# Soliton dynamics and scattering

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**Abstract.** A survey of results and problems of soliton dynamics in dispersive and hyperbolic nonlinear PDE's and the related spectral and scattering theory. I focus on the problem of large time behavior of the nonlinear Schrödinger equation, with both solitary and radiative waves appearing in the solution. The equations are nonintegrable in general and in arbitrary dimension. I will formulate the main conjectures relevant to soliton dynamics.

**Mathematics Subject Classification (2000).** Primary 35Qxx.

**Keywords.** Solitons, NLS, asymptotic-completeness, dispersive waves, asymptotic stability.

## 1. Introduction

The advances in spectral and scattering theory of the last 20 years, combined with the intense physical research based on nonlinear dispersive equations have led to major progress in nonlinear dynamics.

The goal of understanding the large time behavior of all solutions of nonlinear dispersive equations is now pursued on many levels: global existence theory, nonlinear scattering, new solutions, applications.

Furthermore, the new applications of NLPDE in physics generate a host of new research in the physics literature, e.g. experimental [12], [9], [30] and theoretical [57], [2], [29].

It is then fair to say that we are witnessing a new (golden) generation that focusses on complex collective systems behavior, the age of coherent structures.

Consider the following generic form of dispersive NLPDE:

$$i\partial_t u = Hu + F(u)u$$

with initial data in (say) some Hilbert space $\mathcal{H}$, typically (vector valued) Sobolev space.

$H$ is a self-adjoint (matrix) operator in general.

For example NLS (semilinear):

$$H = -\Delta, \quad F(u) \sim \sum_{i=1}^{N} \lambda_i |u|^{p_i},$$

$$\mathcal{H} = H^1(\mathbb{R}^n);$$

*Research partially supported by NSF Grant# DMS-0501043.

Proceedings of the International Congress
of Mathematicians, Madrid, Spain, 2006

and the NLKG equation (hyperbolic type):

$$\partial_t \begin{pmatrix} u \\ \dot{u} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ H_m & 0 \end{pmatrix} \begin{pmatrix} u \\ \dot{u} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ F(u) & 0 \end{pmatrix} \begin{pmatrix} u \\ \dot{u} \end{pmatrix},$$

$$H_m = -\Delta + m^2.$$

In general we expect that if the $\lambda_i > 0$ and the $p_i$ are not too large/small that global existence holds, that uniqueness holds and scattering theory holds and all solutions disperse like free waves for large time [54]. When some $\lambda_i$ are negative $F$ is attractive and blow-up occurs in general, in finite time, for data not too small.

However a fundamental new phenomena appears: the dispersion by $H$ can be exactly cancelled by the focusing of $F$, to create a new type of localized in space solutions, smooth, uniformly in time. I shall refer to these solutions as *coherent structures or solitons*.

**Example 1.1.** NLS solitons, KG/SG kinks, vortices, monopoles, breathers, topological solitons, hedgehogs, skyrmions, blackholes, ... and the more recent more exotic coherent structures: compactons, peakons, noncommutative solitons. [31], [54], [53].

**Definition 1.1.** A localized solution/soliton/coherent structure is a solution of a dispersive equation satisfying

$$\lim_{R\to\infty} \left\{ \sup_t \|\chi(|x| > R) u(x,t)\|_{\mathcal{H}} \right\} = 0.$$

To understand how fundamental are coherent structures we have the following conjecture:

**Grand Conjecture** (Asymptotic Completeness). Generic asymptotic states are given by independently (freely) moving coherent structures and free radiation.

**Comments.** 1) The Grand Conjecture states that besides free waves only these coherent structures can emerge as $t \to \infty$.

2) We are very far from proving such a result for any interesting equation.

In making progress in this direction I will formulate another "simple" conjecture that I expect to play an important role:

**Petite Conjecture.** Localized solutions of NLS are almost periodic in time.

**Comments.** 1) I used "NLS" and not "dispersive" to avoid giving a rigorous definition of "dispersive".

2) This conjecture is a nonlinear analog of the geometric characterisation of bound states in linear theory originally proposed by Ruelle and developed as RAGE theorem.

It also states that coherent structures/solitons are the "bound states" of dispersive wave equations.

3) While such a result follows for data near a stable soliton solution of NLS [40], [37], [35], [10], [45], [46], [4], [5], [36] it is not known for even a single equation in such generality.

4) One can replace the localization assumption by the stronger condition

$$\sup_t \|\chi(|x| > R)u(x,t)\|_{\mathcal{H}} \leq cR^{-m}, \quad R > 1,$$

some $m > 0$, or even exponential decay in $R$ for many equations.

## 2. Asymptotic stability

The kind of problems we do understand are small perturbations of the putative asymptotic states described above. That is, we can prove that if the initial data is close to $N$ solitons moving independently and small perturbation it will propagate as expected when $t \to \infty$.

Next, I shall describe the developments and arguments leading to this result.

So consider the NLS in three or more dimensions, here we follow [40]:

$$i\frac{\partial \psi}{\partial t} = -\Delta\psi - F(|\psi|^2)\psi \quad x \in \mathbb{R}^n \text{ and } n \geq 3. \tag{1}$$

If $F$ has a negative (attractive) part the equation will have, in general, coherent structure solutions, such as solitons. To find (some of) them, we look for time periodic solutions

$$\psi = e^{i\omega t}\phi_\omega(x),$$

which gives

$$-\omega\phi_\omega = -\Delta\phi_\omega - F(|\phi_\omega|^2)\phi_\omega.$$

In general $\phi_\omega$ will be localized (at least as an $L^2$-function) for $\omega > 0$.

Thanks to the pioneering works of [8], [53], [3], [24], [32], [17] we know a great deal of information about such solitons: existence, decay at infinity, uniqueness of the positive solutions, symmetry and more.

By Galilean invariance

$$\psi_\sigma \equiv e^{i\vec{v}\cdot x - i\frac{1}{2}(|v|^2-\omega)t + i\gamma}\phi_\omega(x - \vec{v}t - \vec{a})$$

are all solutions of NLS;

$$\sigma = (\vec{u}, \gamma, \vec{a}, \omega) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}.$$

$\psi_\sigma$ are moving solitons, with velocity $\vec{v}$ localized at $\vec{a}$ (at time 0), with energy $\omega$ and phase $\gamma$. The general conjecture then states that generic solutions of NLS are asymptotic to a sum of such moving solitons plus a free wave (a solution of the free Schrödinger equation corresponding to $F = 0$).

A weaker but useful result is orbital *stability* of solitons: we would like to know if a small perturbation of a soliton (as initial data) leads to a nearby, soliton (in $\sigma$ space) up to a phase, for all times [7], [13], [14], [18], [19], [20], [58], [59].

It turns out that in many cases stability follows from *linear stability* :

First we linearize the equation around a soliton:

$$\psi \equiv e^{i\theta}(\phi_\omega + R)$$

and deriving (the linear part of) the equation of $R$: since there will be both $R$ and $\bar{R}$ terms in the equation we complexify to get:

$$i\frac{\partial}{\partial t}\begin{pmatrix} R \\ \bar{R} \end{pmatrix} = \mathcal{H}\begin{pmatrix} R \\ \bar{R} \end{pmatrix},$$

$$\mathcal{H} \equiv \begin{pmatrix} L+ & W(x) \\ -W(x) & -L_+ \end{pmatrix},$$

$$L_+ = -\Delta + \omega - F(\phi_\omega^2) - F'(\phi_\omega^2)\phi_\omega^2,$$
$$W(x) = -F'(\phi_\omega^2)\phi_\omega^2.$$

Since $\mathcal{H}$ is not self-adjoint we do not have $L^2$ conservation under such $\mathcal{H}$. Linear stability states that on the complement of the root space of $\mathcal{H}$ the solutions are uniformly bounded in $L^2$:

$$\sup_t \|U(t)Pf\|_{L^2} \leq c\|f\|_{L^2}$$

where $P$ is the projection on $(N^*)^\perp$ and $N$ is the root space of $\mathcal{H}$,

$$N \equiv \bigcup_{\ell\geq 1} \ker \mathcal{H}^\ell.$$

Such analysis can be made for other coherent structures in a similar way [vortices, kinks, blackholes,. . . .].

To proceed with the problem of asymptotic stability and completeness we are going to need much more detailed spectral/scattering results for $e^{i\mathcal{H}t}P$. Proving such results for various types of $\mathcal{H}$ that appear in applications is in general difficult, and while there are many works in this direction from the last few years, a lot is left open, see the review [42].

We need to prove

a) Absence of embedded eigenvalues in the continuous spectrum of $\mathcal{H}$.

There are remarkably detailed and complete results for self-adjoint Schrödinger operators on $\mathbb{R}^n$ and manifolds [22], [41].

The extension to matrix operators such as $\mathcal{H}$ is still eluding us, and a general new approach would be of great interest to the field, see however [42].

b) Absence of threshold resonances. This is related to the next condition.

c) Dispersive estimates:

$$\|U(t)P\psi_0\|_{L^\infty} \lesssim t^{-n/2}\|\psi_0\|_{L^1}.$$

Weaker estimates are also of interest, such as

$$\|\langle x\rangle^{-\sigma}U(t)P\langle x\rangle^{-\sigma}\| \lesssim t^{-\sigma}, \quad t > 1.$$

The subject of dispersive estimates is another fast growing field of research inspired by soliton dynamics, see the latest review of Schlag [42].

Such $L^p$-dispersive estimates were proved for

$$H = -\Delta + V \text{ on } \mathbb{R}^n, \quad n \geq 3,$$

for a quite general class of $V$ in [23]. It was then extended (also to $L^p$ boundedness of wave operators) in [60]; further important new extension is due to [38]. For matrix operators see [39], [42]. All the results which are known require enough decay at $\infty$ of the potential perturbations. The decay is determined by the decay of the soliton.

NLS solitons decay exponentially and so they are included.

However other types of coherent structures like vortices, monopoles etc. lead to slowly decaying potentials, and so the dispersive theory for both Schrödinger and matrix operators is still lacking the generality needed.

What is more, the above stated decay also requires the knowledge of absence of threshold resonances, a difficult problem to solve for specific operators. In fact, in some situations (e.g. kink scattering) such resonances are present, and therefore the asymptotic stability theory of kinks requires as yet unknown modifications.

## 3. The method of modulation equations

Presently, the modulation equations approach is the only method that is used to prove asymptotic stability of solitons. The first step is the *Ansatz* and the associated *orthogonality condition*.

**Ansatz.**

$$\psi(t) = S_{\sigma(t)} + R(t).$$

Here $S_\sigma$ is a soliton with parameters $\sigma$.

**Orthogonality condition.** $R(t)$ belongs to the range of $P_t$ so it is orthogonal to the root space adjoint $N^*$.

**Remark 3.1.** When the soliton is small we can replace $P_t$ by $P_{t=0}$ [45], [46]. Under favorable conditions we can derive a closed system of equations for $R(t), \sigma(t)$. The equation for $\mathbb{R} \equiv (R(t), \bar{R}(t))$ then has the form

$$i\partial_t\mathbb{R}(t) = \mathcal{H}\mathbb{R}(t) + \mathcal{NL}(\mathbb{R}(t))$$

where $\mathcal{NL}(\mathbb{R}(t))$ is a sum of quadratic and higher order terms in $\mathbb{R}$, and terms of the form

$$O(\dot{\sigma}\mathbb{R}).$$

The $\sigma$-equation is a system of ODE's:

$$\dot{\sigma}(t) = \mathcal{K}(t)\sigma(t) + \mathcal{NL}(\mathbb{R}, \sigma).$$

The idea for solving such a system is based on bootstrap and smallness. Suppose that we can prove that the solutions of the PDE for $(\mathbb{R}(t))$ is indeed dispersive:

$$|||\mathbb{R}(t)||| = O(t^{-m})$$

where $|||\cdot|||$ is typically a sum of $L^P$ and local decay norms.

For example, for $p = \infty$, and three or more dimensions we expect $m = 3/2$ at least. Then, plugging this estimate for $\mathbb{R}(t)$ into the $\mathcal{NL}(\mathbb{R}, \sigma)$ terms of the ODE we shall be able to prove that

$$\dot{\sigma}(t) \text{ is in } L^1(dt) \text{ (excluding } \dot{\gamma})$$

and perhaps get some pointwise decay.

Then these estimates on $\dot{\sigma}$ should be sufficient to control the $O(\dot{\sigma}R)$ terms in the PDE. Smallness can then be used to close the above self-consistent system of estimates.

The key to implementing this procedure is proving sufficient decay estimates for the PDE, assuming the decay of $\dot{\sigma}$. This leads us to study the following general problem:

$$i\frac{\partial R}{\partial t} = \mathcal{H}PR + P\mathcal{NL}(R)$$

where $P$ is the projection on $(N^*)^{\perp}$.

When $R$ is small, we try to solve this problem by Duhamel type identities

$$R(t) = e^{-i\mathcal{H}Pt}R(0) - i\int_0^t e^{-i\mathcal{H}P(t-s)}P\mathcal{NL}(R(s)\,ds.$$

The starting point is then knowing the basic estimates for

$$e^{-i\mathcal{H}Pt}$$

which is the linear problem described before.

There are many situations where the range of $P$ includes, besides the continuous spectrum of $\mathcal{H}$, some point spectrum. In this case, further decomposition of the solution is needed.

This will be described in the next section.

So consider the case when $P = P_c(\mathcal{H})$. We need, besides the $L^p$ decay estimates for $\mathcal{H}$, to estimate the nonlinear terms.

When the nonlinearities are of sufficient power decay (near zero) this is not difficult.

However, notice that we always need to deal with a term of the form

$$\sum SR^2.$$

When the soliton $S$ is well localized, local decay pointwise in time plus some $L^p$ decay is sufficient. The complications arise in cases where there is not enough decay or when $S$ is not well localized. For example the kink problem leads to $S$ that is of order 1 at infinity! Hence all the decay should come from $R^2$, which is not possible in 1-dimension. For this and other equally subtle reasons the kink asymptotic stability is completely open. In particular, one needs to solve the *long range* nonlinear scattering problem. A lot of work was done on this problem by Ginibre–Velo–Naumkin [16], [21] and collaborators. Recently Delort [11] worked out the problem for NLKG in 1-dimension. A new simpler and general approach was very recently developed in [26], [27], [28] ]

In the simplest cases we end up with estimating

$$R(t) \sim c^{-i\mathcal{H}_c t} R(0) - i \int_0^t e^{-i\mathcal{H}_c(t-s)} O(\chi R^2(s) + R(s)^p + O(s^{-\alpha}) R(s)) ds$$

in $L^p$.

A particularly effective way of doing it, is to use the mixed norm $L^\infty + L^2$.

We then get

$$\|R(t)\|_{L^2+L^\infty} \leq C \|R(0)\|_{L^2+L^1} \langle t \rangle^{-n/2} + c \int_0^t \frac{ds}{\langle t-s \rangle^{n/2}} \|O(\cdot)\|_{L^2+L^1}\, ds.$$

In three or more dimensions, $n/2 > 1$.

Since for all $s$ large enough and any $\varepsilon > 0$,

$$\begin{aligned} &\|\mathcal{O}(\cdot)\|_{L^2+L^1} \\ &\quad \leq C \langle s \rangle^{-n/2} \sup_{0 \leq s' \leq s} \{ \|R(s')\|_{2+\infty}^2 + \|R(s')\|_{2+\infty}^{2+m} + \|R(s')\|_p^p + \varepsilon \|R(s')\|_{2+\infty} \} \end{aligned}$$

provided $\left(\frac{n}{2} - \frac{n}{p}\right) \cdot p \geq \frac{n}{2}$ implies $p \geq 3$. $\|R(t)\|_{L^2+L^\infty}$ is $O(t^{-n/2})$ for $R(o)$ small.

It is possible to improve the estimates to lower values of $p$, by proving only that $\|R\|_q \leq ct^{-1-\varepsilon}$ for

$$q \text{ such that } \frac{n}{2} - \frac{n}{q} = 1 + \varepsilon.$$

A different approach, which applied to nonlocalized perturbations of solitons is based on using Strichartz estimates instead of pointwise bounds. See [15].

## 4. Selection of the ground state, spurious eigenvalues

In general there is more than one (family of) localized states to the nonlinear equation.

Examples include

$$i\frac{\partial\psi}{\partial t} = (-\Delta + V(x))\psi + \lambda F(|\psi|)\psi \tag{2}$$

with $H = -\Delta + V(x)$ having more than one (negative) eigenvalue in its spectrum. Another example is the nonlinear wave equation analog of (2):

$$(\partial_t^2 - \Delta + V(x) + m^2)u = \lambda F(u)u \tag{3}$$

with $-\Delta + V(x) + m^2 > 0$, and having as before one or more eigenvalues.

Other examples include the NLS with excited state solitons etc., spurious eigenvalues, including embedded eigenvalues in the spectrum of the linearization. In these cases the grand conjecture is more complicated to state.

The generic behavior we expect is that the asymptotic states are combinations of ground state (families) solitons *only* plus free radiation.

This was proved for the NLWE (3) in the case of one bound state in 3-dimensions in [49]. The techniques developed in [49] will be briefly described below; they were used to deal with the more involved NLS (2) in the case of two bound states in [51], [56] and some results were obtained for more then two bound states in [55]. It also applied to the *linear* resonance problem in QM [47], [48], [50], [33], [34], [18], [6]. Very recently an experiment was done confirming the predictions below [30], [52].

We use modulation equations again. Consider the problem (2): Time periodic soliton solutions, nonlinear bound states, bifurcate from the linear eigenstates of $-\Delta + V(x)$. For each $E_i$, $i = 0, 1$, we solve

$$(-\Delta + V(x))\psi_{E_i} + \lambda F(|\psi_{E_i}|^2)\psi_{E_i} = E_i\psi_{E_i} \tag{4}$$

such that $E_i \to E_{i*}$ as $\lambda \to 0$, where $(i = 0, 1)$

$$(-\Delta + V(x))\psi_i = E_{*_i}\psi_i \ \|\psi_i\| = 1.$$

We assume the initial data is small in $H^s$, $(s \geq 2)$ and $2E_{1*} - E_{0*} > 0$.

We begin with the Ansatz $\phi(t) \equiv e^{-i\theta(t)}[\psi_0(t) + \psi_1(t) + \phi_2(t)]$ where $\psi_0(t) \equiv \psi_{E_0(t)}$ is a solution of the ground state eigenvalue equation with energy $E_0(t)$, at time $t$. $E_0(t)$ will be determined later by orthogonality conditions [Se 7, 5, 1]. Similarly $\psi_1(t)$ is an excited state eigenvector with eigenvalue $E_1(t)$. $\theta(t) \equiv \theta_0(t) + \tilde{\theta}(t)$; $\theta_0(t) = \int_0^t E_0(s)ds$. $\tilde{\theta}(t)$ will be chosen appropriately; it includes (logarithmic) divergent phase. Substitution of the above Ansatz for $\phi$ into (1), and complexifying the equations $[\phi_2 \to (\phi_2, \bar{\phi}_2) \equiv \Phi_2(t)$, $\psi_j \to (\psi_j, \bar{\psi}_j) \equiv \Psi_j(t)$ etc.] we derive

$$i\partial_t\Phi_2(t) = \mathscr{H}_0(t)\Phi_2(t) - i\partial_t\Psi_0 - \left[((E_0 - E_1) + \partial_t\tilde{\theta})\sigma_3 + i\partial_t\right]\Psi_1 + \vec{F}_{NL},$$

where $\vec{F}_{\mathcal{NL}}$ is nonlinear in $\Phi_2, \Psi_0, \Psi_1, \tilde{\theta}$ and $\mathcal{H}_0(t)$ is given by the matrix operator

$$\sigma_3 \begin{pmatrix} H - E_0(t) + 2\lambda|\psi_0(t)|^2 & \lambda\psi_0^2(t) \\ \lambda\bar{\psi}_0^2(t) & H - E_0(t) + 2\lambda|\psi_0(t)|^2 \end{pmatrix} \tag{5}$$

where $\sigma_3$ is the Pauli matrix diag$(1, -1)$. We consider the spectrum of $\mathcal{H}_0(t)$ for fixed $t$, and $|\psi_0| \equiv |\alpha_0|$ small: (a) The continuous spectrum extends from $-\mu$ to $-\infty$, and $\mu$ to $\infty$ where $\mu \equiv E_1 - E_0 + O(|\alpha_0|^2)$. The discrete spectrum is $\{0, -\mu, \mu\}$, with $0 < |\mu| < |E_0|$ by assumption. (b) Zero is a generalized eigenvalue of $\mathcal{H}_0$, with generalized eigenspace spanned by $\left\{\sigma_3\Psi_0, \partial_{E_0}\Psi_0\right\}$.

The discrete spectral subspace has dimension four. Therefore, $\Psi_2$ which lies in the continuous spectral part of $\mathcal{H}_0(t)$, is constrained by four orthogonality conditions. Furthermore $\partial_t\tilde{\theta}$ is chosen to remove divergent logarithmic phase contributions. In the weakly nonlinear (perturbative) regime, bound states have expressions $\psi_{E_j} = \alpha_j(\psi_{j*}(x) + g|\alpha_j|^2\psi_j^{(1)}(x) + \mathcal{O}(g^2|\alpha_j|^4))$ and $E_g = E_{j*} + \mathcal{O}(|\alpha_j|^2)$. The system for $\Phi_2$ and $\vec{\alpha} = (\vec{\alpha}_0, \vec{\alpha}_1)$ can be written in the form $i\partial_t\vec{\alpha} = \mathcal{A}(t)\vec{\alpha} + F_\alpha$, $i\partial_t\Phi_2 = \mathcal{H}(t)\Phi_2 + F_\Phi$ .

To proceed further we decompose $\Phi_2$ into its continuous spectral (dispersive) part, $\eta \in \mathcal{H}_0(T)$, and its components along the discrete modes. The latter are higher order and controllable. Thus NLS at low energy is equivalent to a system of the form:

$$\begin{aligned} i\partial_t\eta &= \mathcal{H}_0(T)\eta + \mathcal{F}_\eta(t; \alpha_0, \beta_1, \eta) + \sigma(t)\eta, \\ i\partial_t\beta_1 &= 2\lambda\langle\psi_{0*}, \psi_{1*}^3\rangle|\beta_1|^2\alpha_0 e^{i\lambda_+ t} \\ &\quad + 2\lambda\langle\psi_{0*}\psi_{1*}^2, \pi_1\Phi_2\rangle\bar{\beta}_1\alpha_0 e^{2i\lambda_+ t} + \mathcal{R}_0, \\ i\partial_t\alpha_0 &= \lambda\langle\psi_{0*}^2, \psi_{1*}^2\rangle e^{-2i\lambda_+ t}\beta_1^2\overline{\alpha}_0 \\ &\quad + \lambda\langle\psi_{0*}\psi_{1*}^2, \Phi_2\rangle\beta_1^2 e^{-2i\lambda_+ t} + \mathcal{R}_1, \end{aligned} \tag{6}$$

where $\mathcal{R}_j$ denotes corrections of a similar form and higher order.

The above system can be viewed as an infinite dimensional Hamiltonian system consisting of two subsystems: a finite dimensional subsystem governing "oscillators", $(\alpha_0, \beta_1)$, and an infinite dimensional subsystem governing the field, $\eta$.

The coupled system (6) can not be solved or understood by looking at the linear terms only. This is due to the fact that the asymptotic behavior is determined by a process, nonlinear, in which the excited state part of the solution decays into radiation and ground state part. We therefore need to derive effective equations for the ground and excited state parts, which include the dissipative effects due to coupling to radiation. see [49], [44] see also [43], [1], [25].

To arrive at the reduction, we solve the $\eta$-equation, making explicit all terms through second order in $g$, using the Green's function $G(t, t') = e^{-i\mathcal{H}_0(T)(t-t')}$. We focus on the key terms coming from the sources in $\mathcal{F}_\eta$ or the type $\alpha_0^i\alpha_1^j$, $0 \le i, j \le 2$

and having oscillatory phases $e^{im_{ij}t}$. Their contribution to $\eta$ is of the form

$$\sim \int_0^t e^{-i\mathscr{H}_0(T)(t-t')}|\chi\rangle e^{im_{ij}(t')}\alpha_0^i(t')\alpha_1^j(t')dt'$$

where $\alpha_0$, $\alpha_1$ is a component of either $\vec{\alpha}_0$ or $\vec{\alpha}_1$, where $|\chi\rangle$ is an (exponentially localized) function of position expressible in terms of $\psi_0*$ and $\psi_1*$. We insert this solution into the $\alpha_0$-, $\alpha_1$-equations, in place of $\Phi_2$. We obtain integro-differential equations for $\alpha_0$, $\alpha_1$, $(\beta_1)$. The resulting terms of the above form are solutions to a forced linear system and among the forcing terms there are (coupled) oscillatory terms with the frequency $\omega_*$, which is resonant with the continuous spectrum. Internal dissipation resulting in nonlinear resonant energy transfer from the excited state to the ground state and to dispersive radiation is derived from these resonant terms; see also the derivation of internal dissipation in both linear and nonlinear resonance theories recently developed by us [46], [44]. This dissipation coefficient is $\Gamma$, the rate of decoherence and relaxation. The above described scheme gives $i\partial_t\tilde{\alpha}_0 = (-\Lambda + i\Gamma) \times |\tilde{\beta}_1|^2\tilde{\alpha}_0 + \tilde{\mathscr{R}}_0(t)$, $i\partial_t\tilde{\beta}_1 = 2(\Lambda - i\Gamma)|\tilde{\alpha}_0|^2|\tilde{\beta}_1|^2\beta_1 + \tilde{\mathscr{R}}_1(t)$.

Introducing the squared projections of the system's state onto the ground state and excited states, $P_0 \equiv |\tilde{\alpha}_0|^2$, $P_1 \equiv |\tilde{\beta}_1|^2$ we obtain NLME. The system is analyzed in terms of renormalized powers $\mathscr{Q}_0$ and $\mathscr{Q}_1$, for which it is shown that there exist transition times $t_0$ and $t_1$, such that $\mathscr{Q}_0(t)$ decays rapidly on $[0, t_0]$, $\mathscr{Q}_0(t)/\mathscr{Q}_1(t)$, grows rapidly on $[t_0, t_1]$, and then finally on $[t_1, \infty)$ the following system governs: $\partial_t\mathscr{Q}_0 = 2\Gamma\mathscr{Q}_0\mathscr{Q}_1^2$, $\partial_t\mathscr{Q}_1 = -4\Gamma\mathscr{Q}_0 Q_1^2$. This gives $Q_0 \uparrow Q_0(\infty)$ and $Q_1 \downarrow 0$ at rates discussed above.

$$\Gamma = \pi\lambda^2|(e_{\omega*}, \psi_{1*}^2\psi_{0*})|^2$$

for $F(|\psi|^2) \equiv |\psi|^2$. Here $e_{\omega*}$ is the generalized eigenvalue of $H_0$ at energy $\omega* = 2E_{1*} - E_{0*}$.

## 5. Concluding remarks

Nonlinear dispersive equations play a prominent role in many fields of physics, including BEC theory, nonlinear optics, large molecule dynamics (e.g. DNA) and more.

The mathematical aspects of such equations is remarkably rich, and contributed new, challenging directions for PDE, mathematical physics spectral and scattering theory and more. In particular, the problem of large time dynamics of interacting solitons and more general coherent structures is witnessing a major progress in the last 15–20 years. Yet, we are only at the beginning of developing the mathematical tools and theories to deal with the general aspects of soliton dynamics. Besides the many problems I listed in the previous sections, other topics worth mentioning are: systems of equations (e.g. coupled Maxwell–Dirac equations and BEC coupled to vapor at finite temperature), solitons dynamics on curved spaces, like arbirarily shaped optical fibre (monoples and other topological solitons), and discrete space modeles.

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