory. As a result, one can both make rigorous statements about rational conformal field theories and set up efficient algorithms for the computation of observable CFT quantities.

A rich dictionary relating algebraic concepts and physical notions is emerging. It includes in particular the following entries:

- The classification of (oriented) full conformal field theories for given chiral data $\mathcal{C}$ amounts to the classification of Morita classes of Frobenius algebras in $\mathcal{C}$. As a special case, the classification of those theories whose torus partition function is "of automorphism type" amounts to determining the Brauer group of the category $\mathcal{C}$.

- The Picard group of the tensor category $\mathcal{H}om(A, A)$ acts as a symmetry group on the full conformal field theory associated to A, while the fusion ring of this tensor category contains information about Kramers–Wannier like dualities.

- Deformations of the conformal field theory are controlled by the Davydov–Yetter cohomology of the pair $(\mathcal{C}, \mathcal{C}_A)$.

The structure of this dictionary gives us confidence that some of the insights of the TFT approach – though, unfortunately, not most of the proofs – will still be relevant for the study of conformal field theories that are not rational any more.

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