Hypocoercive diffusion operators

Cédric Villani

Abstract. In many problems coming from mathematical physics, the association of a degenerate diffusion operator with a conservative operator may lead to dissipation in all variables and convergence to equilibrium. One can draw an analogy with the well-studied phenomenon of hypoellipticity in regularity theory, and actually both phenomena have been studied together. Now a distinctive theory of "hypocoercivity" is starting to emerge, with already some striking results, and several challenging open problems.

Mathematics Subject Classification (2000). Primary 35B40; Secondary 35K70, 76P05.

Keywords. Hypocoercivity, hypoellipticity, diffusion equations, spectral gap, logarithmic Sobolev inequalities, Fokker–Planck and Boltzmann equations, H Theorem.

Introduction

During the past decade, considerable progress has been achieved in the qualitative study of diffusion equations in large time, be it for linear or nonlinear models. Quantitative functional methods have become especially popular. Here are some of the keywords in the field: spectral gap (Poincaré) inequalities, logarithmic Sobolev inequalities, analysis of entropy production, gradient flows, rescalings. Most of the time, estimates on the rate of convergence are established in the end by means of some Gronwall-type inequality $dE/dt \leq -\Phi(E)$, where E is a Lyapunov functional for the system. Among a large literature, I shall only quote some of my own works: entropy production estimates for the spatially homogeneous Boltzmann equation, in collaboration with Giuseppe Toscani [25], [26], [28]; and for certain nonlinear diffusion equations with a convex mean-field interaction, in collaboration with José Antonio Carrillo and Robert McCann [2].

While these subjects are still very active, in this text I shall focus on a newer direction of research which has emerged only a few years ago, and can be loosely described as "the role of the non-dissipative part in the dissipation process".

Indeed, it happens not so rarely that the dissipative properties of an equation are strongly influenced by some of the conservative terms in this equation. This statement in itself is nothing new, since it is almost obvious in the context of hydrodynamics (dissipativity in Navier–Stokes is certainly considerably more complex than in the heat equation). In the context of diffusion equations, the interaction between dissipative and conservative terms is also well-known, since it is at the basis of the phenomenon of

Proceedings of the International Congress
of Mathematicians, Madrid, Spain, 2006

hypoellipticity. To make the discussion a bit more precise, let me recall a particularly simple theorem of hypoelliptic regularization, which is a direct consequence of Lars Hörmander's celebrated regularity theorem [20]. Let $A_1, \dots, A_k$ and B be C^∞ vector fields on $\mathbb{R}^N$, identified with derivation operators, and let $L = -\sum A_j^2 + B$. If the rank of $(A_1, \dots, A_k)$ is strictly less than N, then the operator L is not elliptic, and there is no a priori reason why the semigroup e^{-tL} would be regularizing in all variables. But if $-\sum [A_j, B]^2 - \sum A_j^2$ is elliptic, where $[A_j, B]$ is the Lie bracket between A_j and B, then e^{-tL} is regularizing in all variables, and the operator L is said to be hypoelliptic. (This is not the classical definition of hypoellipticity, but it will do for the purpose of this presentation.) We see here how the "nondissipative" first-order operator B interacts with the "dissipative part" of L, or more precisely the derivation operators A_j, to produce the missing directions of regularization. Possibly the most important instance of application is to the operator $L = -\Delta_v + v \cdot \nabla_x$, where $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$; in that case $A_j = \partial/\partial v_j$, $B = v \cdot \nabla_x$, $[A_j, B] = \partial/\partial x_j$. The corresponding evolution equation $\partial_t f + Lf = 0$ is degenerate, but still presents some of the typical features of a parabolic equation; the word "ultraparabolic" is sometimes used for it.

Hypoelliptic regularity has been the object of hundreds of works for the past four decades. But what was understood only very recently is that quite similar phenomena arise in the study of rates of convergence to equilibrium. To describe this, I shall use the word "hypocoercivity", which was suggested to me by Thierry Gallay. A typical hypocoercivity theorem will give sufficient conditions on an operator L so that e^{-tL} will converge to equilibrium at a certain rate, even though L is not "coercive", in the sense that the kernel of its dissipative part is much larger than the set of equilibria.

Hypoellipticity and hypocoercivity are often found together, and have been actually studied together, by refined hypoelliptic techniques [6], [7], [15], [16], [19], and sometimes by probabilistic methods [8], [22], [23]. However, these two phenomena are distinct: Each of them can occur without the other; and the structures which underlie them are not exactly the same. This motivates the development of a separate theory of hypocoercivity. In the sequel, I shall present some of the first results in this direction.

Acknowledgement. The ideas exposed in the sequel have benefited from interactions with many people who are quoted within the text. Warm thanks are due to Martin Hairer, Frédéric Hérau and Clément Mouhot for their detailed comments on a preliminary version of these notes; and to Thierry Gallay for illuminating discussions.

1. Motivations

In this section I shall describe some concrete examples which motivate the study of hypocoercivity. All of them come from mathematical physics, and none of them is academic. Of course the list is far from exhaustive.

The kinetic Fokker–Planck equation. In stochastic analysis, Fokker–Planck equations are often encountered as equations satisfied by the time-dependent laws of solutions of first-order stochastic differential equations. In "real life" however, equations of motion are not first-order, but second-order. Consider for instance a particle in $\mathbb{R}^n$, following Newton's equations with a potential force $-\nabla V$, a white noise random forcing, and a linear friction with coefficient $\theta = 1$: Then its position X_t at time t satisfies the second-order stochastic differential equation

$$\frac{d^2X_t}{dt^2} = -\nabla V(X_t) + \sqrt{2}\,\frac{dB_t}{dt} - \frac{dX_t}{dt},$$

where B_t is a standard Brownian motion. (Of course, the coefficient $\sqrt{2}$ is just a convenient normalization, and the writing is formal in the sense that B_t is not differentiable.) To write the associated partial differential equation, define $f_t(x, v)$ as the density of the law of $(X_t, \dot{X}_t)$ in $\mathbb{R}^n \times \mathbb{R}^n$. Then f is a solution of

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla V(x) \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (fv), \tag{1}$$

where Δ_v and $\nabla_v \cdot$ respectively stand for the Laplace and divergence operators in velocity space. Equation (1) is *kinetic* in the sense that it involves not only the position, but also the velocity variable; it is one of the fundamental equations in gas dynamics. It admits many nonlinear variants, among which the Vlasov–Poisson–Fokker–Planck equation, which is accepted as one of the fundamental equations of stellar dynamics.

When V is quadratic, the fundamental solution of (1) is explicit and Gaussian. Its examination shows that there is relaxation to a Gaussian equilibrium (in x and v variables) as $t \to \infty$, and this convergence is exponentially fast, with an explicit rate. Here we see a perfect illustration of the hypocoercivity phenomenon: The differential operator on the left-hand side of (1) is conservative (it describes the trajectories of a classical dynamical system in $\mathbb{R}^n \times \mathbb{R}^n$ with Hamiltonian $V(x) + |v|^2/2$), and the right-hand side alone is diffusive degenerate (it only acts on the velocity variable v, so cannot cause any relaxation to equilibrium with respect to the x dependence); however, their combination leads to an exponential convergence to equilibrium.

For more general potentials, there is still a global equilibrium:

$$f_\infty(x, v) = \frac{e^{-\left[V(x) + \frac{|v|^2}{2}\right]}}{Z},$$

where Z is a normalizing constant. Then it is an obviously natural question whether exponential convergence to f_∞ holds true under adequate assumptions on the potential V, which go beyond the "trivial" quadratic case. Shockingly enough, the first such results were obtained only around 2002, by Frédéric Hérau and Francis Nier [19]. They used a quite sophisticated approach taking roots in Joseph Kohn's approach to hypoellipticity. Since then, their method has been very much simplified, as I shall describe later.

About the choice of functional space. The choice of functional space in which to study the large-time behavior of (1) is not innocent. From a probabilistic or physical point of view, it is most natural to assume that f is an integrable density (or even a measure) with possibly rapid decay at infinity. However, in the majority of mathematical studies on the Fokker–Planck equation (kinetic or not), a different choice is made, namely

$$\int \frac{f^2}{f_\infty} < \infty. \tag{2}$$

The reason is simple: Perform a change of unknown in (1) by writing $h = f/f_\infty$ (from a probabilistic perspective, this amounts to considering the adjoint equation); then (1) turns into

$$\frac{\partial h}{\partial t} + v \cdot \nabla_x h - \nabla V(x) \cdot \nabla_v h = \Delta_v h - v \cdot \nabla_v h. \tag{3}$$

Now the operator appearing on the right-hand side is self-adjoint in the Hilbert space $L^2(f_\infty\, dx\, dv)$, so (3) might lend itself to a spectral treatment. Assumption (2) simply says that h belongs to the above-mentioned Hilbert space.

Of course, formally, equations (1) and (3) are equivalent. But this is misleading, since the additional assumption (2) is a very strong restriction. In fact, it does happen that for certain potentials V, the convergence to equilibrium is exponential under the "L^2-type" assumption (2), but not under a more general "L^1-type" assumption (that is assuming just integrability, and maybe some moment bounds). In this sense, L^1 results are stronger than L^2 results. This is actually one of the reasons of the popularity of logarithmic Sobolev inequalities: They provide a natural functional tool to study convergence to equilibrium in L^1 spaces.

The moral of this discussion is that for physical relevance the discussion of convergence to equilibrium of solutions to (1) should not be limited to an L^2 framework, but also include more general L^1-type assumptions. In the sequel, I shall describe some results in this direction.

Oscillator chains. Even though Fourier's law of conduction of heat is one of the oldest partial differential equations, it is still extremely far from a rigorous theoretical understanding. Many models of statistical physics have been proposed to describe heat conduction. Here is one of them, described in [8]. Each atom in a solid body is labelled (for the sake of this discussion, we may assume that the dimension is 1, so atoms are labelled $0, 1, \dots, N$), and the unknowns are the displacements $X^0, \dots, X^N$ of the atoms with respect to their respective equilibrium positions. Each atom is bound to its equilibrium position with a "pinning potential" V, and it also interacts with its two neighbors by an interaction potential W, assumed to be symmetric ($W(z) = W(-z)$). So the equation for X^k is just

$$\frac{d^2 X_t^k}{dt^2} = -\nabla V(X_t^k) - \nabla W(X_t^k - X_t^{k-1}) - \nabla W(X_t^k - X_t^{k+1}). \tag{4}$$

Of course these equations do not apply to the atoms that are at the extreme left ($k = 0$) and the extreme right ($k = N$) in the chain, since they have only one neighbor. But these extremal atoms are also *shaken* by some external bath, with a temperature of agitation $T^{(\ell)}$ on the left, and $T^{(r)}$ on the right. The corresponding equations for, say, $k = 0$, can be written

$$\begin{cases} \dfrac{d^2 X_t^0}{dt^2} = -\nabla V(X_t^0) - \nabla W(X_t^0 - X_t^1) + \ell, \\ \dfrac{d\ell}{dt} = \lambda^{(\ell)} \sqrt{2T^{(\ell)}}\, \dfrac{dB_t^{(\ell)}}{dt} - \ell + (\lambda^{(\ell)})^2 \ell\, \dfrac{dX_t^0}{dt}. \end{cases} \tag{5}$$

Here $\lambda^{(\ell)}$ is a coefficient describing the strength of the coupling between the particle and the heat bath.

Again, the law of this system is described by a linear partial differential equation in the variables $\ell, r, X^0, \dots, X^N, \dot{X}^0, \dots, \dot{X}^N$. It is very similar to the kinetic Fokker–Planck equation, except that it is much more degenerate, since the diffusion only acts on the variables ℓ and r.

There are now two difficult problems which naturally arise: (i) Show that the solution $f_t(\ell, r, x^0, \dots, x^N, v^0, \dots, v^N)$ approaches some stationary distribution as $t \to \infty$; (ii) Study the properties of this stationary distribution, and in particular the associated energy flux. (In this case, it is better to say "stationary distribution" rather than "equilibrium", precisely because the temperatures are not necessarily equal.) In particular, if $T^{(\ell)} > T^{(r)}$, in the asymptotic regime $N \to \infty$, is it true that energy flows from the left to the right, and what is the relation between the average flux and the difference of temperatures!?

When $T^{(\ell)} = T^{(r)}$, the equilibrium distribution is easy to write down explicitly, and problem (ii) is trivially solved. But as soon as these temperatures are different, the stationary solution is not explicit – except in the case when V and W are quadratic, but then the results are physically irrelevant!! It is conjectured that some anharmonicity is *necessary* to get the Fourier law (ironically enough, the heat equation, although one of the most basic *linear* models in science, needs some dose of microscopic nonlinearity to be explained). Then problem (ii) becomes incredibly difficult.

Even when the two temperatures are equal, problem (i) appears to be quite difficult. It is actually a typical hypocoercive situation: The diffusion on ℓ and r should lead in the end to a relaxation to equilibrium in all variables.

Exponential convergence to the stationary distribution has been proved recently by several authors [8], [7], even for the case when $T^{(\ell)} \neq T^{(r)}$, under various assumptions on the potentials; but the dependence of the estimates upon the number of atoms is just terrible.

The Boltzmann equation. The Boltzmann equation is one of the basic partial differential equations in statistical mechanics. It is a kinetic model for the evolution of a rarefied gas of particles interacting via binary collisions. Historically, it has preceded

the Fokker–Planck equation; but the analytical problems that it raises are considerably more acute. A mathematically-oriented presentation of the Boltzmann equation can be found in my long review paper [27]. The classical Boltzmann equation in n dimensions of space can be written

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f), \quad x \in \Omega_x, \ v \in \mathbb{R}^n, \tag{6}$$

where Ω_x is a bounded connected open spatial domain, and Q is the Boltzmann collision operator, defined by

$$Q(f, f) = \int_{\mathbb{R}^n} \int_{S^{n-1}} B(v - v_*, \sigma)\big[f(x, v')f(x, v'_*) - f(x, v)f(x, v_*)\big]\, d\sigma\, dv_*.$$

Here σ is a variable unit vector in $\mathbb{R}^n$, $B(v - v_*, \sigma)$ is a collision kernel depending on the particular form of the interaction (for instance $B(v - v_*, \sigma) = |v - v_*|$), and the transform $(v, v_*) \to (v', v'_*)$ is computed by the rules of elastic collision:

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma.$$

This equation should of course be supplemented with boundary conditions. To simplify things, one can assume that Ω_x is just the n-dimensional torus $\mathbb{T}^n$ (periodic boundary conditions); another common choice is specular reflexion in a bounded open set.

In spite of hundreds of papers, the mathematical theory of the Boltzmann equation is far from complete; in particular there is still no theory of classical solutions in the large. However, strong regularity results have been obtained in a close-to-equilibrium regime. A complete theory can also be put together as long as there is a pointwise control of certain hydrodynamic fields (density in physical space, mean velocity, temperature, pressure tensor).

It was Boltzmann's beautiful observation that the H functional (negative of the entropy),

$$H(f) = \int f \log f \, dx\, dv$$

is nonincreasing with time along solutions of the Boltzmann equation. Then there is a unique large-time equilibrium, which takes the form of a Maxwellian (Gaussian) distribution:

$$f_\infty(x, v) = \rho \, \frac{e^{-\frac{|v-u|^2}{2T}}}{(2\pi T)^{n/2}},$$

where $\rho \geq 0$ (total mass), $u \in \mathbb{R}^n$ (total mean momentum) and $T \geq 0$ (mean temperature) are constants. This equilibrium is obtained by maximizing the entropy given the conservation laws.

The problem of convergence to equilibrium for the Boltzmann equation is famous for historical reasons (it triggered a hot controversy in the nineteenth century) and also

for theoretical reasons (as a manifestation of irreversibility in the statistical description of a reversible mechanical system; and as the justification of the law of maximum entropy on a basic model). See my lecture notes [29] for an overview of this question.

Of course the complexity of the Boltzmann equation, and its nonlinearity are major difficulties in the study. But behind that, we can recognize once again a hypocoercive situation: The dissipation (collision) operator Q on the right-hand side of (6) is very degenerate since is only acts on the velocity dependence, and it is only its association with the conservative transport operator $v \cdot \nabla_x$ on the left-hand side which can lead to convergence to equilibrium.

Stability of Oseen's vortices. The last example in this gallery comes from hydrodynamics and was brought to my attention by Thierry Gallay. It is a well-documented fact in turbulence theory that the vorticity of a two-dimensional incompressible flow tends to coalesce and form large vortices. Thierry Gallay and Eugene Wayne [11] have studied this phenomenon rigorously for a two-dimensional incompressible viscous fluid in the whole space: If $\omega = \omega_t(x)$ is the vorticity, the equation is just

$$\frac{\partial \omega}{\partial t} + \mathrm{BS}[\omega] \cdot \nabla \omega = \Delta \omega,$$

where $\mathrm{BS}[\omega]$ is the velocity field obtained from the vorticity ω via the Biot–Savart law:

$$\mathrm{BS}[\omega](x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\, \omega(y)\, dy,$$

and $v^\perp$ is obtained from v by rotation of angle $\pi/2$.

If $\omega_0 \in L^1(\mathbb{R}^2)$, then ω_t converges to 0 as $t \to \infty$, due to viscous dissipation. But a refined analysis shows that ω_t is asymptotically close to an explicit self-similar Gaussian solution, which physically corresponds to a unique large vortex, called Oseen's vortex. In fact, in suitably rescaled variables, the vorticity does converge to a stationary Gaussian distribution.

The linear stability analysis of this phenomenon reduces to the spectral analysis of the operator $S + \alpha B$ in $L^2(\mathbb{R}^2)$, where

$$\begin{cases} S\omega = -\Delta\omega + \dfrac{|x|^2}{16}\,\omega - \dfrac{\omega}{2}, \\ B\omega = \mathrm{BS}[G] \cdot \nabla\omega + 2\,\mathrm{BS}[G^{1/2}\omega] \cdot \nabla G^{1/2}. \end{cases} \tag{7}$$

Here G is a Gaussian distribution: $G(x) = e^{-|x|^2/4}/(4\pi)$; and α is the value of the "circulation Reynolds number", which in the present set of conventions is just $\int \omega_0$.

The spectral study of $S + \alpha B$ turns out to be quite tricky. In the hope of getting a better understanding, one can decompose ω in Fourier series: $\omega = \sum_{n\in\mathbb{Z}} \omega_n(r) e^{in\theta}$, where (r, θ) are standard polar coordinates in $\mathbb{R}^2$. For each n, the operators S and B

can be restricted to the vector space generated by $e^{in\theta}$, and can be seen as just operators on a function $\omega(r)$:

$$\begin{cases} (S_n\omega)(r) = -\partial_r^2\omega - \left(\frac{r}{2} + \frac{1}{r}\right)\partial_r\omega - \left(1 - \frac{n^2}{r^2}\right)\omega, \\ (B_n\omega)(r) = i\,n\,(\varphi\omega - g\Omega_n). \end{cases}$$

Here $g(r) = e^{-r^2/4}/4\pi$, $\varphi(r) = (1-e^{-r^2/4})/2\pi r^2$, and $\Omega_n(r)$ solves the differential equation

$$-(r\Omega')' + \frac{n^2}{r}\Omega = \frac{r}{2}\omega.$$

The regime $|\alpha| \to \infty$ is of physical interest and has already been the object of numerical investigations by physicists. There are two families of eigenvalues which are imposed by symmetry reasons; but apart from that, it seems that all eigenvalues converge to infinity as $|\alpha| \to \infty$, and for some of them the precise asymptotic rate of divergence $O(|\alpha|^{1/2})$ has been established by numerical evidence. If that is correct, this means that the "perturbation" of S by αB is strong enough to send most eigenvalues to infinity as $|\alpha| \to \infty$. This is particularly striking when one realizes that S is symmetric in $L^2(\mathbb{R}^2)$, while B is antisymmetric. Obviously, this is again a manifestation of a hypocoercive phenomenon.

Let us simplify things just a bit by throwing away the nonlocal term $g\Omega_n$ in the expression of B_n. After a few manipulations, the problem reduces to the following

Model Problem 1.1. Identify sufficient conditions on $f : \mathbb{R} \to \mathbb{R}$, so that the real parts of the eigenvalues of

$$L_\alpha : \omega \longmapsto (-\partial_x^2\omega + x^2\omega - \omega) + i\alpha f\omega$$

in $L^2(\mathbb{R})$ go to infinity as $|\alpha| \to \infty$, and estimate this rate.

So far this problem has been solved only partially, by Isabelle Gallagher and Thierry Gallay; I shall describe their results later on.

2. A dynamical approach

Together with Laurent Desvillettes [3], [5], I have developed a method to study quite general hypocoercive situations. The method ultimately relies on the analysis of a *system* of coupled differential inequalities of first and second order (instead of just one first-order differential inequality as in Gronwall's lemma). The method was devised with the aim of proving convergence to equilibrium for *uniformly smooth* solutions of the Boltzmann equation, so I shall explain its principle on that particular example. Complete proofs [5] are quite long and technical, so my goal here is only to isolate the main ideas in a sketchy way.

First and second order differential inequalities. As we know, Boltzmann's H functional goes down in time along solutions of (6). A more precise analysis shows that the total entropy production, $-dH/dt$, is always strictly positive, unless f_t is a *hydrodynamical state*:

$$f_t(x, v) = \rho_t(x) \frac{e^{-\frac{|v-u_t(x)|^2}{2T_t(x)}}}{(2\pi T_t(x))^{n/2}}.$$

In words, a hydrodynamical state is a kinetic distribution which is in Maxwellian equilibrium with respect to the velocity variable, but not with respect to the position variable; so it only depends on the fields of local density (ρ), mean velocity (u) and local temperature (T).

With a much more refined analysis, one can establish a *quantitative lower bound* on the entropy production, under adequate (very strong) smoothness, decay and positivity assumptions on f: For any $\varepsilon > 0$ there is a constant $K_\varepsilon > 0$ such that

$$-\frac{d}{dt}[H(f) - H(M)] \geq K_\varepsilon \big[H(f) - H(M^f_{\rho\, u\, T})\big]^{1+\varepsilon}, \tag{8}$$

where M is the global equilibrium, which is a Maxwellian distribution, $M^f_{\rho\, u\, T}$ is the hydrodynamical state with the same (local) density, mean velocity and temperature as f, and the dependence of f, ρ, u and T on time is implicit. (The quantity in the right-hand side of (8) is nonnegative.)

Inequality (8) gives a good lower bound on the entropy production, as long as the unknown f *stays away from hydrodynamical states*. But if f decides to become hydrodynamic, or very close to, then the entropy production vanishes and there is nothing that we can deduce about the convergence to equilibrium. This is where the antisymmetric part of the Boltzmann equation has to help us.

Now the second differential inequality is obtained by introducing a suitable functional measuring the distance of f to the space of hydrodynamical states. On one hand it should be controlled by the quantity $H(f) - H(M^f_{\rho\, u\, T})$; but on the other hand it should be simple enough to make explicit computations. A natural choice is $\mathcal{E}_1(f) = \|f - M^f_{\rho\, u\, T}\|^2_{L^2}$; note that this functional depends on f in a strongly nonlinear way, via ρ, u and T. The point now is to give a *lower bound on the second-order time-derivative* of this new functional.

Differentiating a functional once along the Boltzmann equation is already complicated, but differentiating it twice is a horrendous task; so it better be well motivated. There are in fact two main reasons to consider the second derivative. The first is that the second derivative in time gives a measure of how fast the distance between f and $M^f_{\rho\, u\, T}$ will increase again if it ever vanishes, or becomes very close to. The second reason is that by applying twice the Boltzmann equation, we let the first-order operator $v \cdot \nabla_x$ act twice, and then the resulting computations are somewhat similar to those that would have been obtained by letting a second-order operator *in the x variable* act once. So in this second-order time derivative one will find some of the

terms that would have appeared in a first-order computation along the heat equation in the x variable.

So, after many calculations one can show that for δ_1 small enough,

$$\frac{d^2}{dt^2}\|f - M^f_{\rho\, u\, T}\|^2_{L^2} \geq K_1 \left[\int_{\Omega_x} |\nabla T(x)|^2\, dx + \int_{\Omega_x} |\{\nabla u(x)\}|^2\, dx\right] - \frac{C_1}{{\delta_1}^{1-\varepsilon}}\left(\|f - M^f_{\rho\, u\, T}\|^2_{L^2}\right)^{1-\varepsilon} - \delta_1[H(f) - H(M)], \tag{9}$$

where K_1 and C_1 are constants which only depend on some a priori smoothness and positivity estimates on f, and $\{\nabla u\}$ is the traceless part of the symmetric part of the matrix-valued field ∇u.

To understand what has been achieved, assume for a moment that f becomes hydrodynamical at some time t_0, and forget the error term with δ_1. Then we have $(d^2/dt^2)\|f - M^f_{\rho\, u\, T}\|^2_{L^2} \geq K_1\|\nabla T\|^2_{L^2}$. In particular, $\|f - M^f_{\rho\, u\, T}\|^2$ is strictly convex, as a function of t, unless ∇T is equal to 0, and will grow quadratically for a short time. So this second equation gives us some information about the inhomogeneities in the temperature field T. In a geometric language, what we have shown, more or less, is that the Boltzmann flow is "transverse" to the space of hydrodynamical states, in presence of heterogeneities of the temperature.

Combining this with the first inequality (8), we see that we have some information about how far f is to the space of hydrodynamical states with constant temperature. There is still something missing, but now we can repeat the procedure: Introduce a new functional measuring the distance of f to that space, for instance $\mathcal{E}_2(f) = \|f - M^f_{\rho\, u\, \langle T\rangle}\|^2_{L^2}$, where $M^f_{\rho\, u\, \langle T\rangle}$ is the hydrodynamical state which has the same density ρ and velocity fields u as f, and a constant temperature whose value is computed by averaging the temperature T of f against the density ρ. Then differentiate twice again. New computations yield a result which is very similar to the one in (9), except that the terms $\|\nabla T\|^2$ and $\|\{\nabla u\}\|^2$ are replaced by $\|\nabla^{\mathrm{sym}} u\|^2$, the square L^2 norm of the complete symmetric part of ∇u.

As a general principle (Korn's inequality), a control of the symmetric part of ∇u implies a control on the whole of ∇u, under suitable boundary conditions. In fact the boundary conditions here are not standard, at least for specular reflexion, so the desired estimates do not follow from the classical theory of Korn inequalities; but let us forget this for the moment and assume that we have indeed a good control on the inhomogeneities of the velocity field.

So far there is still no control on the inhomogeneities of the density, but at this stage the reader has probably understood the sequel of the method: Introduce a suitable functional $\mathcal{E}_3$ measuring the distance of f to the space of hydrodynamical states with constant temperature and constant velocity field, and differentiate this expression twice in time. It is possible to choose $\mathcal{E}_3(f) = \|f - M^f_{\rho\, 0\, 1}\|^2_{L^2}$, where $M^f_{\rho\, 0\, 1}$ has the same density as f, the same average temperature and the same total momentum; and then in the resulting computations pops up the desired term $\|\nabla\rho\|^2$.

Closing the system. At this stage we have a system of four differential inequalities (one of first order, and three of second order). To close the system, one can use:

- A *physical input*: The total entropy is the sum of a purely kinetic entropy and a purely hydrodynamical entropy, which controls inhomogeneities of all fields ρ, u and T.
- An *analytical input*: In presence of smoothness bounds, all the norms appearing in the computations are "almost equivalent". More precisely, if $\|f\|_1$ and $\|f\|_2$ are any two Lebesgue, or Sobolev norms, then for any $\varepsilon > 0$ one can find a constant C_ε, only depending on some smoothness estimates on f, such that $\|f\|_1 \leq C_\varepsilon \|f\|_2^{1-\varepsilon}$. This step, obviously based on elementary interpolation theory, is crucial to "get the exponents right"; without it, one would get disastrous rates of convergence, or just no rate at all. This way of *trading smoothness for exponents* is one of the reasons why the method is so greedy in regularity.
- A *geometric/analytical input*: Certain norms of differential quantities ($\|\nabla T\|$, $\|\nabla^{\mathrm{sym}} u\|$, $\|\nabla \rho\|$) imply a control on the departure of the corresponding fields to their mean value. The shape of the domain (connectedness, rotational symmetry, etc.) plays a crucial role here; all of this can be expressed *quantitatively* with some functional inequalities of Poincaré or Korn type (see in particular [4]).

Study of the differential system. Not all the hard job has been done at this stage. The result is a system of first and second-order differential inequalities, coupled together in a quite intricate way, from which one wants to extract estimates about the rates of relaxation. This can be done with the help of the following (definitely not obvious) lemma:

Lemma 2.1. *Let $h(t) \geq 0$ satisfy*

$$h''(t) + A h(t)^{1-\varepsilon} \geq \alpha > 0 \quad \text{for all } t \in (t_1, t_2)$$

for some $\varepsilon < 0.1$. Then,

– either $t_2 - t_1$ is small:

$$t_2 - t_1 \leq 50\, \frac{\alpha^{\frac{\varepsilon}{2(1-\varepsilon)}}}{A^{\frac{1}{2(1-\varepsilon)}}};$$

– or h is large on the average:

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} h(t)\, dt \geq \frac{\alpha^{\frac{1}{1-\varepsilon}}}{100}\ \inf\left(\frac{1}{A}, \frac{1}{A^2}\right).$$

With a repeated use of Lemma 2.1, one can show in the end that the system of differential inequalities implies relaxation to equilibrium at a rate $O(t^{-\infty})$, that is, faster than any inverse power of time. All in all, one can get the following result, which is stated here in a slightly sketchy way (see [5] for more precise statements):

Theorem 2.2. *Let $(f_t)_{t\geq 0}$ be a smooth solution of the Boltzmann equation* (6)*, such that all the derivatives of f are uniformly bounded, and all the moments of f are bounded, uniformly in time. Further assume that f satisfies a pointwise lower bound of the form $f_t(x,v) \geq K_0 e^{-A_0|v|^{q_0}}$. Then, under adequate boundary conditions, f_t converges to global equilibrium as $t \to \infty$, at least as fast as $O(t^{-\kappa})$ for all $\kappa > 0$.*

Further comments. More information about the implementation of this program in the context of the Boltzmann equation can be found in the original research paper [5], or, in a lighter form, in the lecture notes [30], [29]. Before being used on the Boltzmann equation, the dynamical approach had been tried on the Fokker–Planck equation [3] and on some other linear models [1], [9]. It is quite robust and adapted to equations with very little structure.

One of its appealing features is that it seems to provide a good physical intuition of what is going on: The system approaches hydrodynamical state under the influence of collisions, then it is driven out of hydrodynamical state by the influence of the transport, etc. Numerical simulations have corroborated this qualitative analysis surprisingly well. In the diagram below, computed numerically by Francis Filbet, one sees very clearly that the solution of the Boltzmann equation oscillates between states

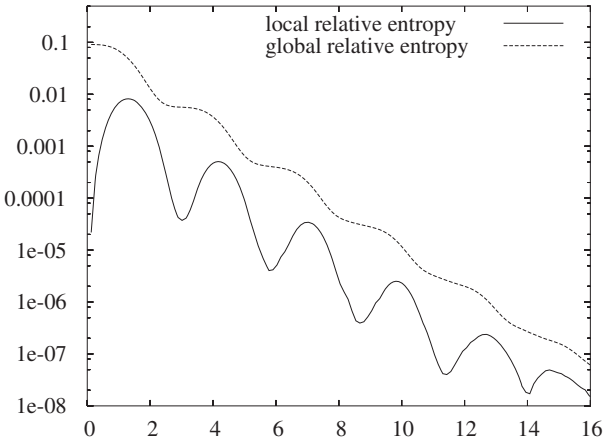

Figure 1. The upper curve is the H functional as a function of time, in semi-log plot; the lower curve is the purely kinetic part of the H functional. When the two curves are far away, the distribution is almost in hydrodynamical state; when they are very close, it is almost homogeneous.

where it is close to hydrodynamical, and states where it is close to homogeneous. In particular, contrary to a widespread belief, the approach to hydrodynamical regime is not faster than the approach to global equilibrium. (All of this is valid only on scales of time on which the Knudsen number is of order 1.)

From the point of view of physics, the discovery of these oscillations may be one of the most noticeable outcomes of the program of hypocoercivity applied to

the Boltzmann equation. (No doubt that one day there will be a simpler analytical way to explain them.) They are not easy to observe, and have even been used as a "benchmark" to test the accuracy of certain numerical schemes (see e.g. [10]).

However, the dynamical method suffers from the complexity of its practical application, and its heavy computational cost. In the next section, I shall describe another method which may be less appealing from the physical point of view, and requires a bit more structure, but has the advantage to be much lighter.

Also, I emphasize that Theorem 2.2 requires strong regularity and decay estimates on the solutions. As discussed in [5], all these estimates can be proven in the *close-to-equilibrium* regime, but remain a major open problem for solutions in the large. Even in a close-to-equilibrium regime, decay estimates based on a linearization method are quite hard to obtain, and were not available at the time when [5] was published; since then this gap has been filled in a series of important works by Yan Guo and Robert Strain [13], [14].

3. A functional approach

In the previous method the resolution of the main degeneracy problem was done via the time-differentiation of certain (relatively) simple functionals. Now the idea is to put as much as possible of the difficulty in a careful choice of the functional; and more precisely to add "correction terms" which are negligible in size, but contribute in an important way to the time-derivative of the functional. This will be more easily explained on the example of the Fokker–Planck equation, in the form (3).

Suppose you want to get a Gronwall inequality for some well-chosen functional, applied to the Fokker–Planck equation. First try the L^2 norm. With the notation $\mu(dx\,dv) = f_\infty(x,v)\,dx\,dv$, and omitting once again the dependence upon time, one has

$$\frac{d}{dt}\int h^2\,d\mu = -\int |\nabla_v h|^2\,d\mu.$$

Since the derivatives in the right-hand side involve only the velocity variables, there is no way to dominate the integral in the left-hand side by the right-hand side (choose $h = h(x)$, then the right-hand side vanishes).

So go to a higher order norm, involving gradients of h. After a bit of work, under suitable assumptions on V, one can find constants $a, c, K > 0$ such that

$$\begin{aligned}\frac{d}{dt}\left(\int h^2\,d\mu + a\int |\nabla_x h|^2\,d\mu + c\int |\nabla_v h|^2\,d\mu\right)\\ \leq -K\left(\int |\nabla_v h|^2\,d\mu + \int |\nabla_v\nabla_x h|^2\,d\mu + \int |\nabla_v\nabla_v h|^2\,d\mu\right).\end{aligned} \tag{10}$$

Again, the right-hand side is not sufficient to control the expression in brackets on the left-hand side. So still nothing!

But now correct the functional on the left-hand side by adding an innocent-looking term $2b \int \nabla_x h \cdot \nabla_v h \, d\mu$. If $b < \sqrt{ac}$, this term does not play any noticeable role in the value of the functional, since

$$\left| 2b \int \nabla_x h \cdot \nabla_v h \, d\mu \right| \leq (1 - \delta) \left[a \int |\nabla_x h|^2 \, d\mu + c \int |\nabla_v h|^2 \, d\mu \right]$$

for some positive constant δ. However, if a, b and c are properly chosen, then we have a differential inequality which is much better than (10):

$$\begin{aligned} &\frac{d}{dt} \left(\int h^2 \, d\mu + a \int |\nabla_x h|^2 \, d\mu + 2b \int \nabla_x h \cdot \nabla_v h \, d\mu + c \int |\nabla_v h|^2 \, d\mu \right) \qquad (11) \\ &\leq -K \left(\int |\nabla_x h|^2 \, d\mu + \int |\nabla_v h|^2 \, d\mu + \int |\nabla_v \nabla_x h|^2 \, d\mu + \int |\nabla_v \nabla_v h|^2 \, d\mu \right). \end{aligned}$$

Now it very easy to close this differential inequality: It suffices that μ satisfies a Poincaré inequality (in the x and v variables).

The algebraic core. I started to work on this approach while struggling to understand the results of Frédéric Hérau and Francis Nier [19], without resorting to the technical hypoelliptic machinery used in their work. After deciding that there should be an elementary approach based on integration by parts and chain rule, I was still flooded by the complex calculations. Then I decided that there should be an even simpler approach with no analysis at all. After going to an abstract formulation of the problem, I found out that there was indeed an extremely simple "algebraic core" which can be presented as follows. Take two operators A and B on a Hilbert space (in the present case A would be the vector-valued differential operator ∇_v, while B would be $v \cdot \nabla_x - \nabla V(x) \cdot \nabla_v$), with $B^* = -B$. Then, at least formally, the time-derivative of

$$\langle Ah, [A, B]h \rangle$$

along the influence of B can be written as

$$\langle ABh, [A, B]h \rangle + \langle Ah, [A, B]Bh \rangle.$$

Pretend that B commutes with $[A, B]$; then the previous expression is

$$\langle ABh, [A, B]h \rangle + \langle Ah, B[A, B]h \rangle,$$

and since $B^* = -B$, this can be rewritten as

$$\langle ABh, [A, B]h \rangle - \langle BAh, [A, B]h \rangle = \langle [A, B]h, [A, B]h \rangle. \qquad (12)$$

In the example of the Fokker–Planck equation, $[A, B] = \nabla_x$, so $\langle Ah, [A, B]h \rangle = \int \nabla_v h \cdot \nabla_x h \, d\mu$, and the right-hand side of (12) is the desired term in $\|\nabla_x h\|^2$.

The advantage to input terms with "mixed derivatives" such as $\int \nabla_x h \cdot \nabla_v h$ had been actually noted before in studies of global in time propagation of the smoothness

for kinetic equations, most notably by Denis Talay [24] and Yan Guo [12]. The simple algebraic core presented above explains why this trick also applies to problems of convergence to equilibrium.

The rest of this section will be devoted to a presentation of some results which have been obtained by pushing further this approach. All the results quoted below are extracted from two preprints by the author [31], [32], and another preprint by Clément Mouhot and Lukas Neumann [21]. There is also an independent series of works by Frédéric Hérau [18], [17], which is based on quite similar tools.

The basic theorem. Two important features of the next theorem are that

– it applies to a general abstract framework: $\mathcal{H}$ is a Hilbert space (think of $\mathcal{H}$ as $L^2(\mu)$, where μ is the equilibrium measure); and $\mathcal{V}$ another Hilbert space (think of $\mathcal{V}$ as $\mathbb{R}^n$, the space of velocities); then A is an unbounded operator $\mathcal{H} \to \mathcal{H} \otimes \mathcal{V}$, and B is an unbounded operator $B \to B$ with $B^* = -B$;

– it considers a linear operator L which is in (abstract) *Hörmander form*, that is $L = A^*A + B$ for some operators A, B as above.

Assume that the semigroup e^{-tL} is well defined and that there is no problem to differentiate the square norms, etc. Systematic tensorization with the identity operator will be used to make sense of notation such as $[A, B] = AB - (B \otimes I)A$. The scalar product in $\mathcal{H}$ will be denoted by $\langle \cdot, \cdot \rangle$, the norm in $\mathcal{H}$ by $\| \cdot \|$, and an operator S will be said to be bounded respectively to a family of operators $T_1, \dots, T_k$ if there is a constant C such that $\|Sy\| \le C(\|T_1 y\| + \cdots + \|T_k y\|)$. The symbol $\Re$ stands for real part.

Theorem 3.1. *With the above notation, write* $[A, B] = C$*, and assume that*

(i) $[A, C] = 0$, $[A^*, C] = 0$;

(ii) $[A, A^*]$ *is bounded relatively to* I *and* A;

(iii) $[B, C]$ *is bounded relatively to* A, A^2, C *and* AC.

Further assume that

(H) $A^*A + C^*C$ *is coercive.*

Then for a suitable choice of constants a, b, c *one has the differential inequality*

$$\frac{d}{dt}\mathcal{F}(e^{-tL}h) \le -K\,\mathcal{F}(e^{-tL}h),$$

where

$$\mathcal{F}(h) = \|h\|^2 + a\,\|Ah\|^2 + 2b\,\Re\,\langle Ah, Ch\rangle + c\,\|Ch\|^2,$$

and K *is a positive constant which only depends on the constants appearing implicitly in assumptions* (ii)*,* (iii) *and* (H).

Remark 3.2. The assumption (H) is obviously an analog in this context of Lars Hörmander's bracket condition.

This theorem applies to the Fokker–Planck equation (3) under simple assumptions on the potential V, and yields exponential convergence to equilibrium for initial data h_0 satisfying $\|\nabla_x h_0\|^2 + \|\nabla_v h_0\|^2 < +\infty$. The latter restriction can finally be removed by an independent study of hypoelliptic regularity [16], [31]. (This is not a standard hypoelliptic estimate since it is global; there would be much to say about it, but this would take us too far.) In the end, one obtains the following theorem, which generalizes and improves the results of [19], [16]. Recall that a measure ν is said to satisfy a Poincaré inequality if one has a functional inequality of the form $\|\nabla h\|_{L^2(\nu)} \geq P\|h - \langle h\rangle\|_{L^2(\nu)}$, $P > 0$, where $\langle h\rangle$ is the average value of h with respect to ν.

Theorem 3.3. *Let $V \in C^2(\mathbb{R}^n)$ with $\inf V > -\infty$, such that*

(a) $|\nabla^2 V| \leq C(1 + |\nabla V|)$;

(b) *the reference measure $\nu(dx) = e^{-V(x)}\,dx$ satisfies a Poincaré inequality with constant P.*

Let $\mu(dx\,dv) = e^{-(V(x)+|v|^2/2)}\,dx\,dv/Z$, where Z is a normalizing constant. Then there are constants $\lambda > 0$ and C', explicitly computable in terms of C and P, such that solutions of the Fokker–Planck equation (3) *satisfies*

$$\|h_t - \langle h_0\rangle\|_{L^2(\mu)} \leq C' e^{-\lambda t}\,\|h_0 - \langle h_0\rangle\|_{L^2(\mu)}.$$

Theorem 3.1, or more precisely its proof, was also used by Isabelle Gallagher and Thierry Gallay to provide a first solution to the Model Problem 1.1, as follows. Set $\mathcal{H} = L^2(\mathbb{R};\mathbb{C})$, $A = \partial_x\omega + x\omega$, $B\omega = (i\alpha f)\omega$. Then $C\omega = i\alpha f'\omega$, so the operator $A^*A + C^*C$ is of Schrödinger type:

$$(A^*A + C^*C)\,\omega = (-\partial_x^2\omega + x^2\omega - \omega) + \alpha^2 f'^2\omega,$$

and the spectrum of $A^*A + C^*C$ can be studied via standard semi-classical techniques. For instance, if $f'(x)^2 = x^2/(1+x^2)^k$, $k \in \mathbb{N}$, then the spectral gap of $A^*A + C^*C$ is bounded below like $O(|\alpha|^{2\nu})$, with $\nu = \min(1, 2/k)$. Then a careful examination of the proof of Theorem 3.1 yields a lower bound like $O(|\alpha|^{\nu})$ on the real part of the spectrum of $A^*A + B$.

Multiple commutators. As in Lars Hörmander's hypoellipticity theorem, multiple commutators are also allowed in hypocoercivity results. But as an important difference, it seems that one only needs to consider commutators with the antisymmetric part. Here is such a theorem:

Theorem 3.4. *With the same notation as before, assume the existence of (possibly unbounded) operators $C_0, C_1, \ldots, C_{N_c+1}$, $R_1, \ldots, R_{N_c+1}$, and $Z_1, \ldots, Z_{N_c+1}$ such that*

$$C_0 = A, \quad [C_j, B] = Z_{j+1}C_{j+1} + R_{j+1} \ \ (0 \leq j \leq N_c), \quad C_{N_c+1} = 0,$$

and, for all $k \in \{0, \dots, N_c\}$,

(i) $[A, C_k]$ *is bounded relatively to* $\{C_j\}_{0\le j\le k}$ *and* $\{C_j A\}_{0\le j\le k-1}$;

(ii) $[A^*, C_k]$ *is bounded relatively to* I *and* $\{C_j\}_{0\le j\le k}$;

(iii) R_k *is bounded relatively to* $\{C_j\}_{0\le j\le k-1}$ *and* $\{C_j A\}_{0\le j\le k-1}$;

(iv) Z_j *is bounded relatively to* I, *and* I *is bounded relatively to* Z_j;

(H) $\sum_{j=0}^{N_c} C_j^* C_j$ *is coercive.*

Then one can choose constants a_k *and* b_k *in such a way that the functional*

$$\mathcal{F}(h) = \|h\|^2 + \sum_{k=0}^{N_c} \big(a_k \|C_k h\|^2 + 2 b_k \, \Re \, \langle C_k h, C_{k+1} h \rangle \big)$$

satisfies the differential inequality

$$\frac{d}{dt} \mathcal{F}(e^{-tL} h) \le -K \mathcal{F}(e^{-tL} h)$$

for some constant K *which can be computed explicitly in terms of the constants appearing implicitly in* (i)–(iv) *and* (H).

This result generalizes Theorem 3.1 in several ways: Multiple commutators are allowed; a remainder R_{j+1}, and a multiplier Z_j are allowed in the identity defining C_{j+1} in terms of C_j; and the various directions C_k are not assumed to commute.

As a simple application of Theorem 3.4, it is possible to prove exponential convergence to equilibrium for the oscillator chain described by equations (4)–(5), under the assumption that V and W are uniformly convex and have a bounded Hessian, *and that the temperatures on the left and on the right are equal*, that is $T^{(\ell)} = T^{(r)}$. Interestingly enough, bounded Hessians are not covered by the results of Jean-Pierre Eckmann and Martin Hairer [7], who impose a superquadratic growth at infinity. Conversely, it is not clear whether Theorem 3.4 can be used to recover the results in [7]. Still some work is required to clarify the situation about these assumptions on the potentials. I shall also come back in the end of these notes to the very unsatisfactory restriction $T^{(\ell)} = T^{(r)}$.

$L \log L$ estimates. Now it is not so difficult to adapt the previous L^2 theory to an $L \log L$ framework, replacing L^2 square norms by *entropies and Fisher informations*. For this I shall have to leave the framework of abstract Hilbert spaces, and replace it by functional spaces on, say, $\mathbb{R}^N$ (or a differentiable manifold). Then the operators $A_1, \dots, A_m$ and B will be vector fields, identified with differentiation operators, the notation A will stand for the vector-valued differential operator $A = (A_1, \dots, A_m)$, and I shall now say that S is bounded relatively to $T_1, \dots, T_k$ if there is a constant C such that $|S(x)| \le C(|T_1(x)| + \cdots + |T_k(x)|)$, where $S(x)$ stands for the value of the vector field S at x. The equilibrium measure will be assumed to take the form $\mu(dX) = e^{-E} \, dX$, where E is a smooth function, dX is the Lebesgue measure in

$\mathbb{R}^N$, and E is normalized so that $\int e^{-E} = 1$. The notation S^* will stand for the adjoint of S *in* $L^2(\mu)$. The linear equation under study will still be $\partial_t h + Lh = 0$, where $L = A^*A + B$ and the unknown is the probability density $f = he^{-E}$.

Theorem 3.5. *With the above conventions, assume that all the assumptions in Theorem* 3.4 *are satisfied, up to the following reinforcements:*

(i′) $[A, C_k]$ *is bounded relatively to* A;

(ii′) $[A, C_k^*]$ *and* $[A, C_k]^*$ *are bounded relatively to* A *and* I;

(iii′) R_k *is bounded relatively to* $\{C_j\}_{0\le j\le k-1}$.

Further assume that there is a positive constant λ *such that* $\sum_k C_k^* C_k \ge \lambda I_N$, *pointwise on* $\mathbb{R}^N$, *and that* μ *satisfies a logarithmic Sobolev inequality. Then there are quadratic forms* $x \to S(x)$, *uniformly positive definite, such that the functional*

$$\mathcal{I}(f) = \int f(\log f + E) + \int f\big\langle S\,\nabla(\log f + E), \nabla(\log f + E)\big\rangle$$

satisfies

$$\frac{d}{dt}\,\mathcal{I}\big((e^{-tL}h)e^{-E}\big) \le -K\,\mathcal{I}\big((e^{-tL}h)e^{-E}\big),$$

for some explicitly computable constant K.

The most noticeable novelty in the assumptions of Theorem 3.5 is that now the reference measure is not required to satisfy a Poincaré inequality, but a logarithmic Sobolev inequality, i.e. for any probability density f one should have

$$\int f\,(\log f + E) \le (2P)^{-1} \int f\,|\nabla(\log f + E)|^2,$$

where P is a positive constant. (The normalization here is such that e^{-E} automatically satisfies a Poincaré inequality with constant P.)

Apart from this, Theorem 3.5 looks very similar to Theorem 3.4. In fact, by writing densities in the form $f = (1+\varepsilon h)e^{-E}$ and letting $\varepsilon \to 0$, one can recover the conclusion of Theorem 3.4 as a perturbative limit regime. Still, there are some subtle things going on, as indicated by the reinforced assumptions (i')–(iii'). In fact there are some tricky additional computations underlying the proof, with rather miraculous simplifications, suggesting that an adequate formalism is still to be found.

Here is an application of Theorem 3.5:

Theorem 3.6. *Let* V *be a* C^∞ *potential on* $\mathbb{R}^n$; *assume that there are positive constants* k, C, $\{C_j\}_{j\in\mathbb{N}}$ *such that*

(a) $|\nabla^j V(x)| \le C_j$ *for all* $j \ge 2$;

(b) e^{-V} *satisfies a logarithmic Sobolev inequality.*

Then for any initial datum $f_0(x, v)$ *with finite moments of all orders, the solution of the Fokker–Planck equation* (1) *converges to equilibrium exponentially fast in the sense of relative entropy and* L^1 *norm, with a rate of exponential convergence that does not depend on* f_0.

This result is obtained by combining Theorem 3.5 with a global hypoelliptic regularization theorem for L^1 initial data. (This again is a highly nonstandard framework for regularization, but I shall not develop this here.)

Beyond the Hörmander form. All the examples treated so far were dealing with linear operators in the form $L = A^*A + B$, where B is antisymmetric. In theory, any operator can of course be cast in this form, but this might be a terrible thing to do in practise; for instance, if the symmetric part of L is an integral operator then A would look horrendous. So it is desirable to prove results under alternative structure assumptions.

It would be illusory to hope for a gain based on commutators like $[L, B]$. Instead, one can introduce an adequate auxiliary operator A into the estimates, in such a way that (i) $A^*A + [A, B]^*[A, B]$ is coercive, and (ii) A "almost commutes" with L.

Recently, Clément Mouhot and Lukas Neumann [21] have derived such a hypocoercivity theorem in the particular framework of kinetic equations; more precisely, $A = \nabla_v$, $B = v \cdot \nabla_x$, $C = \nabla_x$, and

$$L = v \cdot \nabla_x - \mathcal{L}, \quad \mathcal{L} = K - \Lambda, \tag{13}$$

where K and Λ only act on the velocity variable, $-\Lambda$ is "damping" (for instance a multiplication operator) and K is "regularizing" (for instance an integral operator). These assumptions cover many interesting cases in kinetic theory [21]. Here I state the results in a slightly more precise (although not yet fully rigorous) way:

Theorem 3.7. *Let $\mathcal{L}$ be an unbounded operator on $L^2(\mathbb{R}^n_v)$, taking the form $\mathcal{L} = K - \Lambda$, and assume that there exists a Hilbert norm $\|\cdot\|_\Lambda$, with $\|\cdot\|_\Lambda \geq \|\cdot\|_{L^2}$, and constants $\kappa, C > 0$ such that*

(i) $\kappa \, \|h\|_\Lambda^2 \leq \langle \Lambda h, h\rangle_{L^2} \leq C \|h\|_\Lambda^2$;

(ii) $\langle \mathcal{L} h, g\rangle \leq C \|h\|_\Lambda \|g\|_\Lambda$;

(iii) *for all $\delta > 0$ there exists $C_\delta > 0$ such that $\langle \nabla_v K h, \nabla_v h\rangle_{L^2} \leq C_\delta \|h\|_{L^2}^2 + \delta \|\nabla_v h\|_{L^2}^2$ for all h*;

(iv) $\langle \nabla_v \Lambda h, \nabla_v h\rangle_{L^2} \geq \kappa \|\nabla_v h\|_{L^2}^2 - C \|h\|_{L^2}^2$;

(C) $\langle \mathcal{L} h, h\rangle_{L^2} \leq -\kappa \, \|h - \Pi h\|_{L^2}^2$, *where Π is the orthogonal projection onto the kernel of $\mathcal{L}$, assumed to be finite-dimensional.*

Then the operator $L = -v \cdot \nabla_x + \mathcal{L}$ is hypocoercive in $L^2(\mathbb{T}^n_x \times \mathbb{R}^n_v)$. There are constants $a, b, c, \lambda > 0$ such that the functional defined by

$$\mathcal{F}(h) = \|h\|_{L^2}^2 + a \, \|\nabla_v h\|_{L^2}^2 + 2b \, \langle \nabla_v h, \nabla_x h\rangle_{L^2} + c \, \|\nabla_x h\|_{L^2}^2$$

satisfies

$$\frac{d}{dt} \mathcal{F}(e^{-tL} h) \leq -\lambda \, \mathcal{F}(e^{-tL} h)$$

for all $h \in L^2(\mathbb{R}^n_x \times \mathbb{R}^n_v) / \operatorname{Ker} L$.

Some comments on the assumptions: Assumption (i) implies the damping nature of $-\Lambda$, and assumption (iii) is a very weak way to state the regularizing property of K. Assumption (iv) is some way to express the fact that ∇_v and Λ satisfy good commutation relations: The estimate would be trivial if $\nabla_v \Lambda$ were replaced by $\Lambda \nabla_v$. Finally, assumption (C) expresses the coercivity of the operator $\mathcal{L}$ when applied to functions which only depend on the velocity variable.

Theorem 3.7 applies for instance to the linearized Boltzmann equation, or many other linear Boltzmann-type models. (There is also an independent study by Frédéric Hérau [17] which analyzes the hypocoercivity of some such operators with very similar tools.)

By combining Theorem 3.7 with a linearization analysis and some global bounds derived by Yan Guo, Clément Mouhot and Lukas Neumann were able to recover a simple proof of the following theorem of convergence to equilibrium for the *nonlinear* Boltzmann equation. In the next statement, H^k stands for the standard L^2-Sobolev space of order k on the domain $\mathbb{T}^n_x \times \mathbb{R}^n_v$.

Theorem 3.8. *Consider the Boltzmann equation* (6) *in* $\mathbb{T}^n_x \times \mathbb{R}^n_v$*, with the collision kernel* $|v - v_*|$*. Let* f_0 *be a* C^∞ *initial density with associated global equilibrium* $f_\infty(x, v) = M(v) = e^{-|v|^2/2}/(2\pi)^{n/2}$*. If*

$$\|M^{-1/2}(f_0 - M)\|_{H^k} \leq \varepsilon,$$

for some k *large enough and some* $\varepsilon > 0$ *small enough, then the corresponding solution of the Boltzmann equation converges to equilibrium exponentially fast:*

$$\|M^{-1/2}(f_t - M)\|_{L^2} = O(e^{-\lambda t}).$$

Nonlinear equations. To conclude this section, I shall show how to recover fully nonlinear hypocoercivity estimates by a variant of the approach developed above. For general nonlinear operators, it is probably hopeless to try to get anywhere unless one assumes some strong assumptions of smoothness and decay at infinity, to make sure that all norms involved are "almost comparable" (that is, they are comparable if one allows them to be raised to powers that are arbitrarily close to 1). So I will assume that $(f_t)_{t \geq 0}$ satisfies uniform bounds in a scale of weighted Sobolev spaces $(X^s)_{s \in \mathbb{R}}$ of arbitrarily high smoothness and decay, that are in interpolation. (For instance, X^s might be defined as the space of functions f such that $(I - \Delta_v - \Delta_x)^{s/2} f(x, v)(1 + |x|^2 + |v|^2)^{s/2}$ lies in L^2.) Then all the nonlinear operators involved will be assumed to be Lipschitz when restricted on balls of X^s, with values in some higher order space X^{s+k}. In practise, this means that our nonlinearities are not worse than polynomial, with coefficients that do not increase faster than polynomial. Then I shall denote the functional derivative of a functional $\mathcal{F}$ at function f by just $\mathcal{F}'_f$. I shall further assume that there is a unique equilibrium f_∞, and a Lyapunov functional $\mathcal{E}$ satisfying

$$\mathcal{E}(f_t) - \mathcal{E}(f_\infty) \geq K \, \|f_t - f_\infty\|_s^{2(1+\varepsilon)}$$

for some suitable $s = s(\varepsilon)$, $K = K(\varepsilon)$, where ε is arbitrarily small. In words, this means that $\mathcal{E}$ essentially controls the square of the distance to equilibrium.

Theorem 3.9. *With the above notation, let*

$$L = B - \mathcal{C}$$

be a nonlinear differential operator, such that B preserves the Lyapunov functional $\mathcal{E}$ (that is, $\mathcal{E}'_f \cdot Bf = 0$), let $(f_t)_{t\geq 0}$ solve $\partial_t f + Lf = 0$, and let $(\Pi_j)_{1\leq j\leq J}$ be nonlinear operators satisfying

$$\Pi_j \circ \Pi_k = \Pi_{\max(j,k)}, \tag{14}$$

such that, for all $t \geq 1$,

(i) $\mathcal{C} \circ \Pi_1 = 0$; $-\mathcal{E}'_{f_t} \cdot (\mathcal{C} f_t) \geq K_\varepsilon [\mathcal{E}(f) - \mathcal{E}(\Pi_1 f)]^{1+\varepsilon}$;

(ii) $K_\varepsilon \|\Pi_1 f_t - f_\infty\|^{2+\varepsilon} \leq \mathcal{E}(\Pi_1 f) - \mathcal{E}(f_\infty) \leq C_\varepsilon \|\Pi_1 f_t - f_\infty\|^{2-\varepsilon}$;

(iii) $\Pi_J f = f_\infty$; $Bf_\infty = 0$;

(H) $\|(\mathrm{Id} - \Pi_j)'_{\Pi_j f} \cdot (B \Pi_j f)\|^2 \geq K_\varepsilon \|(\Pi_j - \Pi_{j+1}) f\|^{2+\varepsilon}$ *for all* $j \in \{1, \dots, J-1\}$.

Then $\|f_t - f_\infty\| = O(t^{-\infty})$.

This theorem may seem particularly abstract and confusing, so I should give some explanations. First, B plays the role of the antisymmetric part, but this shows only in the assumption that it does not contribute to the decay of $\mathcal{E}$; on the contrary, $\mathcal{C}$ should be thought of as the symmetric, or collisional part, and it does make the Lyapunov functional decay.

Next, the operators Π_j act as a family of "nested projections". The first one, Π_1, sends f to the kernel of the "collision operator" $\mathcal{C}$; then the second one sends f to a smaller subspace, and then each Π_j takes values in a smaller subspace until finally one reaches f_∞. The "concrete" examples are the maps $f \to M^f_{\rho\, u\, T}$, $f \to M^f_{\rho\, u\, \langle T\rangle}$, $f \to M^f_{\rho\, 0\, 1}$, $f \to f_\infty$ which we considered in Section 2 (so for the Boltzmann equation we need four such nonlinear projections).

Finally, the key hypocoercivity condition is (H): It ensures basically that the effect of the "antisymmetric part" B is strong enough to get us out of the image of Π_j, unless we are in the image of Π_{j+1}.

Theorem 3.9 leads to a simplified proof of Theorem 2.2, which does not involve any second-order differential inequality, but just variants of Gronwall's lemma. Once again, the key point is to add a correction to the Lyapunov functional $\mathcal{E}$ into another functional $\mathcal{F}$. The correction is small enough that the value of $\mathcal{F}$ is very close to the value of $\mathcal{E}$; but its structure is such that $\mathcal{F}$ satisfies (almost) a Gronwall-type estimate. More explicitly,

$$\mathcal{F}(f) = \big[\mathcal{E}(f) - \mathcal{E}(f_\infty)\big] + \sum_{j=1}^{J-1} a_j \big\langle (\mathrm{Id} - \Pi_j) f,\ (\mathrm{Id} - \Pi_j)'_f \cdot (Bf) \big\rangle, \tag{15}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in, say, X^0, and $\varepsilon > 0, a_j > 0\,(1 \le j \le J-1)$ are small numbers depending on the smoothness of f_t, on δ, and on estimates on the distance of f to f_∞ (in general $1 \gg a_1 \gg \cdots \gg a_{J-1}$).

If the reader thinks that I am being too abstract and formal here, I invite him or her to write down explicitly what (15) is for the Boltzmann equation: Take $\mathcal{E}(f) = \int f \log f$, $f_\infty = M(v)$, $\Pi_1 f = M^f_{\rho\, u\, T}$, $\Pi_2 f = M^f_{\rho\, u\, \langle T \rangle}$, $\Pi_3 f = M^f_{\rho\, 0\, 1}$, $\Pi_4 f = f_\infty$, and $Bf = v \cdot \nabla_x f$; then the expression of $\mathcal{F}(f)$ would fill up basically a whole page. Expression (15) is not only quite general, it is also the best way to conduct calculations.

Let me conclude this section with another theorem that can be derived from Theorem 3.9: convergence to equilibrium for the nonlinear Vlasov–Fokker–Planck interaction with moderate interaction and small coupling.

Theorem 3.10. *Let $W \in C^\infty(\mathbb{T}^n)$ be an even smooth function with* $\sup W - \inf W$ *small enough, and let f_0 be a probability density on $\mathbb{T}^n_x \times \mathbb{R}^n_v$, with all moments finite. Then there is a unique solution $(f_t)_{t \ge 0}$ to the partial differential equation*

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x (W * \rho) \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (f v), \quad \rho_t(x) = \int f_t(x, v)\, dv;$$

and it does converge to a uniquely determined equilibrium distribution f_∞, with

$$\| f_t - f_\infty \|_{L^1} = O(t^{-\infty}).$$

This theorem also follows directly from Theorem 3.9, now by choosing just $\Pi_1 f = \rho M$, $\Pi_2 f = f_\infty$, where the equilibrium f_∞ is the unique minimizer of the energy functional $H(f) + (1/2) \int \rho(x)\, \rho(y)\, W(x - y)\, dx\, dy$. The assumption on W being smooth and small enough guarantees the uniqueness of the minimizer (it implies that we stay away from phase transitions) and allows to develop a very strong regularity theory for the equation.

4. Perspectives

There seems to be a whole mathematical world to explore behind the hypocoercivity phenomenon, both in nonlinear and linear partial differential equations.

Obvious remaining open problems concern the nonlinear equations such as Boltzmann's equation, for which the convergence to equilibrium is proven only under strong conditional smoothness assumptions; however, Theorem 2.2 says that, in some sense, it all amounts to a good understanding of the Cauchy problem. The situation is more subtle for coupled equations, such as the Vlasov–Fokker–Planck model: Theorem 3.10 solves the problem only for smooth *small enough* potentials, leaving completely open the issue of phase transition for large potentials. Realistic models such as the Vlasov–Poisson–Fokker–Planck equation require further thoughts.

But within the range of linear equations, where one is more demanding about conclusions, there is even much more to say. First, one would like to get a qualitative description of the convergence to equilibrium, and in particular of the oscillations described in the discussion of the Boltzmann equation. As discussed in [29], these oscillations appear in many models, but not always, and their presence or absence should be related to some spectral analysis of the linearized operator, involving conservation laws and hydrodynamical approximations. Recently, Francis Filbet, Clément Mouhot and Lorenzo Pareschi [10] have made some progress on this issue, by combining numerical simulations, linearization and asymptotic analysis; they suggest that for large domains the period of oscillations is given by the imaginary part of the eigenvalues of the linearized compressible Euler system, while the asymptotic rate of decay is determined by the real parts of the eigenvalues of the compressible linearized Navier–Stokes system. For small enough domains, the situation is completely different, and hydrodynamic effects should be negligible. Actually, numerical simulations in small domains show that the gas distribution first becomes spatially homogeneous, and then converges to equilibrium like a solution of the spatially homogeneous Boltzmann equation – a scenario which is somehow opposite to the ideas that seemed to be prevailing among physicists and mathematicians.

Another issue is about the quantitative relevance of the estimates. All the estimates derived from the hypocoercivity theorems in this text are explicit, but this does not mean that they have the correct order of magnitude. For simple equations like the Fokker–Planck equation, the rates of convergence predicted by my method seem to be off the true value by a factor of about 10^2, which is not so bad (and much better than previous estimates). But the rates obtained for the oscillator chain have an incredibly bad dependence on the number of oscillators, leaving motivation for quantitative improvement.

It is important to note that the analysis of operators of the form $A^*A + B$ involved a systematic comparison with the symmetric operator $A^*A + [A, B]^*[A, B]$, or more complicated symmetric operators constructed from brackets with B. The same is true of the hypoelliptic method by Bernard Helffer and Francis Nier [16]. This might look satisfactory, but might also give wrong orders of magnitude. For instance, in the Model Problem 1.1, we have seen that if the eigenvalues for $A^*A + [A, B]^*[A, B]$ grow like $|\alpha|^{2\nu}$, then the real parts of the eigenvalues for $A^*A + B$ grow at least like $|\alpha|^{\nu}$. This behavior is optimal for some forms of the function f, but not for other ones, as pointed out to me by Thierry Gallay. Indeed, if $f(x) = 1/(1 + x^2)$, then $\nu = 1/4$, but numerical simulations suggest that the growth is like $|\alpha|^{1/2}\ldots$. This might indicate a fundamental limitation of present techniques, and motivate the development of a refined analysis.

In the example of the oscillator chain, the application of Theorem 3.4 is so far restricted to the oversimplified case when $T^{(\ell)} = T^{(r)}$. In fact, Theorem 3.4 could be applied if we had some basic qualitative information about the non-explicit equilibrium measure: For instance, some bounds on the Hessian of the logarithm of its density; and a Poincaré inequality. This leads to another challenging topic: deriving

qualitative *global* information about stationary solutions of linear partial differential equations.

Finally, the links and analogies between hypoellipticity and hypocoercivity need to be further explored. The interplay goes in both directions: It is possible to adapt some of the tricks presented here, into elementary methods for the study of hypoelliptic regularization. While these do not apply with such generality as the classical techniques introduced by Lars Hörmander and later by Joseph Kohn, they are quite flexible, in particular to get global estimates, or estimates from L^1 data. The same discovery has been made independently by Frédéric Hérau. Hopefully, all this agitation will lead to a new look at the old field of hypoelliptic regularity.

References

[1] Cáceres, M. J., Carrillo, J. A., and Goudon, T., Equilibration rate for the linear inhomogeneous relaxation-time Boltzmann equation for charged particles. *Comm. Partial Differential Equations* **28** (5–6) (2003), 969–989.

[2] Carrillo, J. A., McCann, R. J., and Villani, C., Kinetic equilibration rates for granular media and related equations: entropy dissipation and mass transportation estimates. *Rev. Mat. Iberoamericana* **19** (3) (2003), 971–1018.

[3] Desvillettes, L., and Villani, C., On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: the linear Fokker-Planck equation. *Comm. Pure Appl. Math.* **54** (1) (2001), 1–42.

[4] Desvillettes, L., and Villani, C., On a variant of Korn's inequality arising in statistical mechanics. *ESAIM Control Optim. Calc. Var.* **8** (2002), 603–619.

[5] Desvillettes, L., and Villani, C., On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation. *Invent. Math.* **159** (2) (2005), 245–316.

[6] Eckmann, J.-P., and Hairer, M., Non-equilibrium statistical mechanics of strongly anharmonic chains of oscillators. *Comm. Math. Phys.* **212** (1) (2000), 105–164.

[7] Eckmann, J.-P., and Hairer, M., Spectral properties of hypoelliptic operators. *Comm. Math. Phys.* **235** (2) (2003), 233–253.

[8] Eckmann, J.-P., Pillet, C.-A., and Rey-Bellet, L., Non-equilibrium statistical mechanics of anharmonic chains coupled to two heat baths at different temperatures. *Comm. Math. Phys.* **201** (3) (1999), 657–697.

[9] Fellner, K., Neumann, L., and Schmeiser, C., Convergence to global equilibrium for spatially inhomogeneous kinetic models of non-micro-reversible processes. *Monatsh. Math.* **141** (4) (2004), 289–299.

[10] Filbet, F., Mouhot, C., and Pareschi, L., Solving the Boltzmann equation in $N \log_2 N$. *SIAM J. Sci. Comput.*, to appear.

[11] Gallay, T., and Wayne, C. E., Global stability of vortex solutions of the two-dimensional Navier-Stokes equation. *Comm. Math. Phys.* **255** (1) (2005), 97–129.

[12] Guo, Y., The Landau equation in a periodic box. *Comm. Math. Phys.* **231** (3) (2002), 391–434.

[13] Guo, Y., and Strain, R., Almost exponential decay near Maxwellian. *Comm. Partial Differential Equations* **31** (3) (2006), 417–429.

[14] Guo, Y., and Strain, R., Exponential decay for soft potentials near Maxwellian. Preprint, 2005.

[15] Hairer, M., and Mattingly, J., Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing. *Ann. of Math*, to appear.

[16] Helffer, B., and Nier, F., *Hypoellipticity and spectral theory for Fokker-Planck operators and Witten Laplacians*, Lecture Notes in Math. 1862, Springer-Verlag, Berlin 2005.

[17] Hérau, F., Hypocoercivity and exponential time decay for the linear inhomogeneous relaxation Boltzmann equation. *Asymptot. Anal.*, to appear; http://helios.univ-reims.fr/Labos/Mathematiques/Homepages/Herau/.

[18] Hérau, F., Short and long time behavior of the Fokker-Planck equation in a confining potential and applications. Preprint (revised version), 2005; http://helios.univ-reims.fr/Labos/Mathematiques/Homepages/Herau/.

[19] Hérau, F., and Nier, F., Isotropic hypoellipticity and trend to equilibrium for the Fokker-Planck equation with a high-degree potential. *Arch. Ration. Mech. Anal.* **171** (2) (2004), 151–218.

[20] Hörmander, L., Hypoelliptic second order differential equations. *Acta Math.* **119** (1967), 147–171.

[21] Mouhot, C., and Neumann, L., Quantitative perturbative study of convergence to equilibrium for collisional kinetic models in the torus. Preprint, 2006.

[22] Rey-Bellet, L., and Thomas, L. E., Asymptotic behavior of thermal nonequilibrium steady states for a driven chain of anharmonic oscillators. *Comm. Math. Phys.* **215** (1) (2000), 1–24.

[23] Rey-Bellet, L., and Thomas, L. E., Exponential convergence to non-equilibrium stationary states in classical statistical mechanics. *Comm. Math. Phys.* **225** (2) (2002), 305–329.

[24] Talay, D., Stochastic Hamiltonian systems: exponential convergence to the invariant measure, and discretization by the implicit Euler scheme. *Markov Process. Related Fields* **8** (2) (2002), 163–198.

[25] Toscani, G., and Villani, C., Sharp entropy dissipation bounds and explicit rate of trend to equilibrium for the spatially homogeneous Boltzmann equation. *Comm. Math. Phys.* **203** (3) (1999), 667–706.

[26] Toscani, G., and Villani, C., On the trend to equilibrium for some dissipative systems with slowly increasing a priori bounds. *J. Statist. Phys.* **98** (5–6) (2000), 1279–1309.

[27] Villani, C., A review of mathematical topics in collisional kinetic theory. In *Handbook of mathematical fluid dynamics*, Vol. I, North-Holland, Amsterdam 2002, 71–305.

[28] Villani, C., Cercignani's conjecture is sometimes true and always almost true. *Comm. Math. Phys.* **234** (3) (2003), 455–490.

[29] Villani, C., Entropy dissipation and convergence to equilibrium. Notes from a series of lectures at Institut Henri Poincaré, Paris 2001 (updated 2004); http://www.umpa.ens-lyon.fr/~cvillani/.

[30] Villani, C., Convergence to equilibrium: Entropy production and hypocoercivity. In *Rarefied Gas Dynamics* (ed. by M. Capitelli), AIP Conference Proceedings 762, American Institute of Physics, 2005, 8–25 .

[31] Villani, C., Hypocoercive diffusion operators in Hörmander form. Preprint, 2006; http://www.umpa.ens-lyon.fr/~cvillani/.

[32] Villani, C., Hypocoercive nonlinear diffusion operators. Preprint, 2006.

UMPA (UMR CNRS 5669), ENS Lyon, 46 allée d'Italie, 69364 Lyon Cedex 07, France
E-mail: cvillani@umpa.ens-lyon.fr