# On Ising droplets

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**Abstract.** One of the fundamental goals of statistical mechanics is to understand the macroscopic effects induced by the random forces acting at the microscopic level. Some satisfactory results are now available for the Ising model at equilibrium in the phase coexistence regime in any dimension: it is rigorously proved that the most likely shapes of the macroscopic droplets of one pure phase floating in the other pure phase are close to the Wulff crystal of the model. However, the dynamical processes leading to the emergence of a droplet are far from being understood. We formulate a classical conjecture: the scaling limit of the Glauber microscopic dynamics should be an anisotropic motion by mean curvature.

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**Keywords.** Ising model, Wulff crystal, Glauber dynamics, mean curvature motion.

## 1. Water and oil

Let us consider a volume of water in absence of gravity at ordinary temperature. We start to pour a very small quantity of oil into the water. First, nothing noticeable happens on the macroscopic scale, i.e., the oil is perfectly dissolved throughout the water and the oil molecules are homogeneously spread within the water: by observing the liquid at the macroscopic level, we cannot even tell that it is a mixture of two distinct types of particles, which nevertheless have the tendency to repel each other. Let us keep pouring oil into the water. We know that the solubility of oil in water is not infinite; at some density threshold (which increases with the temperature), we obtain a solution of water saturated with oil. This solution is still a pure phase, completely homogeneous on the macroscopic level, and it realizes a perfect tradeoff between entropy and energy; we call it the water phase. Let us pour in some more oil. The excess of oil is not dissolved any more and it precipitates: macroscopic droplets of oil emerge. These droplets are not regions where there are only oil molecules, rather in these regions we observe the symmetric pure phase consisting of oil saturated with water, which we call the oil phase. The droplets are delimited by an abrupt change of the local density of water and oil molecules. We wish to understand the law governing the evolution and the shapes of these droplets.

The classical phenomenological theory asserts the existence of a macroscopic surface free energy $\mathcal{I}$ and that the droplets evolve so as to minimize $\mathcal{I}$. For instance, at equilibrium, in case $\mathcal{I}$ is isotropic, one observes a unique spherical droplet of the

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oil phase floating in the sea of the water phase. Our aim is to confirm the predictions of the phenomenological theory starting from a truly microscopic model. We wish to understand how the random forces acting at the atomic level, more precisely the probabilistic repulsive effect between the two types of particles, can induce such deterministic macroscopic effects. One of the most famous results in probability theory is the law of large numbers: If $(X_n)_{n\in\mathbb{N}}$ is a sequence of independent identically distributed random variables with mean $m$, then, with probability one,

$$\lim_{n\to\infty} \frac{1}{n}(X_1 + \cdots + X_n) = m.$$

What we have in mind is a generalization of the law of large numbers, but in a fundamentally new context, which we could state informally as follows:

$$\lim_{\text{number of particles}\to\infty} \begin{pmatrix} \text{global effect of the random} \\ \text{microscopic repulsive forces} \end{pmatrix} = \text{single droplet.}$$

The limiting deterministic object is the shape of the droplet at equilibrium and the problem is intrinsically geometric; we deal with spatially dependent random variables and we leave radically and definitively the independent framework. Hence the geometry enters the problem in a decisive way, in the random interactions and in the formulation of the result itself.

Let us try to set up a simple model of our experiment with water and oil. A convenient choice is a lattice model: each site of the lattice is occupied either by a water particle or by an oil particle, which we indicate respectively by $+$ or $-$. The interaction between different particles is repulsive and occurs when the substances are in immediate contact. Hence a repulsive nearest neighbour interaction is a sensible choice. Since we focus only on the repulsive interaction between different molecules, we can assume that the two substances are symmetric and that their self-interactions are of equal magnitude, or equivalently, equal to zero. We do not assume that the self-interactions between two particles of the same type are negligible compared to the repulsive effect; rather, we say that because of the symmetry, the global effect of the self-interactions cancels out. Thus the total energy of a configuration should be simply the number of all nearest neighbour pairs with different signs. We end up exactly with the Hamiltonian of the famous Ising model (to be defined precisely in the next section). In our experiment the density of oil is fixed, therefore we have a constraint on the possible configurations: the proportion of pluses and minuses has to be fixed. This situation amounts to considering the Ising model with plus boundary conditions (guaranteeing the water dominance) conditioned on the event that the average magnetization is equal to a fixed value smaller than the spontaneous magnetization at the given temperature.

## 2. Definition of the Ising model

For reasons of technical simplicity, it is easier to build our model on a lattice. We will work with the lattice $\mathbb{Z}^d$; each site of the lattice is occupied by one of the two types of particles that we denote by $-$ and $+$. Let $\Lambda \subset \mathbb{Z}^d$ be a cubic box. A configuration in $\Lambda$ is a map $\sigma : \Lambda \to \{-, +\}$ and for $x \in \Lambda$, we denote by $\sigma(x)$ the type of the particle present at $x$. The energy or Hamiltonian $H_\Lambda(\sigma)$ of the configuration $\sigma$ in $\Lambda$ is, up to a constant, twice the number of interfaces between the minuses and the pluses, that is,

$$H_\Lambda(\sigma) = -\frac{1}{2} \sum_{\substack{x,y\in\Lambda \\ |x-y|=1}} \sigma(x)\sigma(y) = - \sum_{\substack{\{x,y\}\in\Lambda^2 \\ |x-y|=1}} \sigma(x)\sigma(y).$$

We use the standard rules to multiply signs: $++ = -- = +$, $-+ = +- = -$. The first sum is above ordered pairs (whence the factor $1/2$) while the second is above unordered pairs. We need also a mechanism to ensure the dominance of one type of particles. This is achieved through boundary conditions. We consider only two types of boundary conditions, by putting either a layer of pluses or of minuses around the box $\Lambda$. The energy or Hamiltonian $H^*_\Lambda(\sigma)$ with boundary conditions $*$ (where $*$ stands for $-$ or $+$) is defined as above for the configurations $\sigma$ such that $\sigma(x) = *$ for all the sites $x$ in $\Lambda$ which are at a distance less than or equal to 1 from the complement of $\Lambda$ and $H^*_\Lambda(\sigma) = +\infty$ otherwise. Next we add some randomness in the model. Let $T > 0$ be the temperature. We build a probability law on the space $\{-, +\}^\Lambda$ of the configurations. This space is huge but finite, hence to define the law we need to specify the individual probability of each possible configuration. The natural way to do this is to use the Boltzmann factor. So, the Gibbs measure $\mu^*_{\Lambda,T}$ in $\Lambda$ at temperature $T$ with boundary conditions $*$ is given by

$$\mu^*_{\Lambda,T}(\sigma) = \frac{1}{Z^*_{\Lambda,T}} \exp -\frac{H^*_\Lambda(\sigma)}{T} \quad \text{for all } \sigma \in \{-, +\}^\Lambda,$$

where the normalizing factor $Z^*_{\Lambda,T}$, called the partition function, is equal to

$$Z^*_{\Lambda,T} = \sum_{\sigma\in\{-,+\}^\Lambda} \exp -\frac{H^*_\Lambda(\sigma)}{T}.$$

Whenever the superscript $*$ is absent, the boundary conditions are not specified and we have the Ising Gibbs measure $\mu_{\Lambda,T}$ with free boundary conditions, associated to the Hamiltonian $H_\Lambda$. Let us take a closer look at this formula. The elements $\Lambda$, $T$, $* \in \{-, +\}$ being fixed, the most likely configurations are those having a small energy, i.e., those for which the contacts between the minuses and the pluses are reduced. Thus we have built a complex probability law with strong spatial correlations. We shall next play a bit with the elements controlling the Gibbs measures $\mu^\pm_{\Lambda,T}$ in order to get some feeling for their influence.

**First asymptotics.** Imagine that we fix the box $\Lambda$ and that we set the boundary conditions to $+$. If we send $T$ to 0, then the measure $\mu^+_{\Lambda,T}$ concentrates on the configuration which realizes the global minimum of the Hamiltonian $H^+_\Lambda$, in this case the configuration where all the sites are pluses. On the contrary, if we send $T$ to $\infty$, the value of the Hamiltonian becomes irrelevant and $\mu^+_{\Lambda,T}$ converges towards the Bernoulli product law where all the sites are independent. The case of $\mu^-_{\Lambda,T}$ being symmetric, we see that

$$\begin{array}{rcccl} \text{Dirac mass at ``all pluses''} & \longleftarrow & \mu^+_{\Lambda,T} & \longrightarrow & \text{i.i.d. Bernoulli} \\ & T \downarrow 0 & & T \uparrow \infty & \\ \text{Dirac mass at ``all minuses''} & \longleftarrow & \mu^-_{\Lambda,T} & \longrightarrow & \text{i.i.d. Bernoulli} \end{array}$$

Something remarkable has already happened: as $T \uparrow \infty$, the boundary conditions are forgotten, while as $T \downarrow 0$, they completely determine the limit. However, we wish to work at a fixed positive temperature $T$. In order to observe a sharp mathematical phenomenon, we consider another kind of limit, namely the thermodynamic limit where the number of particles goes to infinity. This is achieved by letting the box $\Lambda$ grow and invade the whole lattice $\mathbb{Z}^d$. As $\Lambda$ increases to $\mathbb{Z}^d$, the expectation $\mu^+_{\Lambda,T}(\sigma(0))$ decreases and converges towards a limiting quantity $m^*(T)$:

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \mu^+_{\Lambda,T}(\sigma(0)) = m^*(T) = -\lim_{\Lambda \uparrow \mathbb{Z}^d} \mu^-_{\Lambda,T}(\sigma(0)).$$

Here is a heuristic explanation for this monotone convergence. Let us consider a huge box $\Lambda$ and the site at the center of the box $\Lambda$. With free boundary conditions, the law of $\sigma(0)$ under $\mu_{\Lambda,T}$ is symmetric, hence it is the one of a fair coin, i.e.,

$$\mu_{\Lambda,T}(\sigma(0) = +) = 1/2 = \mu_{\Lambda,T}(\sigma(0) = -).$$

If we put $+$ boundary conditions, these boundary conditions start to influence positively the sites at distance 1 from the boundary of the box, which themselves influence the sites at distance 2 from the boundary. This effect propagates and reaches the origin, so that the law of $\sigma(0)$ under $\mu^+_{\Lambda,T}$ is slightly biased towards $+$:

$$\mu^+_{\Lambda,T}(\sigma(0) = +) \; > \; 1/2 \; > \; \mu^+_{\Lambda,T}(\sigma(0) = -).$$

The larger the box $\Lambda$ is, the smaller is the resulting effect at the origin, hence the influence of the boundary conditions decreases as the box increases and the following monotone limit exists:

$$m^*(T) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \mu^+_{\Lambda,T}(\sigma(0)).$$

The fundamental and basic question is whether something of the influence of the boundary conditions still remains after we have sent them to infinity. Equivalently, is $m^*(T)$ equal to 0?

The quantity $m^*(T)$ is called the spontaneous magnetization at temperature $T$. This terminology stems from the fact that the Ising model was originally introduced

as a model of ferromagnetism: under some adequate conditions, a magnet submitted to the influence of a magnetic field will remember the sign of the field even after it has disappeared (see [24] and the references therein for a serious physical introduction to the Ising model).

**Phase transition.** We say that there is a phase transition at temperature $T$ if $m^*(T) > 0$. The first fundamental result concerning the phase transition in the Ising model is the following.

**Theorem 2.1.** *In any dimension $d \geq 2$, there exists a positive and finite critical temperature $T_c(d)$ such that the Ising model exhibits a phase transition for $T < T_c(d)$ and it does not for $T > T_c(d)$.*

It is also possible to take the thermodynamic limit of the finite volume Gibbs measure $\mu^+_{\Lambda,T}$, and not only of the expected value $\mu^+_{\Lambda,T}(\sigma(0))$. As $\Lambda$ increases to $\mathbb{Z}^d$, the measure $\mu^+_{\Lambda,T}$ decreases stochastically and converges weakly towards the infinite volume Gibbs measure $\mu^+_T$, which is a probability measure on the space of infinite volume configurations $\{-,+\}^{\mathbb{Z}^d}$. Similarly, $\mu^-_{\Lambda,T}$ increases weakly towards a measure $\mu^-_T$:

$$\lim_{\Lambda\uparrow\mathbb{Z}^d} \mu^-_{\Lambda,T} = \mu^-_T, \quad \lim_{\Lambda\uparrow\mathbb{Z}^d} \mu^+_{\Lambda,T} = \mu^+_T.$$

The spontaneous magnetization $m^*(T)$ is equal to the expected value of $\sigma(0)$ under $\mu^+_T$ and there is a phase transition at temperature $T$ if and only if $\mu^-_T$ and $\mu^+_T$ are distinct. In other words, we have

$$m^*(T) > 0, \quad \mu^-_T \neq \mu^+_T \qquad \text{for all } T < T_c(d),$$

whereas for $T > T_c(d)$, we have $m^*(T) = 0$ and $\mu^-_T = \mu^+_T$.

## 3. The Wulff crystal

We shall mimic mathematically the initial experiment of Section 1 with the help of the Ising model. Let us consider a box $\Lambda(n)$ of diameter $n$ full of pluses. We take $n$ very large, of the order of the Avogadro number $6.02 \times 10^{23}$. We start deleting pluses and replacing them by minuses, first a small quantity of minuses. It is possible to build a stochastic dynamics in the box which is conservative (i.e., the total numbers of minuses and pluses remain unchanged or equivalently the empirical magnetization $n^{-d}\sum_{x\in\Lambda(n)}\sigma(x)$ remains constant) and whose final equilibrium is the Gibbs measure $\mu^+_{\Lambda(n),T}$ conditioned to have the initial fixed magnetization. The simplest such dynamics is the so-called Kawasaki dynamics: at random exponential times, a pair of neighbouring particles might be exchanged according to a simple local probabilistic rule (see Section 4). As long as the empirical magnetization is larger than $m^*(T)$, the configuration in $\Lambda(n)$ at equilibrium is expected to be spatially homogeneous. If

we keep pouring minuses into the box and removing pluses, we soon reach the value $m^*(T)$, and at this point we obtain the saturated pure phase $\mu_T^+$, i.e., the configuration in $\Lambda(n)$ looks like a finite sample of the infinite volume Gibbs measure $\mu_T^+$. We finally add some more minuses and we cross the threshold $m^*(T)$. We wish to understand the response of the system and the most likely configurations inside the box when there is an excess of minuses.

It turns out that this simple model indeed confirms the prediction of the phenomenological theory. At equilibrium, with probability tending to 1 as $n$ goes to $\infty$, a region emerges inside the box $\Lambda(n)$ where the configuration statistically looks like the minus phase $\mu_T^-$, surrounded by a region filled with the plus phase $\mu_T^+$. When rescaled by a factor $n$, the shape of this region converges as $n$ goes to $\infty$ towards a deterministic shape, called the Wulff crystal of the Ising model. This crystal is convex, it depends on the temperature and on the initial lattice $\mathbb{Z}^d$; it bears the name of Wulff, who studied it one century ago [33].

In order to detect conveniently the Wulff region, we rescale the box $\Lambda(n)$ by a factor $n$ and we send it onto the $d$-dimensional unit cube $[-1/2, 1/2]^d$. Let $\sigma$ be a spin configuration in $\Lambda(n)$. To $\sigma$ we associate a measure $\sigma_n$ on $[-1/2, 1/2]^d$ by setting

$$\sigma_n = \frac{1}{n^d} \sum_{x \in \Lambda(n)} \sigma(x)\, \delta_{x/n}$$

where $\delta_{x/n}$ is the Dirac mass at $x/n$. We call $\sigma_n$ the empirical magnetization. The expectation $b_n$ of $\sigma_n$ is

$$b_n = \frac{1}{n^{d+1}} \sum_{x \in \Lambda(n)} \sigma(x) x.$$

We denote by $\mathcal{L}^d$ or simply by $dx$ the $d$-dimensional Lebesgue measure.

**Theorem 3.1.** *Let $d \geq 2$ and let $T < T_c(d)$. There exists a bounded, closed, convex set $\mathcal{W}$ containing 0 in its interior, called the Wulff crystal of the Ising model such that the following holds.*

*Let $m < m^*$ be close enough to $m^*$ so that the rescaled Wulff crystal*

$$\mathcal{W}(m) = \left(\frac{m^* - m}{2m^*}\right)^{1/d} \frac{\mathcal{W}}{\mathcal{L}^d(\mathcal{W})^{1/d}}$$

*fits into the unit cube $[-1/2, 1/2]^d$. Let $w_n$ be the random measure defined by*

$$w_n(x)\, dx = \Big(1_{[-1/2,1/2]^d}(x) - 2 \cdot 1_{\mathcal{W}(m)}\Big(\frac{b_n}{m^* - m} + x\Big)\Big) m^*\, dx.$$

*This is the measure having density $-m^*$ on $-b_n/(m^* - m) + \mathcal{W}(m)$ and $m^*$ on the complement. Under the conditional probability*

$$\mu_n(\cdot) = \mu^+_{\Lambda(n),T}\Big(\,\cdot\,\Big|\frac{1}{n^d} \sum_{x \in \Lambda(n)} \sigma(x) \leq m\Big)$$

*the difference between the random measures $\sigma_n$ and $w_n$ converges weakly in probability towards* 0*, i.e., for any continuous function* $f : [-1/2, 1/2]^d \to \mathbb{R}$*,*

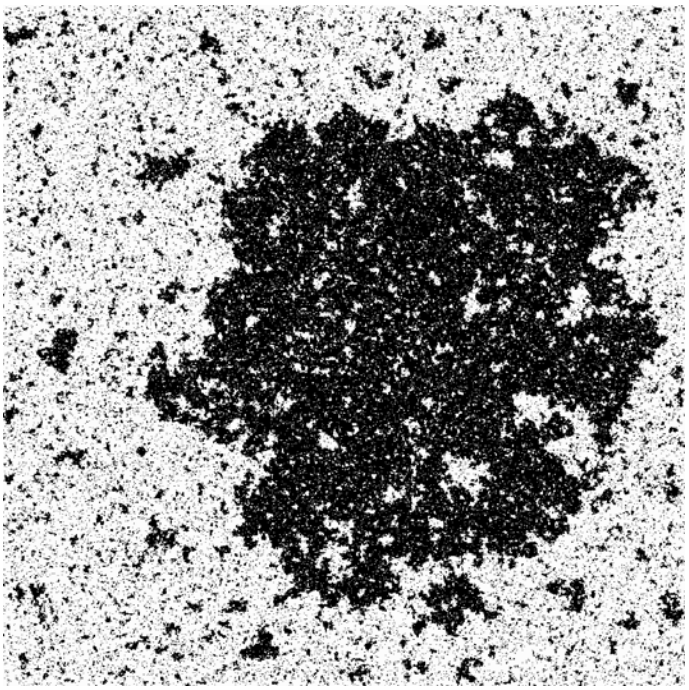
$$\lim_{n\to\infty} \mu_n\big(|\sigma_n(f) - w_n(f)| \geq \varepsilon\big) = 0 \quad \textit{for all } \varepsilon > 0.$$

*The probabilities of the deviations are of order* $\exp -cn^{d-1}$.

The last sentence of the theorem means the following. For any continuous function $f : [-1/2, 1/2]^d \to \mathbb{R}$, any $\varepsilon > 0$, there exist positive constants $b, c$ depending on $d$, $T$, $f$, $\varepsilon$ such that

$$\mu_n\Big(\Big|\frac{1}{n^d} \sum_{x\in\Lambda(n)} \sigma(x)\, f\Big(\frac{x}{n}\Big) + \int_{\mathcal{W}(m)} 2m^* f\Big(-\frac{b_n}{m^* - m} + x\Big)\, dx - \int_{[-1/2,1/2]^d} m^* f(x)\, dx\Big| > \varepsilon\Big) \leq b \exp(-cn^{d-1}).$$

The main assertion of the theorem is that the left-hand quantity goes to 0 for any continuous function $f$ and $\varepsilon > 0$. The objects appearing in the statement, namely the spontaneous magnetization $m^*$ and the Wulff crystal $\mathcal{W}$, are built as the thermodynamic limit of finite volume quantities. These objects can equivalently be defined with the help of the infinite volume Gibbs measure $\mu_T^+$.



Simulation of the Ising Wulff crystal at $T = 2.26$ after 69 days on a 1 Ghz PC.

Theorem 3.1 in dimension 2 is a consequence of the much finer results of Dobrushin, Kotecký, Shlosman [23] and Pfister [29] for low temperatures and Ioffe and Schonmann [25] for all subcritical temperatures. In dimensions 3 and higher, they were proven by Bodineau [7] for low temperatures and by Cerf and Pisztora [17] until the slab percolation threshold for temperatures such that the associated infinite volume FK measure is unique. Recently, Bodineau proved that this slab percolation threshold coincides with the true critical point [8] and that for any subcritical temperature, the associated infinite volume FK measure is indeed unique [9].

In two dimensions the Wulff droplet can be identified with a random region surrounded by a minus spin cluster. Its external boundary is therefore a large contour separating plus and minus spins which follows closely the boundary of the Wulff crystal in the sense of the Hausdorff metric [23], [25]. In dimension $d \geq 3$, it is widely believed that for low temperatures, the Wulff droplet can still be defined by a microscopic contour. However for temperatures close to $T_c$, a fundamentally new situation is expected. The dominant minus spin cluster of the Wulff droplet should percolate all the way to the boundary of the box. More precisely, there should exist two big spin clusters, one of pluses and one of minuses, and they should both be omnipresent in the entire box; the densities of these clusters should undergo an abrupt change at the boundary of the Wulff droplet. In this case the phase boundaries cannot be described directly with contours.

Let us mention the most recent works on the Wulff crystal. The low temperature expansion of the 3D Wulff crystal is computed in [14]. Alexander succeeded recently in deriving cube root fluctuations of the random curve around the Wulff crystal in the FK model in two dimensions [1], [4]. Couronné and Messikh provide a two dimensional version of Pisztora's coarse graining estimates [22]. Messikh analyzes the phase coexistence phenomenon in the 2D Ising model close to criticality: the Wulff crystal then becomes a circle [16], [27] (see also [28] for an application to image segmentation). Couronné has shown that the statistical repartition of the large finite clusters in the FK percolation model can be approximated by a Poisson process [21]. He has also studied the Wulff crystal for oriented percolation in dimension $d \geq 3$ [20]. In this model, the Wulff crystal has a singular point. Bodineau, Schonmann and Shlosman investigate the question of the flatness of the Wulff crystal [12]. Biskup, Chayes and Kotecky study the formation/dissolution of equilibrium droplets in the context of the 2D Ising model [5] and they derive the Gibbs–Thomson formula in the droplet formation regime [6]. Alexander, Biskup and Chayes devise an Ising-based model of a solvent-solute system [2], [3] and they study the associated phase separation phenomenon. The book [13] contains the proof of Theorem 3.1 and the corresponding statements in percolation.

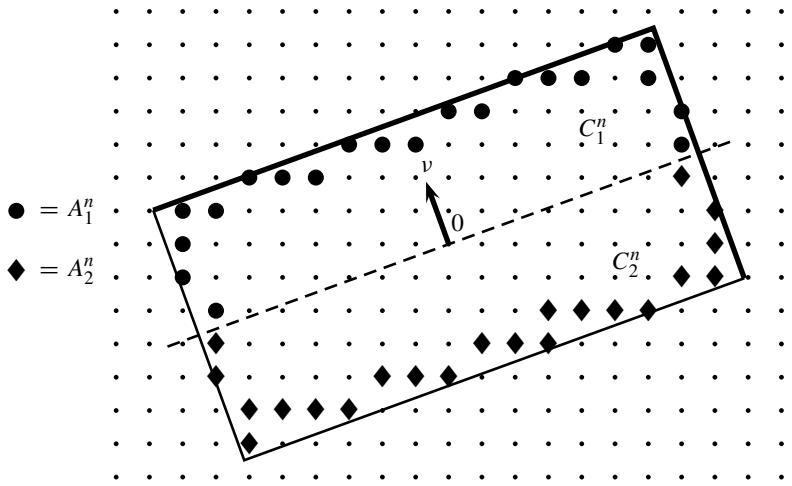
**The Wulff construction.** The Wulff crystal $\mathcal{W}$ appearing in Theorem 3.1 can be defined constructively. First we define the surface tension of the Ising model as follows. Let $\nu \in S^{d-1}$ be a unit vector in $\mathbb{R}^d$ and let $A$ be a unit hypersquare

orthogonal to $\nu$. Let $D_n$ be the cylinder

$$D_n = \{na + t\nu : a \in A,\ |t| \le n\}.$$

The set $D_n \setminus nA$ has two connected components, which we denote by $C_1^n$ and $C_2^n$. For $i = 1, 2$ and $n \in \mathbb{N}$, let $A_i^n$ be the set of the points of $C_i^n \cap \mathbb{Z}^d$ which have a nearest neighbour in $\mathbb{Z}^d \setminus D_n$:

$$A_i^n = \{x \in C_i^n \cap \mathbb{Z}^d : \text{there exists } y \in \mathbb{Z}^d \setminus D_n \text{ such that } |x - y| = 1\}.$$




The Ising Hamiltonian in $D_n$ is

$$H_n(\sigma) = -\frac{1}{2} \sum_{\substack{x,y \in D_n \\ |x-y|=1}} \sigma(x)\sigma(y) \quad \text{for all } \sigma \in \{-,+\}^{D_n}.$$

Let $\mathcal{E}_n$ be the set of the spin configurations $\sigma$ inside $D_n$ such that $\sigma(x) = +$ for $x \in A_1^n \cup A_2^n$. Let $\mathcal{F}_n$ be the set of the spin configurations $\sigma$ inside $D_n$ such that $\sigma(x) = -$ for $x \in A_1^n$ and $\sigma(x) = +$ for $x \in A_2^n$. The partition functions $Z_n^+$, $Z_n^{-,+}$ corresponding to pure $+$ and mixed $-, +$ boundary conditions at temperature $T$ are

$$Z_n^+ = \sum_{\sigma \in \mathcal{E}_n} \exp -\frac{H_n(\sigma)}{T}, \quad Z_n^{-,+} = \sum_{\sigma \in \mathcal{F}_n} \exp -\frac{H_n(\sigma)}{T}.$$

Let $T > 0$. The limit

$$\tau(\nu) = \lim_{n \to \infty} -\frac{1}{n^{d-1}} \ln \frac{Z_n^{-,+}}{Z_n^+}$$

exists in $[0, \infty]$. The function $\tau$ is called the surface tension of the Ising model. It satisfies the weak simplex inequality, it is continuous and invariant under the isometries which leave $\mathbb{Z}^d$ invariant. Moreover $\tau$ is positive in the regime $T < T_c$. The Wulff crystal $\mathcal{W}$ appearing in Theorem 3.1 is the Wulff shape associated to the surface tension $\tau$, called also the crystal of $\tau$, defined by

$$\mathcal{W} = \{x \in \mathbb{R}^d : x \cdot w \leq \tau(w) \text{ for all } w \text{ in } S^{d-1}\}.$$

**Large deviations.** The way to prove Theorem 3.1 is rather long. The key is the analysis of the deviations of the average magnetization from its typical value. We have first a weak law of large numbers:

$$\lim_{n\to\infty} \mu^+_{\Lambda(n),T}\Big(\frac{1}{n^d}\sum_{x\in\Lambda(n)}\sigma(x)\Big) = m^*(T) \quad \text{for all } T > 0.$$

The large deviations from above are similar in nature to what happens for a sum of independent identically distributed random variables.

**Theorem 3.2.** *Let $d \geq 2$ and let $T > 0$. For any $\alpha \in [-1, 1]$, the limit*

$$J(\alpha) = \lim_{n\to\infty} -\frac{1}{n^d} \ln \mu^+_{\Lambda(n),T}\Big(\frac{1}{n^d}\sum_{x\in\Lambda(n)}\sigma(x) \geq \alpha\Big)$$

*exists and is finite. The map $\alpha \in [-1, 1] \mapsto J(\alpha) \in \mathbb{R}^+$ is convex continuous. It vanishes on $[-1, m^*(T)]$ and it is strictly positive on $]m^*(T), 1]$.*

The deviations from above are of volume order. The function $J$ appearing in Theorem 3.2 vanishes on $[0, m^*[$ because on this interval the large deviations are of surface order. As we are in the phase coexistence regime, a new large deviation principle on the surface scale emerges.

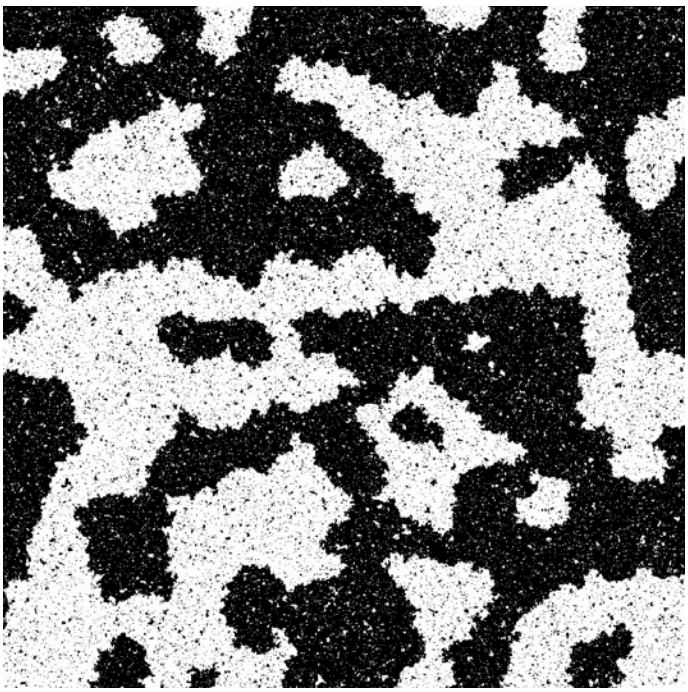
**Theorem 3.3.** *Let $d \geq 2$ and let $T < T_c(d)$. For $m < m^*$ close enough to $m^*$, so that the rescaled Wulff crystal $\mathcal{W}(m)$ fits into the unit cube $[-1/2, 1/2]^d$, we have*

$$\lim_{n\to\infty} \frac{1}{n^{d-1}} \ln \mu^+_{\Lambda(n),T}\Big(\frac{1}{n^d}\sum_{x\in\Lambda(n)}\sigma(x) \leq m\Big) = -d\left(\frac{m^* - m}{2m^*}\right)^{\frac{d-1}{d}} \mathcal{L}^d(\mathcal{W})^{\frac{1}{d}}.$$

Theorems 3.2 and 3.3 provide a complete picture of the large deviation behavior of the average magnetization. The emergence of a droplet is responsible for the asymptotic formula of Theorem 3.3. The full proofs of Theorems 3.2 and 3.3 can be found in the book [13].

## 4. Dynamics

Another very interesting topic is of course the dynamics. One should try to understand the dynamical mechanism leading to the creation of a Wulff crystal. It is naturally expected that several reasonable choices of microscopic dynamics induce the macroscopic dynamics associated to the motion by mean curvature. For instance it seems to be the case for the non-conservative Glauber dynamics, which is relevant for the beautiful theory of metastability (see [30], [31]). However the full understanding of these dynamics seems currently out of reach and only partial results are available [11], [15], [18], [19], [26], [32].



Phase separation under the Glauber dynamics at $T = 2.1$.

Let $d \geq 2$ and let $\Lambda \subset \mathbb{Z}^d$ be a cubic box. We wish to build a stochastic dynamics on the configuration space $\{-,+\}^\Lambda$ which models the microscopic repulsive forces between the particles. The interaction being microscopic, only one site or two neighbouring sites can be altered at each step. For $\sigma : \Lambda \to \{-,+\}$ and $x \in \Lambda$, we denote

$$S(\sigma, x) = \sum_{y:|x-y|=1} \sigma(y)$$

the sum of the spins of the neighbours of $x$.

Let $T > 0$ be a positive temperature. We consider two dynamics, which are built as discrete time Markov chains $(\sigma(k))_{k\geq 0}$ with state space $\{-,+\}^{\Lambda}$. We describe next the transition mechanisms of each dynamics.

**Glauber dynamics.** We suppose that $\sigma(k)$ is known and we explain how to build $\sigma(k+1)$. We first choose a site $x \in \Lambda$ randomly with the uniform law on $\Lambda$. We then compute $\Delta = 2\sigma(k,x)S(\sigma(k),x)$.

- If $\Delta < 0$, we flip the spin at $x$.
- If $\Delta \geq 0$, we flip the spin at $x$ with probability $\exp -(\Delta/T)$.

With the Glauber dynamics, at most one spin is changed at each time step.

**Kawasaki dynamics.** We suppose that $\sigma(k)$ is known and we explain how to build $\sigma(k+1)$. We first choose two neighbouring sites $x, y \in \Lambda$ randomly with the uniform law on the pairs of neighbours in $\Lambda$. We then compute $\Delta = (\sigma(k,x) - \sigma(k,y))(S(\sigma(k),x) - S(\sigma(k),y))$.

- If $\Delta < 0$, we exchange the spins of the sites at $x$ and $y$.
- If $\Delta \geq 0$, we exchange the spins of $x$ and $y$ with probability $\exp -(\Delta/T)$.

With the Kawasaki dynamics, at most two spins are changed at each time step.

A fundamental problem is to understand the scaling limits of these dynamics. For the Glauber dynamics, the adequate scaling is expected to converge to an anisotropic motion by mean curvature. More precisely, to the Markov chain $(\sigma(k))_{k\geq 0}$ we associate a process $(\sigma_n(t))_{t\geq 0}$ taking its values in the space of the Borel measures on $[-1/2,1/2]^d$ by setting

$$\sigma_n(t) = \frac{1}{n^d} \sum_{x\in\Lambda(n)} \sigma(\lfloor n^2 t \rfloor, x)\, \delta_{x/n} \quad \text{for all } t > 0,$$

where $\delta_{x/n}$ is the Dirac mass at $x/n$. We call $\sigma_n(t)$ the stochastic empirical magnetization.

Let $A$ be a subset of $[-1/2,1/2]^d$ having smooth boundary. We take as initial condition at step $n$ the configuration defined by

$$\sigma(0,x) = 1 - 2\cdot 1_{nA}(x) \quad \text{for all } x \in \mathbb{Z}^d.$$

Let $\mu$ be a function defined on the unit sphere $S^{d-1}$ of $\mathbb{R}^d$ with values in $\mathbb{R}^+$. Let $(A(t))_{t\geq 0}$ be the anisotropic mean curvature motion starting from $A$ associated to the function $\mu$, that is the solution (in some weak sense) of the equation:

$$v(x) = -\mu(\nu(x))\,\kappa(x)\,\nu(x) \quad \text{for all } t > 0,\ x \in \partial A(t),$$

where $v(x)$ is the speed at $x$, $\nu(x)$ is the normal vector to $A(t)$ at $x$ and $\kappa(x)$ is the curvature of $A(t)$ at $x$. Let $(w(t,x)\,dx)_{t\geq 0}$ be the measure valued process defined by

$$w(t,x)\,dx = m^*\big(1 - 2\cdot 1_{A(t)}(x)\big)\,dx \quad \text{for all } t \geq 0.$$

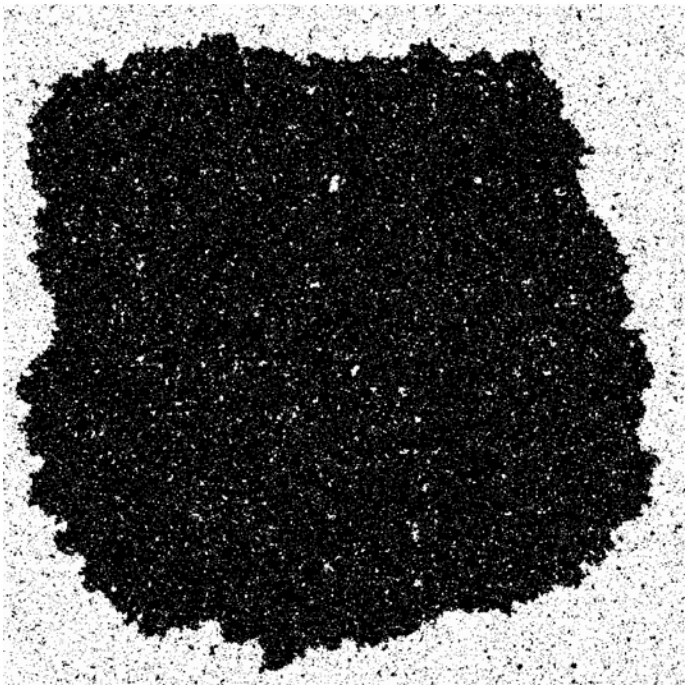
The big conjecture is that there exists a function $\mu\colon S^{d-1} \to \mathbb{R}^+$ such that the stochastic empirical magnetization $\sigma_n(t)$ converges weakly to the anisotropic mean curvature motion associated to $\mu$: there exists $T > 0$ such that, for any continuous function $f\colon [-1/2, 1/2]^d \to \mathbb{R}$,

$$\lim_{n\to\infty} P\big(|\sigma_n(t)(f) - w_n(t)(f)| \geq \varepsilon\big) = 0 \quad \text{for all } t \in [0,T],\ \varepsilon > 0.$$

We are still very far from the proof of such a result. In the two dimensional case at zero temperature, some specific computations [15], [32] show that, if this conjecture is correct, then the function $\mu$ governing the anisotropic motion by mean curvature must be equal to

$$\mu(x) = \frac{1}{2(|\cos\theta| + |\sin\theta|)^2},$$

where $\theta$ is the angle between the horizontal axis and the tangent to the boundary at $x$. The scaling limit of the conservative Kawasaki dynamics seems even more challenging to understand.



Evolution of a square droplet under Glauber dynamics at $T = 2.1$.

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