Simple random covering, disconnection, late and favorite points

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Abstract. We review recent advances in the study of the fractal nature of certain random sets, the key to which is a multi-scale truncated second moment method. We focus on some of the fine properties of the sample path of the most basic stochastic processes such as the simple random walk and the Brownian motion. As we shall see, probability on trees inspires many of our proofs, with trees used to model the relevant correlation structure. Along the way we also mention a few open problems.

Mathematics Subject Classification (2000). Primary 60J15; Secondary 28A80, 60G17, 60J65, 82C41.

Keywords. Random walk, Brownian motion, Gaussian free field, cover time, late points, favorite points, thick points, multi-fractal analysis, disconnection, intersection local time.

1. Introduction

The simple random walk (SRW) on a graph $G = (V, E)$ of finite degrees tracks the movement on the set V of vertices by a particle which at each time step jumps with equal probability to any one of the nearest neighbors of its current position, independently of all previous positions. In particular, the SRW on $\mathbb{Z}^d$ is a fundamental object in probability theory. More than forty years ago, Erdős and Taylor posed in [ET60] the following problem about the SRW on $\mathbb{Z}^2$: What is the maximal number of visits by the walk to one lattice site during its first n steps? More formally, denote by $L_n(x)$ the number of visits to x by the SRW during its first n steps, and set $L_n^* := \max_{x \in \mathbb{Z}^2} L_n(x)$. Then, it was conjectured in [ET60, (3.11)] and proved in [DPRZ01] that with probability one,

$$\lim_{n \to \infty} \frac{L_n^*}{(\log n)^2} = \frac{1}{\pi}. \tag{1.1}$$

As illustrated in Section 3, the key to proving (1.1) is a multi-scale truncated second moment method, inspired by the study of the corresponding problem for SRW on finite, regular trees. As detailed in Section 2, the same approach provides information about the location in $\mathbb{Z}^2$ where L_n^* is attained, the number of sites $x \in \mathbb{Z}^2$ for which $L_n(x)$ is exceptionally large, and the fractal dimension of the corresponding object for the sample path of the planar Brownian motion.

Proceedings of the International Congress
of Mathematicians, Madrid, Spain, 2006

The *cover time* C_G for a SRW on a finite graph G is the number of steps till the walk has visited all sites of G at least once. It has been studied intensively by probabilists, statistical physicists, combinatorialists and computer scientists (e.g. [Ald89], [Bro90], [BH91], [NCF91], [MP94]). In particular, the SRW on a finite graph is a time-reversible Markov chain and the asymptotics of C_G is an important aspect of the general theory of reversible Markov chains, see [AF01]. The problem of determining the asymptotics of the cover time $C_n := C_{\mathbb{Z}_n^2}$ for the two dimensional lattice torus $\mathbb{Z}_n^2 = \mathbb{Z}^2/n\mathbb{Z}^2$ of side length n was posed by Wilf, see [Wil89], and more formally by Aldous, who conjectured in [Ald89] that

$$\lim_{n\to\infty} \frac{C_n}{(n\log n)^2} = \frac{4}{\pi} \quad \text{in probability,} \tag{1.2}$$

and proved the upper bound of $4/\pi$ in (1.2). A lower bound $2/\pi$ was later proved in [Law92] and the conjecture (1.2) resolved in [DPRZ04]. Whereas the general theory of reversible Markov chains provides the correct growth order of C_n it fails to provide the multiplying constant in (1.2). As described in Section 4, the proof of (1.2) relies on the same multi-scale truncated second moment method used en-route to (1.1). Further, this approach provides information about the number and spatial distribution of the sites in $\mathbb{Z}_n^2$ for which the time of first visit by the SRW is of the order of the cover time. In addition, it allows us to answer the following questions of Révész about (discrete) discs covered by the random walk on $\mathbb{Z}^2$ till time n, namely, where every site of the lattice within the disc is visited by the walk at least once.

- What is the radius ρ_n of the largest disc, centered at the origin that is covered during the first n steps of the walk? It is shown in [DPRZ04] that

$$\lim_{n\to\infty} \mathbb{P}\Big(\frac{(\log\rho_n)^2}{\log n} \geq y\Big) = e^{-4y}, \tag{1.3}$$

  for all $y > 0$ (in 1989 Révész derived upper and lower bounds for (1.3) with non-matching constants 120 and $1/4$; these have been improved to 4 and 2 in [Law92] which also quotes (1.3) as a conjecture of Kesten).

- What is the radius R_n of the largest disc (of arbitrary center) that is covered during the first n steps of the walk? It is shown in [DPR07] that with probability one,

$$\lim_{n\to\infty} \frac{\log R_n}{\log n} = \frac{1}{4} \tag{1.4}$$

  (non-matching bounds for (1.4) with constants other than $1/4$ are given in [Rév93] where the existence of the limit is also conjectured).

In Section 5 we detail additional results for intersection local times and for the two dimensional Gaussian free field that have been obtained by the same approach we review here.

Statements such as (1.1) and (1.2) are easier to handle for the SRW on $\mathbb{Z}^d$, $d \geq 3$ whose transience enables us to effectively localize the relevant occupation measures, in contrast with the case of $d = 2$.

Let $G_n = \mathbb{Z}_n^d \times \mathbb{Z}$ denote the infinite discrete cylinder based on the d-dimensional discrete torus of side length n. We say that a finite subset Γ disconnects G_n if, for large r the sets $\mathbb{Z}_n^d \times [r, \infty)$ and $\mathbb{Z}_n^d \times (-\infty, -r]$ are contained in two distinct connected components of $G_n \setminus \Gamma$. Consider the time D_n till the range of the SRW on G_n disconnects the cylinder. Clearly, the *disconnection time* D_n is between the cover time C_n of the base $\mathbb{Z}_n^d$ by the projection of the SRW and the cover time $\widehat{C}_n$ of the slice $\mathbb{Z}_n^d \times \{0\}$ by the SRW on G_n. It was shown in [DS06] that

$$\lim_{n\to\infty} \frac{\log D_n}{\log n} = \lim_{n\to\infty} \frac{\log \widehat{C}_n}{\log n} = 2d \quad \text{in probability.} \tag{1.5}$$

That is, the disconnection time is roughly of order n^{2d} and comparable to $\widehat{C}_n$, but in contrast to the case of $d = 1$, when $d \geq 2$ it is substantially larger than the cover time C_n (which is roughly of order $n^{\max(d,2)}$, up to logarithmic correction terms). We outline in Section 6 the geometric argument which is the key to the proof of (1.5) and explain in what sense (1.5) implies for $d \geq 2$ a massive clogging of the truncated cylinders of height $n^{d-\varepsilon}$ by the SRW before it disconnects the infinite cylinder. See also [Szn06] for recent universality results about the asymptotic of the disconnection time for the SRW on $H_n \times \mathbb{Z}$ (under mild conditions on the finite graph H_n).

For earlier surveys of parts of this body of work see [Per03], [Dem05], [Shi06]. Many additional interesting examples of random fractals as well as numerous references to earlier works on such problems are provided in the survey [Tay86]. See also [LeG92] for more about the planar Brownian path and [Lyo05] for probability on trees.

2. Favorite and thick points

2.1. Favorite points for SRW on $\mathbb{Z}^d$. Erdős and Révész in [ER84] call a site $x \in \mathbb{Z}^d$ for which $L_n(x) = L_n^*$ a *favorite point* of the SRW. In a similar manner, for any $0 < \alpha < 1$ we say that $x \in \mathbb{Z}^2$ is an α*-favorite point* of the walk if $L_n(x) \geq (\alpha/\pi)(\log n)^2$. The size of the set $\mathcal{F}_n(\alpha)$ of α-favorite points can be estimated by the same approach leading to (1.1). More precisely, it is shown in [DPRZ01] that for each $\alpha \in (0, 1]$,

$$\lim_{n\to\infty} \frac{\log |\mathcal{F}_n(\alpha)|}{\log n} = 1 - \alpha \quad \text{a.s.} \tag{2.1}$$

In other words, the n^β-most visited point during the first n steps of the walk is visited approximately $\frac{1-\beta}{\pi}(\log n)^2$ times. This is in contrast with typical points on the path of the walk, each of which has order $\log n$ visits.

It is also shown in [DPRZ01] that any random sequence $\{x_n\}$ in $\mathbb{Z}^2$ such that $L_n(x_n)/L_n^* \to 1$ must satisfy

$$\lim_{n\to\infty} \frac{\log \|x_n\|}{\log n} = \frac{1}{2} \quad \text{a.s.} \tag{2.2}$$

In particular, the favorite points, i.e. those x_n^* where L_n^* is attained, are consistently located near the frontier of the set of visited points, at least on a logarithmic scale.

For the SRW on $\mathbb{Z}$ the analog of the statement (2.2) is contained in the results of Bass and Griffin [BG85]. See also [Rév05, page 160] for a list of unsolved problems about favorite points of the SRW on $\mathbb{Z}$, taken from [ER84]. Tóth provided recently a partial answer to one of these questions, showing that the SRW on $\mathbb{Z}$ has for sufficiently large time n at most three different favorite points x_n^* (cf. [Tót01] for this result and its history). However, not much is known about x_n^*. For example,

Open Problem 2.1.
- Determine for which $d \geq 1$ the SRW on $\mathbb{Z}^d$ has with probability one at most two favorite points x_n^* for all n sufficiently large.
- Describe the evolution of $n \mapsto x_n^*$ in any dimension $d \geq 2$.
- How is the growth of time between the first and last visits to x_n^* prior to n, affected by the dimension d?

2.2. Thick points for planar Brownian motion. Let $w^{(m)}(t) = \sqrt{d/m}\, S_{\lfloor mt \rfloor}$ denote a time-space rescaled image of the SRW $(S_k, k \geq 0)$ on $\mathbb{Z}^d$. Donsker's functional CLT tells us that the distribution of $(w^{(m)}(t), t \geq 0)$ converges as $m \to \infty$, to that of the Brownian motion $(w(t), t \geq 0)$. The latter is a continuous in time, $\mathbb{R}^d$-valued Gaussian stochastic process, of independent coordinates, each starting at zero and having zero mean increments of variance $|t - s|$.

It is thus not surprising that the continuous time analogs of (1.1) and (2.1) can be expressed in terms of the Brownian *occupation measure*

$$\mu_t^w(A) = \int_0^t \mathbf{1}_A(w(s))ds, \quad \text{for all } A \subseteq \mathbb{R}^d \text{ Borel}, \tag{2.3}$$

in the planar case, that is, when $d = 2$. To this end, let $D(x, r)$ denote the open disc in $\mathbb{R}^2$, centered at x and of radius r, and let $\bar{\theta} = \inf\{t \geq 0 : \ \|w(t)\| \geq 1\}$ be the exit time of the planar Brownian motion from the unit disc $D(0, 1)$. Since $w([0, \bar{\theta}]) := \{w(t) : 0 \leq t \leq \bar{\theta}\}$ is a compact set, it follows that $\mu_{\bar{\theta}}^w(D(x, r)) = 0$ for any $x \notin w([0, \bar{\theta}])$ and all r small enough. Further, it is not hard to show that for almost all Brownian paths, the pointwise Hölder exponent of the random measure $\mu_{\bar{\theta}}^w$, namely,

$$\lim_{r\to 0} \frac{\log \mu_{\bar{\theta}}^w(D(x, r))}{\log r},$$

takes the same value 2 for all points $x \in w([0, \bar{\theta}])$ (see also [Ray63, Theorem 1] for the precise lim sup decay rate of $\mu_{\bar{\theta}}^w(D(0, r))$ when $r \to 0$). Therefore, standard multi-fractal analysis must be refined in order to capture the delicate fluctuations of the Brownian occupation measure and obtain a non-degenerate dimension spectrum. Indeed, it is shown in [DPRZ01] that for any $0 < a \le 2$,

$$\dim\{x : \lim_{r\to 0} \frac{\mu_{\bar\theta}^{w}(D(x,r))}{r^2(\log r)^2} = a\} = 2 - a \quad \text{a.s.} \tag{2.4}$$

(where throughout $\dim(A)$ denotes the Hausdorff dimension of the set A). For a typical x on the Brownian path $\mu_{\bar\theta}^{w}(D(x,r)) \asymp r^2|\log r|$ (e.g. see [DPRZ01, Lemma 2.1]), so the *a-thick points*, i.e. those in the set considered in (2.4), correspond to unusually large occupation measure.

The identity (2.4), together with the appropriate upper bound, yields that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^2} \frac{\mu_{\bar\theta}^{w}(D(x,r))}{r^2(\log r)^2} = 2 \quad \text{a.s.} \tag{2.5}$$

as conjectured by Perkins and Taylor. It is not hard to show then that both (2.4) and (2.5) hold when $\bar\theta$ is replaced by any deterministic $0 < T < \infty$ and it is in the latter form that (2.5) was stated as [PT87, Conjecture 2.4]. As shown in [DPRZ02, Theorem 1.2], both (2.4) and (2.5) apply even when the discs $D(x,r) = x + rD(0,1)$ are replaced by the sets $x + rK$, provided the set K is normalized to have area (Lebesgue measure) π and its boundary has zero Lebesgue measure.

See also [PPPY01] for the application of (2.5) to the problem of reconstructing the range of spatial Brownian motion from the occupation measure projected to the sphere.

2.3. From Brownian motion to SRW. The passage from (2.5) and (2.4) to the corresponding results (1.1) and (2.1) for the discrete setting is based on the celebrated strong approximation theorem of Komlós, Major and Tsunády [KMT75] which constructs in an enlarged probability space a one dimensional Brownian motion w and a SRW $(S_k, k \ge 0)$ on $\mathbb{Z}$ such that $\mathbb{P}(\sup_{t\le 1} |w(t) - w^{(n)}(t)| \ge c(\log n)/\sqrt{n}) \to 0$ when $n \to \infty$. A simple geometric argument extends this conclusion to the SRW on $\mathbb{Z}^2$. Further, applying Einmahl's multidimensional version of this strong approximation theorem, the same argument allows [DPRZ01, Theorem 5.1] to establish (1.1), (2.1) and (2.2) for a wide collection of two-dimensional lattice valued random walks whose increments are of zero mean and finite moments. Here is an outline of such an argument, revealing the source of the factor 2π between (1.1) and (2.5). Taking $r = r(n) = n^{\eta-1/2}$ for fixed $\eta > 0$ small, and fixing $0 < a < 2$, one predicts from (2.4) that there are about r^{a-2} discs of radius r, which are r-separated of each other, each having a Brownian occupation measure of about $ar^2(\log r)^2$. By strong approximation a similar result applies for the occupation measure of $w^{(n)}$. Since $t \mapsto w^{(n)}(t)$ is piecewise constant on intervals of length $1/n$, this translates to $nar^2(\log r)^2$ visits by $w^{(n)}$ to discs whose radius is approximately r. Further, with each of these discs having about $\pi r^2 n/2$ of the sites of $\sqrt{2/n}\mathbb{Z}^2$, we see that to these discs correspond distinct (random) points in $\mathbb{Z}^2$ having at least $2a(\log r)^2/\pi$ visits during the first n steps of the SRW. So, for $\alpha = 2a(1/2-\eta)^2$ these $r^{a-2} = n^{1-2\eta-\alpha/(1-2\eta)}$ points are α-favorite, and considering first $n \to \infty$ then $\eta \to 0$ gives the lower bound in (2.1) and consequently also in (1.1).

As shown in [Ros05], (2.1) and (1.1) can also be proved without reference to the Brownian motion results, by directly applying the multi-scale truncated second moment approach of Section 3 to the SRW on $\mathbb{Z}^2$.

3. The multi-scale truncated second moment

Fixing a positive integer $b \geq 2$ let Γ_h denote the b-ary rooted regular tree of height h, that is, the degree of each vertex of Γ_h is $b+1$, except for the root, denoted $\mathbf{o}$, whose degree is b and the b^h leaves of this tree, each of whom has degree one. Starting a SRW (X_i) on Γ_h at its left-most leaf let $\tau_{\mathbf{o}} = \inf\{i \geq 0 : X_i = \mathbf{o}\}$ denote the hitting time of the root of Γ_h and L_x the number of visits to $x \in \Gamma_h$ by $\{X_i, i \leq \tau_{\mathbf{o}}\}$. Fixing $0 < \alpha < 1$ we call a leaf x of Γ_h *α-favorite* if $L_x \geq \alpha h^2 \log b$ and let $\mathcal{F}_h(\alpha)$ denote the set of α-favorite leaves. In Subsection 3.1 we outline the multi-scale truncated second moment method in the context of proving that

$$\lim_{h\to\infty} \frac{1}{h} \log |\mathcal{F}_h(\alpha)| = (1-\alpha)\log b \quad \text{in probability.} \tag{3.1}$$

As demonstrated in Subsection 3.2, this is the core of the computations leading to (2.4) and (2.5), thereby also to (1.1), (2.1) and (2.2), where the trees Γ_h serve in revealing the hidden correlation structure across scale (i.e., the radius of discs), and space (i.e., their centers).

3.1. Favorite points for SRW on regular trees. Let $\partial\Gamma_h$ denote the set of leaves of Γ_h and $\mathbf{o} \leftrightarrow x$ denote the shortest path in Γ_h between $\mathbf{o}$ and $x \in \partial\Gamma_h$, also called the ray of x. Fixing x and projecting the SRW on its ray, we see that L_x has the same law as the number of visits to a reflecting boundary at h prior to absorption at 0, for a SRW (Y_i) on $\{0, 1, \ldots, h\}$. In particular, $\mathbb{P}(L_x \geq t) \leq \left(1-\frac{1}{h}\right)^{t-1}$ for any $x \in \partial\Gamma_h$, $t \geq 1$. Taking $t = t_h$ for $t_h := \alpha h^2 \log b$ yields the upper bound in (3.1) by an application of the first moment bound $\mathbb{P}(Z \geq 1) \leq \mathbb{E}Z$ for $Z = Z_h = b^{-\beta h}|\mathcal{F}_h(\alpha)|$ and $\beta > 1-\alpha$. With a little more work we find that as $h \to \infty$ also $\mathbb{E}(Z_h) \to \infty$ for $\beta < 1-\alpha$, supporting the validity of (3.1). To take advantage of diverging expectations one usually relies on the second moment method. That is, applying the classical bound $\mathbb{P}(Z \geq \delta\mathbb{E}Z) \geq (1-\delta)^2(\mathbb{E}Z)^2/\mathbb{E}Z^2$ for some $0 < \delta < 1$ and the preceding $Z = Z_h$. However, at least for $\alpha > 1/2$ this approach fails to work here. Indeed, as shown in [Dem05, Lemma 4.2], then $\mathbb{E}Z_h^2/(\mathbb{E}Z_h)^2 \to \infty$ since for such α and any $0 < \xi < 2\alpha - 1$ there exists $\eta = \eta(\alpha, \xi) > 0$ such that for large enough h

$$b^{-2(1-\alpha)h} \sum_{(x,y)\in\partial_\xi\Gamma_h} \mathbb{P}(L_x \geq t_h, L_y \geq t_h) \geq b^{\eta h}, \tag{3.2}$$

where $\partial_\xi \Gamma_h$ denote the collection of pairs $x, y \in \partial\Gamma_h$ for which $\mathbf{o} \leftrightarrow x$ and $\mathbf{o} \leftrightarrow y$ separate at distance ξh from $\mathbf{o}$.

Starting the SRW $\{Y_i\}$ at $Y_0 = h$ and assuming t_h returns to h prior to its absorption at 0, the expected number of excursions between $k-1$ and k till the t_h-th return to h is about $\alpha k^2 \log b$ when $k \gg 1$ and $h - k \gg 1$ (cf. [Dem05, Lemma 4.3]). This suggests that for a typical $x \in \mathcal{F}_h(\alpha)$ the SRW (X_i) has prior to $\tau_{\mathbf{o}}$ about $\alpha k^2 \log b$ visits to a vertex on $\mathbf{o} \leftrightarrow x$ which is at distance k from $\mathbf{o}$. The analysis leading to (3.2) reveals also that the main contribution of the collection $\partial_\xi \Gamma_h$ to the second moment of $|\mathcal{F}_h(\alpha)|$ is via the rare events of having prior to $\tau_{\mathbf{o}}$ a sufficiently excessive number of SRW excursions between the vertex z where $\mathbf{o} \leftrightarrow x$ and $\mathbf{o} \leftrightarrow y$ separate and the leaves of the sub-tree rooted at z. The typical number of such excursions for α-favorite leaves is only a fraction of what these events require, so they contribute little to $\mathbb{E}Z_h$ when h is large. However, the occurrence of such rare event yields too many α-favorite leaves at the sub-tree rooted at z, hence resulting with the excessive growth of the second moment of Z_h.

Since this problem occurs for any separation height ξh, $\xi < 2\alpha - 1$, one should pursue a multi-scaling truncation strategy. That is, apply the second moment method for $Z_h = b^{-\beta h}|\mathcal{S}_h(\alpha)|$, replacing $\mathcal{F}_h(\alpha)$ by a subset $\mathcal{S}_h(\alpha)$ of leaves along the rays of which various excursion counts are kept within a relatively small distance from the typical excursion count profile for an α-favorite leaf. Of course we are to do so while not changing much the mean of Z_h, that is, keeping

$$\mathbb{P}(x \in \mathcal{S}_h(\alpha)) = b^{-\alpha h(1+o(1))} \tag{3.3}$$

for all $x \in \partial\Gamma_h$. To attain (3.3) only $o(h)$ excursion counts are to be controlled along each ray of Γ_h. Specifically, fixing $c > 0$ large enough, we set heights $h_0 = 0$, $h_1 = 1$ and $h_k = [ck \log k]$ for $k = 2, \ldots, m$, taking $h = h_m$, and consider the number N_k^x of complete excursions between vertices x_{k-1} and x_k at distances h_{k-1} and h_k from $\mathbf{o}$ along $\mathbf{o} \leftrightarrow x$ which occur between the first visit of the SRW to x_1 and its first successive visit to $x_0 = \mathbf{o}$. Recall that for a typical α-favorite leaf x, the SRW (X_i) makes about $\alpha h_k^2 \log b$ visits to x_k during this time interval. With $\Delta_k = h_k - h_{k-1} = c \log k(1 + o(1))$ for large k, this translates to N_k^x being near $n_k = \alpha h_k^2 \log b/(c \log k)$ for a typical $x \in \mathcal{F}_h(\alpha)$. Consequently, we take as $\mathcal{S}_h(\alpha)$ those $x \in \partial\Gamma_h$ such that $|N_k^x - n_k| \le k$ for $k = 2, \ldots, m$. Per $x \in \partial\Gamma_h$, the sequence $(N_k^x, k = 1, \ldots, m)$ is the realization of a non-homogeneous Markov chain on $\mathbb{Z}_+$, starting at $N_1^x = 1$. The transition probabilities of this chain are given by explicit hyper-geometric distributions. As $n_{k+1} = n_k(1 + 2/k)(1 + o(1))$, for large k one gets by normal approximation that the probability of the transition from $N_k^x = [n_k]$ to $N_{k+1}^x = [n_{k+1}]$ is about $p_k = b^{-\alpha\Delta_k}/\sqrt{n_k}$. With $k/\sqrt{n_k} \to 0$, further analysis shows that for some finite constant C which depends only on (α, b, c) and for any $|\ell_k - n_k| \le k$ and $|\ell_{k+1} - n_{k+1}| \le k + 1$, the probability of the transition from $N_k^x = \ell_k$ to $N_{k+1}^x = \ell_{k+1}$ is between $C^{-1} p_k$ and $C p_k$ (cf. [Dem05, Lemma 4.6]). Note that $q_m = \mathbb{P}(x \in \mathcal{S}_{h_m}(\alpha))$ is the same for any $x \in \partial\Gamma_{h_m}$. Also, in the definition of the event $\{x \in \mathcal{S}_{h_m}(\alpha)\}$, for each k, the random variable N_k^x can take any one of

$2k+1$ possible values. Hence, by the preceding analysis $m^{-1} \log \left(q_m / \left[\prod_{k=1}^{m-1} k p_k \right] \right)$ is bounded. Moreover, as $n_k = \zeta k^2 \log k$ for some positive constant ζ, it follows that $kp_k = b^{-\alpha \Delta_k (1+o(1))}$ and with $m = o(h_m)$, it is now easy to verify that (3.3) holds (cf. [Dem05, Proposition 4.4]).

We turn next to study the correlation structure of $\{x \in \mathcal{S}_{h_m}(\alpha)\}$ across $x \in \partial\Gamma_{h_m}$. Specifically, let $B_{l,x} = \{|N_k^x - n_k| \le k, l < k \le m\}$ for $l = 1, \dots, m-1$, noting that $B_{1,x} = \{x \in \mathcal{S}_{h_m}(\alpha)\}$, so the second moment of $|\mathcal{S}_{h_m}(\alpha)|$ is the sum of $\mathbb{P}(B_{1,x} \cap B_{1,y})$ over $x, y \in \partial\Gamma_{h_m}$. Let $q_{m,l}$ denote the maximum of these probabilities over pairs (x, y) such that $x_1 = y_1$ and the rays $\mathbf{o} \leftrightarrow x$ and $\mathbf{o} \leftrightarrow y$ separate at a vertex z whose distance from $\mathbf{o}$ is between h_{l-1} and h_l. Since $B_{1,y} \subseteq B_{l,y}$, it thus suffices for us to get an upper bound on $\mathbb{P}(B_{1,x} \cap B_{l,y})$. To this end, note that given the value of N_{l+1}^y, the event $B_{l,y}$ is independent of $B_{1,x}$. Consequently,

$$q_{m,l} \le q_m \sum_{|\ell - n_{l+1}| \le l+1} \mathbb{P}(B_{l,y} | N_{l+1}^y = \ell).$$

By a similar reasoning,

$$q_m \ge q_{l+1} \inf_{|\ell - n_{l+1}| \le l+1} \mathbb{P}(B_{l,y} | N_{l+1}^y = \ell).$$

As we have already seen, the above terms $\mathbb{P}(B_{l,y} | N_{l+1}^y = \ell)$ are almost constant (up to a factor C^2) with respect to ℓ, leading to the bound

$$q_{m,l} \le C^2 (2l+3) \frac{q_m^2}{q_{l+1}} \tag{3.4}$$

(cf. [Dem05, Lemma 4.8]). This is effectively the same correlation structure as for independent percolation on the tree Γ_{h_m} projected to skeleton heights $\{h_k\}$ and with level depending edge probabilities that are about p_k. In particular, using the bound of (3.4) leads to $\mathbb{E}[|\mathcal{S}_h(\alpha)|^2] \le K(\mathbb{E}|\mathcal{S}_h(\alpha)|)^2$ for some $K = K(\alpha)$ finite and any $h \in \{h_k\}_{k=1}^\infty$ (cf. [Dem05, Lemma 4.9]). Combining this with the first moment estimate of (3.3), an application of the second moment method yields that the probability of ${h_m}^{-1} \log |\mathcal{S}_{h_m}(\alpha)| \ge (1-\alpha) \log b (1+o(1))$ is bounded away from zero (being about $1/K$).

It remains to improve this result to one that holds with probability approaching one, and to connect the fact that N_m^x is near n_m with the event $\{x \in \mathcal{F}_{h_m}(\alpha)\}$. To this end, for $h \in [h_{m+1}, h_{m+2})$ and any vertex $v \in \Gamma_h$ of height $h - h_m + 1$ let $\mathcal{S}_{h_m}^v(\alpha)$ be defined in analogy to $\mathcal{S}_{h_m}(\alpha)$, but for the subtree rooted at the ancestor of v and consisting of those $u \in \Gamma_h$ with v on the shortest path from u to $\mathbf{o}$. Next let $\mathcal{S}_h^*(\alpha)$ be the union of the sets $\mathcal{S}_{h_m}^v(\alpha)$ over the R_h vertices v of height $h - h_m + 1$ that the SRW on Γ_h visits by time $\tau_{\mathbf{o}}$. The preceding bound applies for the SRW within any regular subtree of depth h_m, hence for each of the sets $\mathcal{S}_{h_m}^v(\alpha)$. Thus, as $R_h \to \infty$ with high probability, and the restrictions of the SRW to within the subtrees of Γ_h rooted at these vertices are independent of each other and of R_h, it follows that

$$\lim_{h\to\infty} \frac{1}{h} \log |\mathcal{S}_h^*(\alpha)| = (1-\alpha)\log b \quad \text{in probability}$$

(cf. [Dem05, Lemma 4.11]). Finally, a simple concentration argument shows that during $n_m - m$ excursions between $x \in \partial\Gamma_h$ and a vertex on its ray at distance Δ_m from x, the SRW visits x less than $n_m\Delta_m(1-\delta)$ times with probability that decays to zero exponentially in n_m (cf. [Dem05, Lemma 4.7]). This leads for any $\beta < \alpha$ to $\mathbb{P}(\mathcal{S}_h^*(\alpha) \subseteq \mathcal{F}_h(\beta)) \to 1$ and consequently completes the proof of (3.1).

3.2. From trees to Brownian motion. The approach of [DPRZ01] in proving (2.4) and (2.5), which goes back to [Ray63], is to control Brownian occupation measures using excursions between concentric discs. Specifically, fixing $R' > R > r$, the total occupation measure of $D(x, r)$ during the first N excursions of the sample path between $D(x, R)$ and the complement of $D(x, R')$ is of the form $\sum_{i=1}^N \tau_i$, where τ_i denotes the occupation measure of $D(x, r)$ accumulated during the i-th such excursion. Since these Brownian excursions are independent of each other, so are the random variables $\{\tau_i\}$. Further, the events of interest here involve exceptionally large occupation measures that translate into having numerous excursions around the same point. Consequently, for the range of N values relevant here, the resulting total occupation measure is highly concentrated around its mean $N\mathbb{E}\tau_1$.

From the corresponding elliptic PDE we have that $\mathbb{E}\tau_1 = r^2\log(R'/R)$ which by the strong Markov property of the Brownian motion implies also that $\mathbb{P}(\tau_1 \geq t[r^2\log(R'/r)+1])$ decays exponentially in t (cf. [Dem05, Lemma 5.5]). The statement (2.5) can be shown to concern the maximum of the occupation measures $\mu_{\bar\theta}^w(D(x_j, r))$ for a suitable non-random discrete net of about r^{-2} points $x_j \in D(0,1)$. To upper bound such maximum, we take $R' = 2$ in which case $\mu_{\bar\theta}^w(D(x,r)) \leq \tau_1$. Hence, for $a > 2$, we see from the tail probabilities of τ_1 (at $t = a\log(1/r)$), that the expected number of discs with centers in this net and occupation measure exceeding $ar^2(\log r)^2$ decays to zero at rate r^{a-2}. By the first moment method we get the upper bound in (2.5), where the almost sure statement is attained by using the monotonicity of $r \mapsto \mu_{\bar\theta}^w((D(x,r))$ to interpolate between radii r_n such that $\sum_n r_n^{a-2}$ is finite.

The same argument shows that the expected number of such discs with occupation measure exceeding $t_r = ar^2(\log r)^2$ diverges at rate r^{a-2} when $a < 2$, supporting the validity of (2.5). Unfortunately, the second moment method fails to work here since the recurrence of the planar Brownian motion precludes a fast decay of the correlation between the events $\{\mu_{\bar\theta}^w((D(x,r)) \geq t_r\}$ even for x and y of distance r^δ of each other (and δ a fixed small positive constant). A natural truncation of the second moment is by localizing the occupation measures. For example, considering only the Brownian occupation measure of $D(x, r)$ during the interval between the first hitting time of this disc by the Brownian path and its successive exit of $D(x, R)$. For a suitable $R(r) \to 0$ such that $R(r)/r \to \infty$ the second moment of the number of discs with localized occupation measures exceeding t_r is much smaller than the corresponding second moment for $\mu_{\theta}^{w}(D(x,r))$. However, it comes at the cost of a substantial drop in the corresponding first moment, so this strategy results with non-matching lower and upper bounds (cf. [PT87]).

We note in passing that the same problem arises in the context of the SRW on $\mathbb{Z}^2$, that is, in [ET60] treatment of (1.1), and more generally in all statements we make in this paper.

We mitigate this problem by employing a multi-scale truncation, as done in Subsection 3.1. To this end, recall that the counts of Brownian excursions between concentric discs of radii $\{e^{-j}\}$ have the same law as the bond occupation measure for the SRW on $\mathbb{Z}$, which plays a key role in proving (3.1). Indeed, for $x \notin D(0,r_1)$ taking $r_k = r_1 e^{-h_k}$ for $h_k = [ck \log k]$, $k = 2, \dots, m$, the numbers N_k^x of Brownian excursions from $\partial D(x, r_{k-1})$ to $\partial D(x, r_k)$ till hitting $\partial D(x, r_0)$ have the same joint law as the random variables N_k^x for the SRW on the regular tree Γ_{h_m} (to make this precise, take $r_0 = er_1$). Thus, setting $n_k = ah_k^2/(c \log k)$ and partitioning the square $S(1,1) = [2r_1, 4r_1]^2$ into e^{2h_m} non-overlapping squares $S(m,i)$ of edge length $2r_m$ each, we now consider the random set $\mathcal{S}_{h_m}(a)$ that consists of centers $x = x_{m,i}$ of squares $S(m,i)$ for which $|N_k^x - n_k| \le k$, $k = 2, \dots, m$. As explained above, the estimate (3.3) applies here (with $\alpha = a$, $b = e$ and $h = h_m$), resulting with $\mathbb{E}|\mathcal{S}_{h_m}(a)| = e^{(2-a)h_m(1+o(1))}$. We can further uniformly bound correlation terms of the form $\mathbb{P}(B_{1,x} \cap B_{l,y})$ for $x = x_{m,i}$ and $y = x_{m,j}$ such that $|x - y| \in (2r_l, 2r_{l-1}]$ in a similar manner to that for the SRW on regular trees, apart from two complications. First, to assure that $\partial D(x, r_k)$ does not intersect $\partial D(y, r_l)$ we have to exclude $k = l-1$. That is, remove from $B_{1,x}$ the constraints on the values of N_{l-1}^x and N_l^x. Even after this is done, given $N_{l+1}^y = \ell$, the event $B_{l,y}$ still depends on $B_{1,x}$ via the locations of the initial and final points of the ℓ Brownian excursions from $\partial D(y, r_{l+1})$ to $\partial D(y, r_l)$. Using the Poisson kernel for the density of the exit location at $z \in \partial D(y, R')$ for a Brownian path starting at some $z' \in D(y, R')$, it can be shown that the probability of $B_{l,y}$ given $N_{l+1} = \ell$ *and the terminal points* of these ℓ excursions is at most $(1 + \kappa r_{l+1}/r_l)^{\ell} \mathbb{P}(B_{l,y} | N_{l+1}^y = \ell)$ (cf. [DPRZ01, Lemma 7.4]). Taking c sufficiently large ($c = 3$ will do), provides enough separation for $n_{l+1} \ll r_l/r_{l+1}$ resulting with a bound of the form of (3.4), apart from replacing $C^2(2l+3)$ by a larger polynomial factor (cf. [DPRZ01, Lemma 8.1]). This is enough for deducing that $\mathbb{E}[|\mathcal{S}_{h_m}(a)|^2] \le K(\mathbb{E}|\mathcal{S}_{h_m}(a)|)^2$, for some $K = K(a)$ and all m. Since $\mathbb{E}|\mathcal{S}_{h_m}(a)| \to \infty$, by the second moment method also $\liminf_m \mathbb{P}(\mathcal{S}_{h_m}(a) \neq \emptyset) \ge 1/K$. By Fatou's lemma, this implies the existence with positive probability of some random $m(j) \to \infty$ and $y_j \in \mathcal{S}_{h_{m(j)}}(a)$. By compactness of $S(1,1)$, the sequence y_j has at least one limit point $y_* \in S(1,1)$.

Let $M_{x,r} = \mu_{\theta}^{w}(D(x,r))/[r^2(\log r)^2]$. We claim that $M_{y_*,r} \to a$ for $r \to 0$. That is, y_* is an a-thick point, see (2.4). To this end, recall that for $R' = r_{k-1}$ and $R = r = r_k$ we have $\mathbb{E}\tau_1 = r_k^2 \Delta_k$, whereas $n_k \Delta_k$ is about $ah_k^2 = a(\log r_k)^2$. With $\{\tau_i\}$ of exponential tail probabilities, it follows that $\mathbb{P}(|M_{x,r_k} - a| > \eta, |N_k^x - n_k| \le k) \le e^{-\gamma n_k}$ for any $\eta > 0$, some $\gamma = \gamma(a,\eta) > 0$, all k and $x \in S(1,1)$. By the monotonicity of the non-negative measure μ_{θ}^{w} and the almost sure continuity properties of $M_{x,r}$

in x and r, we further deduce the existence of $\delta(r, w) \to 0$ such that with probability one, if $x \in S(1, 1)$ and $|N_k^x - k| \le n_k$ for all $k \le \ell$, then $|M_{x,r} - a| \le \delta(r)$ for all $r \ge r_\ell$ (cf. [DPRZ01, Section 6]). Recall that for any fixed ℓ, if j is sufficiently large then $|N_k^{y_j} - n_k| \le k$ for $k = 2, \dots, \ell$. Consequently, $\limsup_j |M_{y_j,r} - a| \le \delta(r)$ for any $r > 0$. Since $|y_j - y_*| \to 0$, the monotonicity of $\mu_{\bar\theta}^w$ leads to $|M_{y_*,r} - a| \le \delta(r)$, and so we deduce that with some positive probability there exists an a-thick point.

Fixing $\beta < 2 - a$, by a slightly more involved argument, the squares $S(m, i)$ whose centers are in $\mathcal{S}_{h_m}(a)$ support the density with respect to Lebesgue measure of a random measure ν_m such that a non-zero weak limit point ν_∞ of $\{\nu_m\}$ is supported on a closed set of a-thick points and has a finite β-energy

$$\mathcal{E}_\beta(\nu_\infty) := \int |x - y|^{-\beta} d\nu_\infty(x) d\nu_\infty(y) \tag{3.5}$$

(cf. [DPRZ01, Section 3]). This in turns implies that with positive probability the dimension of the set of a-thick points is at least β. Using Brownian scaling, it follows by Blumenthal's zero-one law that the latter property holds with probability one, establishing the stated lower bound in (2.4) and hence also that of (2.5).

4. Fractal geometry: late points and covered discs

4.1. Cover times and covered discs. For $0 \le \gamma < 1$ let $C_n(\gamma)$ denote the time it takes until the largest disc unvisited by the SRW on the two dimensional lattice torus $\mathbb{Z}_n^2$ has radius n^γ. It is shown in [DPRZ04] that

$$\lim_{n\to\infty} \frac{C_n(\gamma)}{(n \log n)^2} = \frac{4}{\pi}(1-\gamma)^2 \quad \text{in probability,} \tag{4.1}$$

with (1.2) corresponding to the special case of $\gamma = 0$. In other words, at time βC_n the radius of the largest disc within the set unvisited by the SRW is $n^{(1-\sqrt{\beta})(1-o(1))}$.

In [PR04], Peres and Revelle deduce from (4.1) that the total variation convergence to the stationary measure for the SRW on the lamplighter group over $\mathbb{Z}_n^2$ requires at least $(4/\pi)(n \log n)^2(1 + o(1))$ steps. This is based on their observation that starting at a zero lamp configuration, the lamps remain identically zero on the sites of $\mathbb{Z}_n^2$ which the lamplighter did not visit, while under the stationary (uniform) measure of this SRW the probability of having a disc of radius $\log n$ in $\mathbb{Z}_n^2$ with a zero lamp configuration tends to zero with n. By general considerations, the asymptotic growth in n of the total variation mixing time for the SRW on the lamplighter group over $\mathbb{Z}_n^d$, $d \ge 2$, is between $\mathbb{E}C_{\mathbb{Z}_n^d}$ and half its value, but it is not known whether this upper bound is tight also for $d \ge 3$ (see [PR04]).

Let $\{w_{\mathbb{T}^2}(t)\}$ denotes a Brownian motion on the two dimensional unit torus $\mathbb{T}^2$, with the corresponding hitting times, $\tau(x, \varepsilon) = \inf\{t > 0 : w_{\mathbb{T}^2}(t) \in D_{\mathbb{T}^2}(x, \varepsilon)\}$, and the ε-cover time,

$$\boldsymbol{C}_\varepsilon = \sup_{x\in\mathbb{T}^2} \{\tau(x, \varepsilon)\}. \tag{4.2}$$

Equivalently, $\boldsymbol{C}_r$ is the amount of time needed for the Wiener sausage of radius r to completely cover $\mathbb{T}^2$. It is shown in [DPRZ04] that

$$\lim_{r\to 0} \frac{\boldsymbol{C}_r}{(\log r)^2} = \frac{2}{\pi} \quad \text{a.s.} \tag{4.3}$$

Similarly to the argument we outline in Subsection 2.3, (4.1) is an immediate consequence of (4.3). Indeed, we identify the vertices of $n^{-1}\mathbb{Z}_n^2$ with the corresponding subset of $\mathbb{T}^2$, noting that up to scaling by n, distances in $\mathbb{Z}_n^2$ match the corresponding distances in $\mathbb{T}^2$. Further identifying the latter with $[0, 1)^2$, we represent $w_{\mathbb{T}^2}(t)$ as the image of the planar Brownian motion $w(t)$ via the non-expansive mapping $x \mapsto x \bmod \mathbb{Z}^2$. Similarly, if S_k denotes the SRW on $\mathbb{Z}^2$, then $X_k = (n^{-1}S_k) \bmod \mathbb{Z}^2$ is the SRW on $n^{-1}\mathbb{Z}_n^2$. By Einmahl's [Ein89, Theorem 1] multidimensional version of KMT strong approximation theorem, we can construct $\{S_k\}$ and $\{w(t)\}$ on the same probability space such that with probability approaching one as $n \to \infty$, the distance in $\mathbb{T}^2$ between $w_{\mathbb{T}^2}(t)$ and $X_{\lfloor mt \rfloor}$ is for $m = 2n^2$ and $t \le 2(\log n)^2$ at most $\delta_n = (\log m)^3/\sqrt{m}$. Fixing $\gamma > 0$, since $\delta_n \ll n^{\gamma-1} = \varepsilon_n$ and $w_{\mathbb{T}^2}$ completely misses some disc $D_{\mathbb{T}^2}(x, \varepsilon_n)$ in $\mathbb{T}^2$ till time $\boldsymbol{C}_{\varepsilon_n}$, we deduce that (X_k) completely misses a disc whose radius is about ε_n till time $2n^2\boldsymbol{C}_{\varepsilon_n}$. That is, $C_n(\gamma)$ is about $2n^2\boldsymbol{C}_{\varepsilon_n}$. Taking $\gamma \to 0$ then provides the lower bound in (1.2). The matching upper bound on C_n is easily obtained upon considering the expected number of sites that are unvisited during the first $\alpha(4/\pi)(n \log n)^2$ steps of the SRW on $\mathbb{Z}_n^2$, with $\alpha > 1$ fixed and $n \to \infty$.

The derivation of (1.3) follows a similar path, where setting $\varepsilon_m = m^{\gamma-1}$ for $\gamma > 0$ one first checks that the lower bound in (4.3) applies also for the ε_m-cover time of the disc $D_{\mathbb{T}^2}(0, br_m)$ when $r_m = c(\log m)^{-3}$ (with b, c fixed and m large). For $R > 0$ sufficiently small, the expected time it takes $w_{\mathbb{T}^2}$ to complete an excursion between the discs of radii r_m and R, centered at the origin, is about $(1/\pi)\log(R/r_m)$. By a concentration argument the number of such excursions made by $w_{\mathbb{T}^2}$ till its ε_m-cover time of $D_{\mathbb{T}^2}(0, br_m)$ is thus about $(2/\pi)(\log \varepsilon_m)^2/[(1/\pi)(\log R/r_m)]$. For $1/2 < b < 1$ and $c < R/b$ this implies by a strong approximation argument that with high probability the planar SRW makes at least $(1-\gamma)^3 2(\log m)^2/(3 \log\log m)$ excursions between the discs of radii $m(\log m)^3$ and $2m$, centered at the origin, till it covers the concentric disc of radius m (cf. [DPRZ04, Lemma 5.1]). A similar argument leads to the matching upper bound. The limit distribution of (1.3) then follows upon studying the tail probabilities for the time it takes the planar SRW to complete one such excursion for large value of m (cf. [Law92]). By a similar technique, [HP06] establishes the law of the iterated logarithm which corresponds to (1.3). That is,

$$\limsup_{n\to\infty} \frac{(\log \rho_n)^2}{\log n \log_3 n} = \frac{1}{4}, \quad \text{a.s.}$$

(where $\log_3$ denotes three iterations of the log function).

Jonasson and Schramm proved in [JS00] that if $G = (V, E)$ is a planar graph of maximal degree D then there are constants k_D and K_D depending only on D such that $k_D|V|(\log|V|)^2 \leq \mathbb{E}[C_G] \leq K_D|V|^2$. As illustrated already, (1.2) is a direct consequence of (4.3), with the same argument applicable for finding the asymptotics of the cover time for SRW on different planar lattices. As explained in [DPRZ04, Section 9], this leads to

Open Problem 4.1. Is the lattice $\mathbb{Z}^2$ asymptotically the easiest to cover when $D = 4$? That is, does $\liminf \mathbb{E}[C_G]/(|V|(\log|V|)^2) = 1/\pi$, where the lim inf is over all planar graphs $G = (V, E)$ of maximal degree $D = 4$ and for $|V| \to \infty$?

The limit distribution of the cover time C_n for the SRW on the lattice torus $\mathbb{Z}_n^2$ and similar random fluctuations are likely related to behavior such as that of branching Brownian motion, and hence to KPP-type partial differential equations (cf. [Ald91], [Bra83], [Bra86], [McK75]). However, very little is known about it. For example,

Open Problem 4.2. Does there exist a non-random sequence b_n such that $b_n(\sqrt{C_n} - \text{Med}\{\sqrt{C_n}\})$ converges in distribution to a non-degenerate random variable, and if so what is the limit distribution?

Even the existence of a non-random normalizing sequence b_n that results with a tight, yet non-degenerate collection, is not obvious. See [BZ06] for a proof of tightness for the simpler problem in the context of SRW on the regular trees Γ_h.

For any $0 < \alpha < 1$, let $R_n(\alpha)$ denote the radius of the largest disc (of arbitrary center) consisting of α-favorite points for the SRW on $\mathbb{Z}^2$. The proof of (1.4) combines the ideas behind the proofs of (1.1) and (1.2). Similarly, based on (2.1) and (4.1), it is also shown in [DPR07] that for any $0 < \alpha < 1$, with probability one,

$$\lim_{n\to\infty} \frac{\log R_n(\alpha)}{\log n} = \frac{1-\sqrt{\alpha}}{4}. \tag{4.4}$$

Indeed, with $h_k = [ck \log k]$ and $r_k = e^{h_k}$, it takes about r_m^2 steps for the SRW to first exit $D(0, r_m)$. Hence, taking $r_{m,k} = e^{h_m - h_k}$, (4.4) follows by showing that with probability one, if $\zeta = (b\beta - \sqrt{\alpha})/(1-\beta) > 1$ for some $b < 1$ and m is large enough, there exists $x \in D(0, r_m)$ such that each $z \in D(x, r_{m,\beta m})$ is visited at least $(4\alpha/\pi)h_m^2$ times prior to the first exit of the SRW from $D(0, r_m)$, whereas no such x exists if $\zeta < 1$ for some $b > 1$. To this end, for each $a < 2$, by strong approximation and the outline of the proof of the lower bound for (2.5), we have that for some $x \in D(0, r_m)$ and $n_k(a) = ah_k^2/(c \log k)$, prior to exiting $D(0, r_m)$ the SRW completes at least $n_k(a) - k$ excursions between $D(x, r_{m,k})$ and the complement of $D(x, r_{m,k-1})$. Thus, considering $k = \beta m - 1$ and $b < \sqrt{a/2}$ for which $\zeta = \zeta(b, \beta, \alpha) > 1$, for m large enough and any $z \in D(x, r_{m,\beta m})$, the SRW completes at least $K = 2(b\beta)^2 h_m^2/(c \log m)$ excursions between $D(z, R)$ and the complement of $D(z, R')$ prior to exiting $D(0, r_m)$ (say, for $R = 2r_{m,\beta m-1}$ and $R' = 0.5r_{m,\beta m-2}$). Let $\widehat{\mathcal{L}}_K(\alpha)$ denote the collection of lattice sites $z \in D(x, r_{m,\beta m})$ visited less than $(4\alpha/\pi)h_m^2$ times during the first K excursions of the SRW between $D(z, R)$ and the complement of $D(z, R')$. Since there are only $\exp(2(1-\beta)h_m(1+o(1)))$ lattice sites in $D(x, r_{m,\beta m})$, upon showing that

$$\mathbb{P}(z \in \widehat{\mathcal{L}}_K(\alpha)) \leq e^{-2(1-\beta)h_m\zeta^2(1+o(1))}, \tag{4.5}$$

it follows that $\mathbb{E}|\widehat{\mathcal{L}}_K(\alpha)| \to 0$ as $m \to \infty$ sufficiently fast to produce the lower bound in (4.4).

The bound (4.5) is obtained by a large deviations estimate of the following type. If T_i are i.i.d. random variables, such that $\mathbb{P}(T_1 > t) \leq Qe^{-t/M}$ for all $t > 0$, then for $\gamma < 1$ and $\lambda = (\gamma^{-1} - 1)/M > 0$,

$$\mathbb{P}\Big(\sum_{i=1}^{K} T_i \leq \gamma^2 KQM\Big) \leq e^{\lambda\gamma^2 KQM}\mathbb{E}[e^{-\lambda T_1}]^K \leq e^{-(1-\gamma)^2 KQ}. \tag{4.6}$$

The K excursions of the SRW between $D(z, R)$ and the complement of $D(z, R')$ are approximately independent of each other. Further, by potential theory estimates for the SRW (cf. [Law91]), the probability Q that during such an excursion the SRW visits z is about $(\log(R'/R))/\log R'$ which in turn is about $(c \log m)/((1-\beta)h_m)$. By similar reasoning, upon visiting z at least once, the mean number of returns to z by the SRW during such an excursion, denoted M is about $\frac{2}{\pi}\log R'$, and the number T_i of such returns to z during the i-th excursion is such that $\mathbb{P}(T_i > t) \leq Qe^{-t(1+o(1))/M}$ for all $t > 0$. Thus, with KQ being about $2(b\beta)^2 h_m/(1-\beta)$ and KQM about $(4(b\beta)^2/\pi)h_m^2$, setting $\gamma = \sqrt{\alpha}/(b\beta) < 1$ in (4.6) leads to (4.5).

As for the matching upper bound in (4.4), by the preceding reasoning, for each $a > 2$, with probability that is fast approaching one (in m) for every $x \in D(0, r_m)$ the SRW completes fewer than $n_k(a)$ excursions between $D(x, r_{m,k})$ and the complement of $D(x, r_{m,k-1})$ by the time it exits $D(0, r_m)$. Further, if $\zeta < 1$ for some $b > 1$ then for $k = \beta m - 1$ the expected size of the set $\widehat{\mathcal{L}}_K(\alpha)$ of $z \in D(x, r_{m,k+1})$ with less than $(4\alpha/\pi)h_m^2$ visits by the SRW during its first $K = n_k(2b^2)$ excursions between $D(x, r_{m,k})$ and the complement of $D(x, r_{m,k-1})$ diverges as $m \to \infty$. A truncated multi-scale second moment argument similar to that of Subsection 4.2 then allows us to deduce that for each fixed x, with probability approaching one (in m), the set $\widehat{\mathcal{L}}_K(\alpha)$ is non-empty, thus completing the proof of (4.4).

4.2. Late points for SRW on regular trees. The cover time problem (4.3) is, in a sense, dual of (2.5), in that it replaces "extremely large" occupation measure by "extremely small" occupation measure. Indeed, the derivation in [DPRZ04] of the lower bound for (4.3) is based on another toy problem involving the SRW on a regular tree. In this case it is the asymptotics of the number $\widetilde{C}_h$ of returns to $\mathbf{o}$ by the SRW on Γ_h, starting at $\mathbf{o}$, till it visits all leaves of the tree.

Both [Ald91] and [Per03] show that $\mathbb{E}\widetilde{C}_h = h^2 b(1 + o(1)) \log b$ as $h \to \infty$. However, these proofs rely on an embedded branching process argument exploiting the tree structure of Γ_h and as such are not suitable to deal with the corresponding

Brownian result (4.3). We explain next how the multi-scale truncated second-moment provides another derivation of the asymptotic of $\widetilde{C}_h$ which is *robust enough* to be adapted in [DPRZ04] to deal with the Brownian motion setting of (4.3).

Turning to deal with $\widetilde{C}_h$, fixing $\alpha > 0$ we say that a leaf x of Γ_h is α*-late* if the number $\widetilde{R}_x$ of returns to $\mathbf{o}$ by the SRW on Γ_h till its first visit to $x \in \partial\Gamma_h$ is at least $\alpha h^2 b \log b$. Starting at $\mathbf{o}$, the probability that the SRW visits a specific leaf x before returning to $\mathbf{o}$ is $1/(bh)$. Hence $\mathbb{P}(\widetilde{R}_x \geq t) = \left(1 - \frac{1}{bh}\right)^t$, yielding the first moment estimate

$$\mathbb{E}(|\mathcal{L}_h(\alpha)|) = b^{h(1-\alpha)(1+o(1))},$$

for the set $\mathcal{L}_h(\alpha)$ of α-late leaves of Γ_h. By the first moment method, this shows that for any $\alpha > 1$ the set $\mathcal{L}_h(\alpha)$ is empty with high probability. Since $\widetilde{C}_h$ is the maximum of $\widetilde{R}_x$ over $x \in \partial\Gamma_h$, we deduce that $\widetilde{C}_h$ is about $h^2 b \log b$ upon showing that for each $0 < \alpha < 1$,

$$\lim_{h\to\infty} \frac{1}{h} \log |\mathcal{L}_h(\alpha)| = (1-\alpha)\log b \quad \text{in probability.} \tag{4.7}$$

As in the derivation of the asymptotic (3.1) for the number of α-favorite leaves, the second moment of $|\mathcal{L}_h(\alpha)|$ is much larger than $b^{2h(1-\alpha)(1+o(1))}$. Let $h_k = [ck \log k]$ and $n_k = \alpha h_k^2 \log b/(c \log k)$. Then, adapting the approach taken in Subsection 3.1, the appropriate way of truncating $\mathcal{L}_h(\alpha)$ is by considering for $h \in [h_{m+\rho}, h_{m+\rho+1})$ and $v \in \Gamma_h$ of height $h - h_m + 1$, the subset $\mathcal{S}^v_{h_m}(\alpha)$ of leaves x in the subtree rooted at v such that $N^x_2 = 0$ and $|N^x_k - n_k| \leq k$ for $k = 3, \ldots, m-1$. In duality with the case of α-favorite leaves, here we set $x_1 = x$ and for $k = 2, \ldots, m$ the vertex x_k is at distance $h_k - 1$ from the *leaf* x along the ray $v \leftrightarrow x$ (so that now $x_m = v$). We then have N^x_k count the number of complete excursions between x_{k-1} and x_k which occur during the first n_m excursions of the SRW between vertices x_{m-1} and x_m.

Omitting the detailed computations for this case, we note in passing that similarly to the derivation of (3.3), here $\bar{q}_m = \mathbb{P}(x \in \mathcal{S}^v_{h_m}(\alpha)) = b^{-\alpha h_m(1+o(1))}$, yielding the desired asymptotic growth of $\mathbb{E}(|\mathcal{S}^v_{h_m}(\alpha)|)$. Further, similarly to the derivation of (3.4), here we have that

$$\bar{q}_{m,l} := \sup_{x_{l-1}\neq y_{l-1}} \mathbb{P}\big(x \in \mathcal{S}^v_{h_m}(\alpha) \;\text{ and }\; y \in \mathcal{S}^v_{h_m}(\alpha)\big) \leq Cl\bar{q}_m\bar{q}_{l-1},$$

for some $C < \infty$ and $l = 2, \ldots, m$ (with the convention of $\bar{q}_1 = \bar{q}_2 = 1$). Although $m \mapsto \mathbb{E}(|\mathcal{S}^v_{h_m}(\alpha)|^2)/\mathbb{E}(|\mathcal{S}^v_{h_m}(\alpha)|)^2$ is now unbounded, its polynomial growth is easily accommodated by considering the sum $|\mathcal{S}^*_h(\alpha)|$ of the b^{h-h_m+1} i.i.d. random variables $|\mathcal{S}^v_{h_m}(\alpha)|$, provided ρ is large enough.

To complete the derivation of (4.7) it thus remains only to show that with high probability $\mathcal{S}^*_h(\alpha) \subseteq \mathcal{L}_h(\beta)$ for $\beta < \alpha$. To this end, recall that the condition $N^x_2 = 0$ guarantees that $x \in \mathcal{S}^v_{h_m}(\alpha)$ is not visited by the SRW during its first n_m excursions between the vertex v at level $h - h_m + 1$ and a specific descendent $u = u(x)$ of v which is Δ_m levels further from $\mathbf{o}$. A concentration argument similar to that of

[Dem05, Lemma 4.7] shows that the probability that the number of returns to **o** during the first n_m excursions between such pair v and u is less than $(1-\delta)n_m b\Delta_m$, decays exponentially in n_m. Since $n_m b\Delta_m = \alpha b h^2(1+o(1))\log b$ it follows that $\mathbb{P}(\mathcal{S}_h^*(\alpha) \subseteq \mathcal{L}_h(\beta)) \to 1$ for any $\beta < \alpha$ and $h \to \infty$, as required.

4.3. Clustering of late points on $\mathbb{Z}_n^2$. Simulations of the SRW on $\mathbb{Z}_n^2$ reveal that the points that are visited late by the walk appear in clumps of various sizes. Motivated by [BH91], the geometric characteristics of these clumps are studied in [DPRZ06]. More precisely, with τ_x denoting the first hitting time of x by the SRW on $\mathbb{Z}_n^2$, it is shown in [DPRZ06] that for $\alpha \in (0,1]$ the set

$$\mathcal{L}_n(\alpha) = \{x \in \mathbb{Z}_n^2 : \ \tau_x \geq \alpha(4/\pi)(n\log n)^2\},$$

of α-*late points* for the SRW has typical size $n^{2(1-\alpha)+o(1)}$ and that for any fixed x in $\mathbb{Z}_n^2$ and $0 < \beta < 1$,

$$\lim_{n\to\infty} \frac{\log|\mathcal{L}_n(\alpha) \cap D(x, n^\beta)|}{\log n} = 2\beta - 2\alpha/\beta \quad \text{in probability} \tag{4.8}$$

(with $D(x,n^\beta)$ the disc of radius n^β, centered at $x \in \mathbb{Z}_n^2$). If the points of $\mathcal{L}_n(\alpha)$ were approximately evenly spread out in $\mathbb{Z}_n^2$, then the number of α-late points in $D(x,n^\beta)$ would be $n^{2\beta-2\alpha+o(1)}$, whereas (4.8) shows that there are significantly less of them (as $2\beta - 2\alpha/\beta < 2\beta - 2\alpha$). The clustering pattern of $\mathcal{L}_n(\alpha)$ is confirmed by another result of [DPRZ06], showing that for any $0 < \alpha, \beta < 1$, if Y_n is chosen uniformly in $\mathcal{L}_n(\alpha)$ then,

$$\lim_{n\to\infty} \frac{\log|\mathcal{L}_n(\alpha) \cap D(Y_n, n^\beta)|}{\log n} = 2\beta(1-\alpha) \quad \text{in probability.} \tag{4.9}$$

Counting pairs of late points one is tempted to apply the approximations

$$\begin{aligned}
&\mathbb{E}[\#\text{ of pairs of }\alpha\text{-late points within distance } n^\beta \text{ of each other}] \\
&\quad\simeq n^2 n^{2\beta}\mathbb{P}(x, y \ \text{ are } \alpha\text{-late when } |x-y| \simeq n^\beta) \\
&\quad\simeq (\text{Typical value of such number of pairs}) \\
&\quad\simeq (\#\text{ discs in } n^\beta \text{ grid with } \alpha\text{-late points}) \\
&\qquad \times (\text{Typical value of } |\mathcal{L}_n(\alpha) \cap D(x,n^\beta)| \text{ when } x \text{ is } \alpha\text{-late})^2.
\end{aligned} \tag{4.10}$$

However, as seen in [DPRZ06], such approximations fail to hold and these three quantities exhibit different power growth exponents.

All preceding results about the clustering of α-late points are derived in [DPRZ06] by the same truncated multi-scale second moment method as in Subsection 4.2. In contrast to the results of [DPRZ04] about cover times and covered discs, the derivation of (4.9) requires conditioning upon the first hitting time at a *point* $x \in \mathbb{Z}_n^2$ for which strong approximation theorems are ineffective. This is handled in [DPRZ06] by appealing to potential theory estimates for SRW (cf. [Law91]) and relying on the fact that only rough approximations of the probability that $|N_k^x - n_k| \leq k$ are required. The derivation of the power growth exponent for (4.10) is more technically challenging since the mean of this object is already off its typical value. Further, the dominant contribution is from pairs of α-late points having significantly less excursions between discs at the intermediate scale n^β (in comparison with the typical excursion count profile for α-late points), forcing an accumulation of many α-late points inside such a disc. Thus, the evaluation of this power growth exponent is by a large deviations analysis of various excursion counts, similar in spirit to that of (4.5).

Not much else is known about the late points. For example,

Open Problem 4.3.
• In [DPRZ06] the power growth exponent of pairs of α-late points within distance n^β of each other is computed. Extend this to a "full multi-fractal analysis". For example, find the power growth exponent of triplets (x_1, x_2, x_3) of α-late points, such that x_i is within distance n^β of x_j for $i, j = 1, 2, 3$.
• What is the distribution of the distance between the last two points to be covered by the SRW in $\mathbb{Z}_n^2$? In particular, does the chance that they are adjacent go to zero as n grows?

Open Problem 4.4. Adapting the proof of (4.3) one can show that for any $a \leq 2$,

$$\dim \left\{ x \in \mathbb{T}^2 : \limsup_{\varepsilon \to 0} \frac{\tau(x, \varepsilon)}{(\log \varepsilon)^2} = \frac{a}{\pi} \right\} = 2 - a \quad \text{a.s.} \tag{4.11}$$

It is not clear what to do when the lim sup in (4.11) is replaced by a limit or a lim inf, since in this case we can no longer avoid considering the stochastic behavior of $\tau(x, \varepsilon)$ across different scales (i.e. ε values) which are highly dependent when the scales are close to each other.

5. Intersection local times and Gaussian free fields

5.1. Intersections and processes with jumps. While having the Markov property is of much help for the results described here, [DPRZ02] deals with thick points for intersection of planar sample path, whereby it is partially lost. For example, [DPRZ02, Theorem 1.4] provides the analog of (2.4), showing that for any $0 < a \leq 1$,

$$\dim \left\{ x : \lim_{r \to 0} \frac{\mathcal{I}_{\bar{\theta}, \bar{\theta}'}(D(x, r))}{r^2 (\log r)^4} = a^2 \right\} = 2 - 2a \quad \text{a.s.} \tag{5.1}$$

where $\mathcal{I}_{T,T'}(A)$ denotes the projected intersection local time of two independent planar Brownian motions $(w(t), 0 \leq t \leq T)$ and $(w'(t'), 0 \leq t' \leq T')$, normalized by factor π (see [LeG92, Chapter VIII] for more on $\mathcal{I}_{T,T'}(\cdot)$ and its properties). By strong approximation this leads to the analogs of (1.1) and (2.1) for the intersections of two independent SRW on $\mathbb{Z}^2$ (cf. [DPRZ02, Theorem 1.1]). The lower bound in (5.1) is proved by first constructing for $\beta < 2-a$, along the lines of Subsection 3.2, a non-zero random measure ν'_∞ of finite β-energy (cf. (3.5)), that is supported on a closed set of a-thick points for $w'([0,\bar{\theta}'])$. The same construction is then repeated for $w([0,\bar{\theta}])$, now using the squares $S(m,i)$ whose centers are in $\mathcal{S}_{h_m}(a)$ to define the density of the random measure ν_m with respect to ν'_∞, instead of with respect to Lebesgue measure. Fixing $\gamma < \beta - a < 2-2a$, the non-zero weak limit point of $\{\nu_m\}$ is then of finite γ-energy and shown in [DPRZ02, Section 4.1] to be supported on a closed subset of the a^2*-thick intersection points* of (5.1). This strategy works since $\mathcal{I}_{\bar{\theta},\bar{\theta}'}(A)$ is a continuous additive functional for $w([0,\bar{\theta}])$ with Revuz measure $\pi\rho$ such that $\rho(B) = \mu^{w'}_{\bar{\theta}'}(A\cap B)$; for almost every path $w'([0,\bar{\theta}'])$, the accumulation on $D(x,r)$ of such an additive functional during one excursion of $w(t)$ between $D(x,R)$ and the complement of $D(x,R')$ has a mean value $\rho(D(x,r))\log(R'/R) \pm Cr^2(\log r)^2$ and exponentially decaying tail probabilities (cf. [DPRZ02, Lemma 2.3]).

With a slightly different approach of directly controlling the excursion counts for two random walks, [DPR07, Theorem 1.3] shows that the radius of the largest discrete disc in the intersection of the sample path of two independent SRW on $\mathbb{Z}^2$, each run for n steps, is $R_{n,2} = n^{1/(2+2\sqrt{2})+o(1)}$. However,

Open Problem 5.1. The growth rate of the diameter $D_{n,2}$ of the largest connected component of the intersection of two independent planar simple random walk path, each run for n steps, is not known. A related open problem is to determine whether the intersection of two independent planar Brownian motion path, each run for a unit time, is almost surely a totally disconnected set.

The results described here depend mostly on the local properties of the stochastic processes considered. They are thus not limited to Brownian motion or to random walks. In particular, sample path continuity is not essential. For instance, Daviaud [Dav05] considers the Cauchy process on $\mathbb{R}$, that is, a stochastic process $X(t)$ with $X(0) = 0$ and stationary independent increments $X(t+s) - X(t)$ each of whom has the Cauchy density $s/(\pi(s^2+x^2))$ with respect to Lebesgue measure. This is a recurrent process, whose Green's function has a logarithmic behavior, similar to that of the planar Brownian motion (cf. [Dav05, Proposition 1.3]), but which has infinitely many jumps. In analogy with (2.4), [Dav05] shows that for $0 \le a \le 1$,

$$\dim\{x : \lim_{r\to 0} \frac{\mu^X_{\bar{\theta}}(D(x,r))}{r(\log r)^2} = \frac{2}{\pi}a\} = 1-a \quad \text{a.s.}$$

where $D(x,r)$ is an interval of radius r and center $x \in \mathbb{R}$ and $\bar{\theta} = \inf\{t : |X(t)| \ge 1\}$. To avoid the technical difficulties due to jumps, [Dav05] relies on the representation of the Cauchy process $X(t)$, up to a well understood time change, as the intersection of the planar Brownian motion $w(t)$ and, say, the x-axis, thereby adapting the strategy of [DPRZ02] to the case at hand.

5.2. The Gaussian free field. The discrete d-dimensional Gaussian free field (abbreviated GFF) on the square $V_n = \{1, \dots, n\}^d \subset \mathbb{Z}^d$ is a Gaussian random vector $(\phi_x,\ x \in V_n)$ of zero mean and covariance given by the Green's function of the SRW restricted to V_n. That is, $\mathbb{E}[\phi_x \phi_y]$ is the expected number of visits to y by the SRW on $\mathbb{Z}^d$, starting at x and run till its first exit from V_n. The GFF (also called the harmonic crystal) is a special case of the *solid on solid* model used in statistical physics to describe the effective interface between two phases at low temperature (cf. [Gia01], [Fun05] and the references therein).

In this context, the presence of a hard wall is manifested by the *entropic repulsion* conditioning on the non-negativity constraint $\Omega^+_{n,\varepsilon} = \{\phi_x \geq 0 : x \in V_{n,\varepsilon}\}$, for small $\varepsilon > 0$, where $V_{n,\varepsilon}$ denotes the subset of points in V_n of distance at least $n\varepsilon$ to the boundary of V_n. For $d \geq 3$ it is well known that entropic repulsion pushes the GFF far from the wall, while asymptotically making no other changes to its law (cf. [Gia01, Chapter 3] and the references therein). Both [BDG01] and [Dav06] consider the effect of entropic repulsion on the two dimensional GFF. To this end, [BDG01, Theorem 2] shows that $\max_{x \in V_n} \phi_x$ grows with n like $g_n = 2\sqrt{2/\pi} \log n$, from which they deduce that upon conditioning on $\Omega^+_{n,\varepsilon}$, the GFF is shifted by g_n, that is, $g_n^{-1} \phi_x \to 1$ uniformly on $V_{n,\varepsilon}$.

Theorem 2 of [BDG01] is derived by a multi-scale truncated second moment approach which is motivated by the similarity between the GFF and a branching random walk type model on regular trees. Whereas the profile of excursion counts along each ray is the key object of study in Section 3.1, the approach of [BDG01] is to consider a notion of success shared by all vertices of Γ_h at a given height (i.e. distance from $\mathbf{o}$). Adapted to the context of (3.1), the h_k-th level of Γ_h is successful if for enough vertices x_k of Γ_h at height h_k, the SRW completed by time $\tau_{\mathbf{o}}$ at least $n_k - k$ excursions between x_k and its ancestor x_{k-1} (at height h_{k-1}). Starting at distance δh from $\mathbf{o}$ with enough independence to have sufficiently many such vertices at distance h_2 further from $\mathbf{o}$, the success of the h_k-th level propagates (in k) by counting the excursions to vertices at height h_{k+1} during the first $n_k - k$ excursions between their ancestors at heights h_{k-1} and h_k. By controlling the union over the probabilities of failure at the different steps $k = 2, 3, \dots, m$ of this process, one concludes that with high probability, the last step, consisting of $\partial \Gamma_h$, is successful.

Pursuing the same approach, [Dav06] goes further in relating the conditioned GFF with the shifted GFF by deriving in this context the analogs of the results of [DPRZ06] and [DPR07]. For example, it is shown in [Dav06, Theorem 1.1] that conditioned on $\Omega^+_{n,\varepsilon}$, the largest disc within $V_{n,\varepsilon}$ for which all values of ϕ_x are below ηg_n is of radius $n^{\eta/2(1+o(1))}$. Indeed, for the unconditional GFF this is the radius of the largest disc for which all values of ϕ_x exceed $(1-\eta) g_n$ (see [Dav06, Theorem 1.7]), a result which is the analog of (4.4). The transformation $\eta = \sqrt{\alpha}$ between the two has to do with the isomorphism between $\phi_x^2/2$ and the local time at x of a continuous time planar SRW.

Open Problem 5.2. The results of [Dav06] suggest the possibility of simpler proofs in the SRW world by proving the corresponding results for the GFF and applying an isomorphism theorem. Can you find such a proof for (1.1)?

We note in passing that "level lines" of the continuous two dimensional GFF are intimately related to the conformally invariant Schramm–Loewner evolution (abbreviated SLE). See [She06] for a survey of the continuous GFF, [Wer04] for the SLE and its application for computing intersection exponents of independent Brownian motions and [SS06] for the convergence of the zero level interface of an interpolated GFF with appropriate boundary values to variants of the SLE(4) process. However, though the path of the planar Brownian motion is also conformally invariant, we do not know of any direct relation between the results presented here and the SLE.

6. Disconnection of cylinders by random walks

Let (X_k) denote the SRW on the infinite discrete cylinder $G_n = \mathbb{Z}_n^d \times \mathbb{Z}$ (endowed with its natural graph structure), which starts at $X_0 = 0$. As X_k is an irreducible, recurrent Markov chain, it is easy to see that the disconnection time $D_n = \inf\{k \geq 0 :\ X_{[0,k]}$ disconnects $G_n\}$ is almost surely finite. Further, $C_n \leq D_n \leq \widehat{C}_n$, where C_n denotes the first time the projection of $X_.$ has visited all points of the base $\mathbb{Z}_n^d$ and $\widehat{C}_n$ denotes the cover time of the slice $\mathbb{Z}_n^d \times \{0\}$ by the SRW on G_n. When $d = 1$, it is straightforward to argue that D_n is roughly of order n^2 and comparable to C_n (and to $\widehat{C}_n$). Indeed, by the general theory of Markov chains one knows that for any $d \geq 1$, the sequence $\log C_n / \log n$ converges in probability to $\max(d, 2)$ (as we have seen already, much more in known about C_n). From (1.5) we see a different behavior for $d \geq 2$ in which case $D_n = n^{2d+o(1)}$ is much larger than C_n.

The upper bound in (1.5) is quite simple to prove. It is based on the fact that $D_n \leq \widehat{C}_n$ and a relatively crude upper bound on the cover time $\widehat{C}_n$ of the slice $\mathbb{Z}_n^d \times \{0\}$. Indeed, fixing $\beta > d - 1$, though the hitting times of the sites on this slice are of infinite mean, the first moment method works for the number Z_n of non-visited sites on it during the first n^β excursions of the SRW between the truncated cylinders $\mathbb{Z}_n^d \times [-n, n]$ and $\mathbb{Z}_n^d \times [-2n, 2n]$. For large n it gives with high probability an upper bound on $\widehat{C}_n$ by the time it takes the walk to make these n^β excursions, which in turn is bounded above by n^γ for fixed $\gamma > 2(\beta + 1)$.

Somewhat surprisingly, this rather primitive strategy of replacing D_n by $\widehat{C}_n$ captures the correct rough order of magnitude of D_n. However, the lower bound is more delicate because a direct enumeration over the huge collection of possible disconnecting subsets of G_n seems to lead nowhere. Instead, we find a robust geometric property that every disconnecting $\Gamma \subseteq G_n$ must have, then show that with high probability $X_{[0,n^{2d-\delta}]}$ lacks this property. More precisely, [DS06, Lemma 2.4] shows that for $\gamma \in (0, 1)$ fixed and any n large enough, if Γ disconnects G_n then there exists a box of side length n^γ in G_n that contains at least order of $n^{d\gamma}$ points of Γ. This is a purely combinatorial argument, based on the following isoperimetric inequality [DP96, (A.3)]:

For any $\varepsilon > 0$ and finite box $B \subset \mathbb{Z}^{d+1}$,

$$|A \cap B| \le (1-\varepsilon)|B| \implies |\partial_B(A \cap B)| \ge \delta |A \cap B|^{d/(d+1)}, \tag{6.1}$$

where $\partial_B(U)$ denotes the points of $B \setminus U$ within distance one of U and the positive constant δ depends only on ε and $d \ge 1$. As shown in [DP96], the inequality (6.1) is a direct consequence of the Loomis–Whitney inequality bounding the size of any finite set $A \subset \mathbb{Z}^{d+1}$ by the d-th root of the product of sizes of the $(d+1)$ projections of A on the hyperplanes perpendicular to the coordinate axes (cf. [LW49, Theorem 2]).

The preceding combinatorial argument is complemented by a probabilistic analysis involving the excursions between two concentric boxes of side length n^γ and $2n^\gamma$. Fixing $1 > \delta > 3(d-1)\gamma > 0$, it shows that for some finite c_0 the probability that during its first $n^{2d-\delta}$ steps the SRW makes more than $c_0 \log n$ such excursions for some pair of boxes, decays to zero as $n \to \infty$. Scaling the occupation measure of the smaller box during one such excursion by $n^{-2\gamma}$ yields a random variable whose moment generating function is bounded uniformly in n and the excursion's starting point. Consequently, for some finite c_1 depending only on δ and γ, during its first $n^{2d-\delta}$ steps, the number of visits by the SRW to any box of side length n^γ does not exceed $c_1(\log n)\, n^{2\gamma}$. For $d \ge 3$ and large n, this is substantially less than the order of $n^{d\gamma}$ points that the walk must have visited for at least one such box by time D_n (since by definition $X_{[0,D_n]}$ disconnects G_n), producing in this case the lower bound on D_n as stated in (1.5).

Though the argument for $d = 2$ is of a similar flavor, it requires a considerable refinement in order to utilize the much smaller differences, of only a logarithmic growth in n, that we have here. To this end, by similar isoperimetric controls, [DS06, Lemma 2.5] shows that for some finite, positive constants c_i, $i = 2, 3, 4$, and n large enough, if Γ disconnects G_n, then for one of the three two-dimensional coordinate projections there exists a box of side length n^γ and a collection of $c_2(\log n)^{2\alpha}$ disjoint sub-boxes, each of side length $\ell_n = n^\gamma(\log n)^{-\alpha}$, whose centers lie on a common $c_3\ell_n$-sub-grid of this box, such that the projection of the intersection of Γ with any of these sub-boxes, contains at least $c_4\ell_n^2$ points. The stated lower bound on D_n in case $d = 2$ is thus the result of a more careful probabilistic analysis which shows that for γ small and $\alpha < 3/4$ the probability that the set $\Gamma = X_{[0,n^{4-\delta}]}$ has this property, tends to zero as $n \to \infty$.

One consequence of (1.5) is that when $d \ge 2$ and n is large, by the time D_n the walk pretty much fills up the truncated cylinders of height $n^{d-\varepsilon}$. More precisely, with $\rho(x, A)$ denoting the minimal length of a nearest neighbor path from $x \in G_n$ to $A \subseteq G_n$, for any $d \ge 2$, $\varepsilon > 0$ and $\eta > 0$,

$$\lim_{n\to\infty} n^{-\eta} \max_{x \in \mathbb{Z}_n^d \times [-n^{d-\varepsilon}, n^{d-\varepsilon}]} \rho(x, X_{[0,D_n]}) = 0 \quad \text{in probability.} \tag{6.2}$$

This is in contrast with the situation when $d = 1$, where with non-vanishing probability there are points in such a truncated cylinder which are at distance n from $X_{[0,D_n]}$. The clogging effect of (6.2) is a direct consequence of the lower bound $n^{2d-\delta}$ on D_n, due to (1.5), as one can show that within $n^{2d-\delta}$ steps, in a uniform fashion for $\mathbb{Z}_n^d \times [-n^{d-\varepsilon}, n^{d-\varepsilon}]$, the walk comes "often enough" within distance n of x, giving it each time an opportunity to come even closer to x.

Acknowledgment. I thank Yuval Peres, Jay Rosen and Ofer Zeitouni, for valuable feedback on a preliminary version of this manuscript, and the National Science Foundation for funding the research on which it is based.

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